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OPTIMAL FOURTH ORDER FAMILY OF ITERATIVE METHODS

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Abstract. In this work, we construct a family of optimal fourth order iterative methods requiring three evaluations. During each iterative step, methods need evaluation of two derivatives and one function. According to the Kung and Traub conjecture an optimal iterative method without memory based on 3 evaluations could achieve an optimal convergence order of 4. The proposed iterative family of methods are especially appropriate for finding zeros of functions whose derivative is easy to evaluate. For example, polynomial functions and functions defined via integrals.

1. Introduction

According to the Kung and Traub's conjecture an optimal iterative method without memory based on n evaluations could achieve optimal convergence order 2^{n-1} [10]. One of the best known optimal second order method based on two functional evaluations for solving the equation $f(x) = 0$ is the Newton's method which is given as follows (NM)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, 3, \dots, \quad \text{and} \quad |f'(x_n)| \neq 0, \tag{1}$$

[3, 5, 15]. There exists numerous modifications of the Newton's method which improve the convergence rate (see [3, 4, 5, 10, 15] and references therein). We are interested in finding zeros of functions defined via integrals: $f(x) := \int_a^x g(t)\,dt + G(x)$. Clearly for such functions numerical evaluation of derivatives is easier than evaluating function itself if $G(x)$ is a constant or a polynomial function [5].

Let us consider iterative methods which require more derivative evaluations than functions. Many iterative methods are developed by considering various quadrature rules in the Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\,dt. \tag{2}$$

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Weerakoon et al. obtained the following cubically convergent iterative method (WF)

$$x_{n+1} = x_n - \frac{2f(x_n)}{(f'(x_n) + f'(x_n - f(x_n)/f'(x_n)))}, \tag{3}$$

by using the trapezoidal rule in the equation (2) [15]. While Frontini et al. considered the midpoint quadrature rule in (2) and Homieier et al. considered properties of vanishing derivative to obtain the following cubical iterative method (FH)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2\,f'(x_n)))}, \tag{4}$$

[3, 4]. Furthermore Homeier derived the following cubically convergent iterative method (HH)

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right), \tag{5}$$

by using the Newton's theorem (2) for the inverse function $x = f(y)$ [5]. We notice that the quadratic convergent Newton's method require evaluations of one function and one derivative while the cubic methods (3)–(5) require evaluations of two derivatives and one function during each iteration. It was shown by the Homeier et al. that finding zeros of functions defined via integrals is easier by the method (4) than by the optimal Newton method [5]. The cubic methods (3)–(5) require three evaluations. Therefore by the Kung and Traub conjecture they are not optimal [10]. In this work, we develop an optimal family of quartically convergent methods requiring two evaluations of derivatives and one evaluation of function during each iteration. The next section presents our contribution.

2. New optimal fourth order iterative methods

To develop optimal methods requesting evaluations of two derivatives and one function during each iteration, we consider the iterative method

$$\left.\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \sum_{j=1}^{4}\alpha_j\left(\frac{f'(y_n)}{f'(x_n)}\right)^j\right], \end{aligned}\right\} \tag{6}$$

where $\alpha_j \in \mathbb{R}$. We prove that the methods of the preceding family are fourth order convergent through the theorem.

THEOREM 1. *Let a sufficiently smooth function $f\colon D \subset \mathbb{R} \to \mathbb{R}$ has a simple root $\gamma \in D$ in the open interval D. Then the methods of the iterative family (6) are at least fourth order convergent if $\alpha_1 := 21/8 - \alpha_4$, $\alpha_2 := -9/2 - 3\alpha_4$ and $\alpha_3 := 15/8 - 3\alpha_4$. Here, α_4 is a free real parameter. The methods of the family satisfies the error equation*

$$e_{n+1} = \left[\left(\frac{64}{27}\,\alpha_4 + \frac{85}{9}\right){c_2}^3 - c_2c_3 + \frac{1}{9}\,c_4\right]e_n^4 + O(e_n^5).$$

Proof. The Taylor series expansion of $f(x)$ and $f'(x_n)$ around the solution γ is given as

$$f(x_n) = f'(\gamma)\left(e_n + c_2{e_n}^2 + c_3{e_n}^3 + c_4{e_n}^4\right) + O\left({e_n}^5\right), \tag{7}$$

$$f'(x_n) = f'(\gamma)\left(1 + 2\,c_2 e_n + 3\,c_3{e_n}^2 + 4\,c_4{e_n}^3\right) + O\left({e_n}^4\right). \tag{8}$$

