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NUMERICAL SOLUTIONS OF THIRD KIND INTEGRAL-ALGEBRAIC EQUATIONS

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Abstract. In this paper, collocation and discretized collocation methods for solving third kind integral-algebraic equations numerically are developed. The global convergence analysis using the spline polynomial space $S_{m-1}^{-1}(\Pi_N)$ is given. We exhibit the methods and tabulate the results for several numerical test cases.

1. Introduction

The general form of the linear semi-explicit integral-algebraic equations is

$$A(t)X(t) = q(t) + \int_0^t K(t,s)X(s)\,ds, \quad t \in I = [0,T], \tag{1.1}$$

where $q(t) = (q_1(t), q_2(t))^t$ and A is a singular matrix with

$$A(t) = \begin{pmatrix} a(t) & d(t) \\ c(t) & b(t) \end{pmatrix},\ K(t,s) = \begin{pmatrix} K_{11}(t,s) & K_{12}(t,s) \\ K_{21}(t,s) & K_{22}(t,s) \end{pmatrix} \text{ and } X(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}.$$

System (1.1) has been widely applied in engineering and physics; particularly, it arises in a number of important problems of the theory of elasticity, neutron transport, and scattering of particles; see for example [1, 8, 9]. System (1.1) has many forms according to the algebraic form of the matrix $A(t)$. The matrix $A(t)$ has two eigenvalues 0 and $\lambda(t)$. If $\lambda(t) \neq 0$, $\forall t \in I$, then there exists a matrix $P(t)$ such that

$$P^{-1}(t)A(t)P(t) = \mathrm{diag}[\lambda(t), 0],$$

thus equation (1.1) can be written in the form

$$\mathrm{diag}[\lambda(t), 0]P^{-1}(t)X(t) = P^{-1}(t)q(t) + \int_0^t P^{-1}(t)K(t,s)X(s)\,ds.$$

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Defining $W(t) = P^{-1}(t)X(t) = (u(t), v(t))^t$, we get

$$\begin{aligned} u(t) &= \widetilde{q}_1(t) + \int_0^t (\widetilde{K}_{11}(t,s)u(s) + \widetilde{K}_{12}(t,s)v(s))\,ds, \\ 0 &= \widetilde{q}_2(t) + \int_0^t (\widetilde{K}_{21}(t,s)u(s) + \widetilde{K}_{22}(t,s)v(s))\,ds \end{aligned} \tag{1.2}$$

which is called semi-explicit linear integral-algebraic equation of index 1. Equation (1.2) has been investigated by Kauthen [6]. He applied the spline collocation method to approximate the solution of (1.2). This paper deals with the following form of (1.1):

$$\begin{aligned} y(t) - a(t)z(t) &= f(t) + \int_0^t (K_{11}(t,s)\,y(s) + K_{12}(t,s)\,z(s))\,ds \\ 0 &= g(t) + \int_0^t (K_{21}(t,s)\,y(s) + K_{22}(t,s)\,z(s))\,ds. \end{aligned} \tag{1.3}$$

We call this type: third kind semi-explicit linear integral-algebraic equation where the data functions $a, f, g, K_{ij}; i, j = 1, 2$, are sufficiently smooth. Furthermore, we suppose that $g(0) = 0,\ |a(t)| \geq a_0 > 0,\ |K_{22}(t,t) + a(t)K_{21}(t,t)| \geq k_0 > 0$ for all $t \in I$. It then follows that equation (1.3) has a unique continuous solutions y and z on I. This can be easily seen as follows: we first differentiate the second equation of (1.3) so that we get a second kind Volterra integral equation which is known to have a unique continuous solution (Yosida [10]). Note that equation (1.3) can be obtained from equation (1.1) by assuming $\lambda(t) = 0, \forall t \in I$ and $a(t) \neq 0$ for all $t \in I$. Moreover, equation (1.3) can be considered as a general case of equation (1.2).

Recently, there has been a growing interest in developing approximate numerical techniques for integral equations of the third kind; see for example [3, 4].

The solutions of (1.3) are to be approximated in the space $S_{m-1}^{(-1)}(\Pi_N)$; the space of discontinuous polynomial spline functions of degree $m-1$. Our goal is to generalize the convergence theory that has been proved by Kauthen [6] to approximate the solution of equation (1.3).

In Section 2, polynomial spline collocation method is used to solve equation (1.3) numerically. A convergence analysis is established in Section 3. In Section 4, we briefly study the discretized collocation method. Numerical illustrations are provided in Section 5.