Using the equations (7) and (8) into the first step of the proposed method (6), we obtain

$$y_n - \gamma = 1/3 e_n + 2/3\,c_2{e_n}^2 + \left(4/3\,c_3 - 4/3\,{c_2}^2\right){e_n}^3 + O\left({e_n}^4\right), \tag{9}$$

the Taylors series expansion of $f'(y_n)$ around the solution γ

$$f'(y_n) = \sum_{k=1}^{\infty} c_k (y_n - \gamma)^k, \tag{10}$$

from the preceding equation and the equation (9), we obtain

$$f'(y_n) = f'(\gamma) + \frac{2}{3} f'(\gamma) c_2 e_n + \frac{1}{3} f'(\gamma)\left(4\,{c_2}^2 + c_3\right){e_n}^2 - \frac{4}{27} f'(\gamma)\left(-27\,c_3c_2 + 18\,{c_2}^3 - c_4\right){e_n}^3 + O\left({e_n}^4\right). \tag{11}$$

Dividing equations (11) and (12) yields

$$\frac{f'(y_n)}{f'(x_n)} = 1 - \frac{4}{3}\,c_2 e_n + \left(4\,{c_2}^2 - \frac{8}{3}\,c_3\right){e_n}^2 + \left(\frac{40}{3}\,c_3c_2 - \frac{32}{3}\,{c_2}^3 - \frac{104}{27}\,c_4\right){e_n}^3 + O\left({e_n}^4\right). \tag{12}$$

Finally substituting from equations (7), (8) and (12) into the second step of the proposed method (6)

$$\begin{aligned} e_{n+1} = {} & \gamma + (-\alpha_1 - \alpha_2 - \alpha_4 - \alpha_3)\,e_n + \frac{1}{3}\,c_2\,(19\,\alpha_4 + 7\,\alpha_1 + 15\,\alpha_3 + 11\,\alpha_2 + 3)\,{e_n}^2 \\ & + \Big(\Big(\frac{38}{3}\,\alpha_4 + 10\,\alpha_3 + 14/3\,\alpha_1 + \frac{22}{3}\,\alpha_2 + 2\Big)c_3 \\ & + \Big(-34\,\alpha_4 - \frac{130}{9}\,\alpha_2 - \frac{22}{3}\,\alpha_1 - \frac{70}{3}\,\alpha_3 - 2\Big){c_2}^2\Big){e_n}^3 \\ & + \Big(\frac{497}{27}\,\alpha_4c_4 + \frac{289}{27}\,\alpha_2c_4 + \frac{131}{9}\,\alpha_3c_4 + 3\,c_4 + 4\,{c_2}^3 - \frac{77}{3}\,\alpha_1c_2c_3 + \frac{185}{27}\,\alpha_1c_4 \\ & - 7\,c_3c_2 + \frac{2584}{27}\,\alpha_3{c_2}^3 + \frac{460}{9}\,\alpha_2{c_2}^3 + \frac{4252}{27}\,\alpha_4{c_2}^3 + \frac{64}{3}\,\alpha_1{c_2}^3 - \frac{253}{3}\,\alpha_3c_3c_2 \\ & - \frac{463}{9}\,\alpha_2c_3c_2 - \frac{373}{3}\,\alpha_4c_3c_2\Big){e_n}^4 + O({e_n}^5). \end{aligned} \tag{13}$$

In the preceding error equation the second and the third order terms are zero if and only if: $\alpha_1 = 21/8 - \alpha_4$, $\alpha_2 = -9/2 + 3\alpha_4$ and $\alpha_3 = 15/8 - 3\alpha_4$. Therefore the proposed method is at least fourth order convergent for any real choice of the parameter α_4. This completes our proof. ∎

As positively remarked by the reviewer: the first-step of the proposed method is similar to the first-step of the Jarratt's method, may be referred to as the Jarratt's correction, and to construct optimal fourth order methods Jarratt was the first to introduce the ratio $f'(y_n)/f'(x_n)$. Jarratt's method is given as (JR)

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 - \frac{3}{2}\frac{f'(y_n) - f'(x_n)}{3f'(y_n) - f'(x_n)}\right), \tag{14}$$

[6]. For other fourth and higher order optimal methods, we refer to the admirable literature [2, 8, 9, 11, 12, 13, 14].

4. Numerical Example

All the computations reported here are done in the programming language C^{++} on the Acer Aspire 5720ZG with Intel dual-core processor T2310 and with 2GB of memory. For numerical precision, we are using the C^{++} library ARPREC [1]. The ARPREC library supports arbitrarily high level of numeric precision. In the program, the precision in decimal digits is set at 2005 with the command "mp::mp init(2005)". To evaluate the integral the Tanh − Sinh quadrature scheme is used. For evaluating quadrature to high precision, in the package ARPREC the primary user working precision is set at 600 digits while the secondary working precision is set at 1000 digits. For convergence, it is required that the distance of two consecutive approximations ($|x_{n+1} - x_n|$) be less than ϵ. And, the absolute value of the function ($|f(x_n)|$) also referred to as residual be less than ϵ. Apart from the convergence criteria, our algorithm also uses maximum allowed iterations as stopping criterion. Thus our algorithm stops if (i) $|x_{n+1} - x_n| < \epsilon$, (ii) $|f(x_n)| < \epsilon$, (iii) itr > maxitr. Here, $\epsilon = 1 \times 10^{-300}$, itr is the iteration counter for the algorithm and maxitr = 100. We test the methods (1), (3) and the proposed method (6) for the following functions