2. Discontinuous collocation approximations

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n = nh$, $n = 0, 1, \ldots, N$ where the stepsize is given by $h = T/N (N > 0)$. Let $0 < c_1 < c_2 < \cdots < c_m \leq 1 (m \geq 1)$ be the collocation parameters and $t_{n,j} = t_n + c_j h, j = 1, \ldots, m,\ n = 0, \ldots, N-1$ the collocation points. Moreover, denote by π_{m+d} the set of all real polynomials of degree not exceeding $m+d$. We define the real polynomial spline space of degree m and of continuity class d as follows:

$$S_{m+d}^{(d)}(\Pi_N) = \{u \in C^d(I) : u \in \pi_{m+d} \text{ on } \sigma_n,\ n = 0, 1, \ldots, N-1\};$$

where $\sigma_n = (t_n, t_{n+1}]$ and $-1 \leq d \leq m+d$, and for $d = -1$

$$S_{m-1}^{(-1)}(\Pi_N) = \{u : u_n = u|_{\sigma_n} \in \pi_{m-1},\ n = 0, \ldots, N-1\}.$$

The exact solutions y, z of (1.3) will be approximated on I by elements $u, v \in S_{m-1}^{(-1)}(Z_N)$ respectively, (called collocation solutions). These approximations satisfy the collocation equations

$$u\,(t) - a(t)v(t) = f\,(t) + \int_0^t (K_{11}\,(t,s)\,u\,(s) + K_{12}\,(t,s)\,v\,(s))\,ds \tag{2.1}$$

$$0 = g\,(t) + \int_0^t (K_{21}\,(t,s)\,u\,(s) + K_{22}\,(t,s)\,v\,(s))\,ds \tag{2.2}$$

for $t = t_{n,j}, j = 1, \ldots, m,\ n = 0, \ldots, N-1$.

Let $Y_{n,j} = u_n\,(t_{n,j})$ and $Z_{n,j} = v_n\,(t_{n,j})$. Since $u_n, v_n \in \pi_{m-1}$, it holds for $\tau \in (0,1]$,

$$u_n\,(t_n + \tau h) = \sum_{j=1}^{m} L_j\,(\tau)\,Y_{n,j},\ \ v_n\,(t_n + \tau h) = \sum_{j=1}^{m} L_j\,(\tau)\,Z_{n,j} \tag{2.3}$$

where $L_j(\tau) = \prod_{k\neq j}(\tau - c_k)/(c_j - c_k),\ j = 1, \ldots, m$, denote the fundamental Lagrange polynomials. Substituting (2.3) into (2.1) and (2.2), we obtain for each $n = 0, \ldots, N-1$, a linear system for the unknowns $Y_{n,j},\ Z_{n,j}, j = 1, \ldots, m$

$$\begin{aligned} Y_{n,j} - a(t_{n,j})Z_{n,j} = f\,(t_{n,j}) &+ h\sum_{k=1}^{m}(\int_0^{c_j} (K_{11}\,(t_{n,j}, t_n + \tau h)\,L_k\,(\tau))\,d\tau)Y_{n,k} \\ &+ h\sum_{k=1}^{m}(\int_0^{c_j} (K_{12}\,(t_{n,j}, t_n + \tau h)\,L_k\,(\tau))\,d\tau)Z_{n,k} \\ &+ h\sum_{i=0}^{n-1}\sum_{k=1}^{m}(\int_0^{1} (K_{11}\,(t_{n,j}, t_i + \tau h)\,L_k\,(\tau))\,d\tau)Y_{i,k} \\ &+ h\sum_{i=0}^{n-1}\sum_{k=1}^{m}(\int_0^{1} (K_{12}\,(t_{n,j}, t_i + \tau h)\,L_k\,(\tau))\,d\tau)Z_{n,k} \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} 0 = g\,(t_{n,j}) &+ h\sum_{k=1}^{m}(\int_0^{c_j} (K_{21}\,(t_{n,j}, t_n + \tau h)\,L_k\,(\tau))\,d\tau)Y_{n,k} \\ &+ h\sum_{k=1}^{m}(\int_0^{c_j} (K_{22}\,(t_{n,j}, t_n + \tau h)\,L_k\,(\tau))\,d\tau)Z_{n,k} \end{aligned}$$

$$+ h\sum_{i=0}^{n-1}\sum_{k=1}^{m}(\int_0^1 (K_{21}(t_{n,j}, t_i+\tau h)\, L_k(\tau))\, d\tau) Y_{i,k}$$

$$+ h\sum_{i=0}^{n-1}\sum_{k=1}^{m}(\int_0^1 (K_{22}(t_{n,j}, t_i+\tau h)\, L_k(\tau))\, d\tau) Z_{i,k}. \tag{2.5}$$

Then the algebraic system defining Y_n and Z_n can be written as

$$\begin{pmatrix} I - hK_{11}^{(n,n)} & -A_n - hK_{12}(n,n) \\ K_{21}^{(n,n)} & K_{22}^{(n,n)} \end{pmatrix} \begin{pmatrix} Y_n \\ Z_n \end{pmatrix} = \begin{pmatrix} f_n + G_n^{(1)} \\ -h^{-1}[g_n + G_n^{(2)}] \end{pmatrix} \tag{2.6}$$