$$\begin{aligned}
f_1(x) &= 2\int_0^x \exp t\, \cos t\, dt + 1 + \exp \pi, && \gamma = \pi,\\
f_2(x) &= \int_0^x t \log(t+1)\, dt - 1/4, && \gamma = 1,\\
f_3(x) &= \int_0^x [\exp -t^3/2 - \exp t^8/2]\, dt + 1/10, && \gamma \approx 0.9054.
\end{aligned}$$

To minimize the asymptotic error constant, we choose $\alpha_4 = -255/64$ in the proposed method (PM) (6). Performance of the present method (6), the Newton method (1), the cubic method (3) and well known Jarratt's method (14) is compared in the Tables 1 and 2. The Tables 1 and 2 present distance between two consecutive iteration and CPU time required by the three methods. Computational results show that the proposed method and the Jarratt's method require substantial less time to converge than the optimal Newton method (1) or the third order method (3). Which is expected because evaluation of derivatives for the functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ is computationally less expensive than evaluating functions.

Consequently the optimal methods which require more derivative evaluations than function evaluations are suitable for functions defined via integrals. To compute the CPU time required, we are using the ARPREC command time(null) [1].

$f_1(x)$ with $x_0 = 3.1$				$f_2(x)$ with $x_0 = 2.0$			
NM	**PM**	**WF**	**JR**	**NM**	**PM**	**WF**	**JR**
4.2×10^{-2}	1.4×10^{-1}	8.1×10^{-2}	4.1×10^{-1}	3.7×10^{-1}	8.8×10^{-1}	8.2×10^{-1}	9.3×10^{-1}
9.0×10^{-4}	2.3×10^{-4}	1.2×10^{-5}	1.1×10^{-7}	1.2×10^{-1}	1.1×10^{-1}	1.6×10^{-1}	6.7×10^{-2}
4.0×10^{-7}	2.7×10^{-17}	2.2×10^{-14}	2.3×10^{-27}	1.2×10^{-2}	1.1×10^{-5}	2.9×10^{-3}	8.6×10^{-6}
8.1×10^{-14}	2.0×10^{-80}	5.9×10^{-34}	1.1×10^{-122}	1.3×10^{-4}	3.0×10^{-20}	2.1×10^{-8}	2.4×10^{-24}
1.1×10^{-27}	1.1×10^{-320}	2.0×10^{-101}	2.0×10^{-488}	1.4×10^{-8}	4.4×10^{-80}	7.0×10^{-24}	3.2×10^{-96}
5.3×10^{-54}	*******	7.1×10^{-258}	*******	1.6×10^{-16}	9.4×10^{-320}	5.9×10^{-70}	4.8×10^{-312}
1.6×10^{-107}	*******	4.1×10^{-674}	*******	2.6×10^{-65}	*******	1.9×10^{-159}	*******
1.4×10^{-210}	*******	*******	*******	3.7×10^{-130}	*******	6.9×10^{-457}	*******
9.8×10^{-429}	*******	*******	*******	5.3×10^{-254}	*******	*******	*******
*******	*******	*******	*******	1.2×10^{-534}	*******	*******	*******
time = 113	time = 46	time = 76	time = 41	time = 122	time = 36	time = 79	time = 44

Table 1. $|x_{n+1} - x_n|$ with $n \geq 1$ and total CPU time required for the function $f_1(x)$ and $f_2(x)$.

NM	**PM**	**WF**	**JR**
2.1×10^{-1}	1.0×10^{-1}	7.1×10^{-1}	3.2×10^{-2}
4.9×10^{-2}	1.2×10^{-4}	7.4×10^{-2}	1.2×10^{-6}
6.2×10^{-4}	2.6×10^{-16}	6.6×10^{-7}	8.3×10^{-23}
9.2×10^{-8}	1.2×10^{-65}	3.1×10^{-21}	4.4×10^{-72}
4.1×10^{-16}	9.1×10^{-260}	2.1×10^{-63}	6.1×10^{-280}
7.1×10^{-32}	4.1×10^{-1010}	8.1×10^{-189}	1.4×10^{-1090}
9.1×10^{-64}	*******	4.3×10^{-534}	*******
7.1×10^{-128}	*******	*******	*******
8.1×10^{-264}	*******	*******	*******
5.1×10^{-528}	*******	*******	*******
time = 142	time = 44	time = 71	time = 42

Table 2. $|x_{n+1} - x_n|$ with $n \geq 1$ and total CPU time required for the function $f_3(x)$.

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