where the matrix $K_{lr}^{(n,n)} = (\int_0^{c_j} K_{lr}(t_{n,j}, t_n+\tau h)\, L_k(\tau)\; d\tau)\,(l,r = 1,2)$, $Y_n = (Y_{n,1},\dots,Y_{n,m})^T$ and $Z_n = (Z_{n,1},\dots,Z_{n,m})^T$. We also define $F_n^{(r)}(t_{n,j}) = \int_0^{t_n}(K_{r1}(t_{n,j},s)\,u(s) + K_{r2}(t_{n,j},s)\,v(s))\,ds$, $f_n = (f(t_{n,1}),\dots,f(t_{n,m}))^T$, $g_n = (g(t_{n,1}),\dots,g(t_{n,m}))^T$, $G_n^{(r)} = (F_n^{(r)}(t_{n,1}),\dots,F_n^{(r)}(t_{n,m}))^T$ $(r = 1,2)$, and $A_n = \mathrm{diag}[a(t_{n,1}),\dots,a(t_{n,m})]$.

3. Convergence analysis

To study the convergence of the collocation method for equation (1.3), we need the following lemmas:

Lemma 1. *Let B be an $n\times n$ matrix such that $\|B\|_\infty < 1$. Then the matrix $(I-B)$ is invertible and*

$$(I-B)^{-1} = I + B + B^2 + \cdots.$$

Lemma 2. *Let Q, B, C, D, E be $n\times n$ matrices such that Q and $E+DQ$ are invertible, then there exists $\overline{h} > 0$ such that for all $h\in[0,\overline{h}]$, the block matrix $\begin{pmatrix} I-hB & -Q-hC \\ D & E\end{pmatrix}$ is invertible.*

Proof. We have by Lemma 1, that there exists $h_1 > 0$ such that for all $h \in [0,h_1]$ the matrix $I-hB$ is invertible. Hence by Leibniz formula, for all $h\le h_1$

$$\begin{pmatrix} I-hB & -Q-hC \\ D & E\end{pmatrix} = \begin{pmatrix} I-hB & 0 \\ D & I\end{pmatrix}\begin{pmatrix} I & (I-hB)^{-1}(-Q-hC) \\ 0 & E - D(I-hB)^{-1}(-Q-hC)\end{pmatrix}.$$

Thus,

$$\det\begin{pmatrix} I-hB & -Q-hC \\ D & E\end{pmatrix} = \det(I-hB)\det(E + D(I-hB)^{-1}(Q+hC))$$
$$= \det(I-hB)\det(B_1),$$

where

$$B_1 = E + D(h(I-hB)^{-1}B + I)Q + D(I-hB)^{-1}hC$$
$$= E + DQ + hD(I-hB)^{-1}(BQ + C).$$

From Lemma 1, we have

$$\left\|(I-hB)^{-1}\right\|_\infty \le \sum_{n=0}^{\infty}\left\|(hB)^n\right\|_\infty = \frac{1}{1-h\left\|B\right\|_\infty} \le \frac{1}{1-h_1\left\|B\right\|_\infty},$$

then $D(I-hB)^{-1}(BQ+C)$ is bounded and since $E+DQ$ is invertible we obtain by using Lemma 1 that the matrix $\begin{pmatrix} I-hB & -Q-hC \\ D & E \end{pmatrix}$ is invertible for all $h \in [0,\overline{h}]$, for some $\overline{h} \in [0,h_1]$. ∎

We now return to system (2.6). Let $D = K_{21}^{(n,n)}$, $E = K_{22}^{(n,n)}$, and $Q = A_n$, then Q is invertible because $|a(t)| \ge a_0 > 0$, on the other hand, we have

$$\begin{aligned} E+DQ &= \left(\int_0^{c_j}\left(K_{22}\left(t_{n,j},t_n+\tau h\right)+a(t_{n,j})K_{21}\left(t_{n,j},t_n+\tau h\right)\right)L_i\left(\tau\right)\,d\tau\right)_{j,i=1,\ldots,m} \\ &= \left(K_{22}\left(t_n,t_n\right)+a(t_n)K_{21}\left(t_n,t_n\right)\right)A+O(h) \end{aligned}$$

where the matrix $A = (a_{ji})_{j,i=0}^m$ with $a_{ji} = \int_0^{c_i} L_j(\tau)\,d\tau$. Since $|K_{22}\left(t_n,t_n\right) + a(t_n)K_{21}\left(t_n,t_n\right)| \ge k_0 > 0$ and the matrix A is invertible by [7], then there exists h_1 such that for all $h \in [0,h_1]$ the matrix $E+DQ$ is invertible. Thus according to Lemma 2 there exist $h_2 \le h_1$ such that the left-hand side block matrix of equation (2.6) is also invertible for all $h \in [0,h_2]$ and the linear algebraic system (2.6) has a unique solutions Y_n, Z_n for $n = 0,1,\ldots,N-1$ and $h \in [0,h_2]$.

We are now in a position to prove the following global convergence result:

Theorem 1. *Consider the polynomial spline approximations u,v in $S_{m-1}^{(-1)}(\Pi_N)$ to the solutions y,z of equation (1.3) and defined by (2.3), (2.4), and (2.5). If $c_m = 1$ then for every choice of c_i ($i = 1,2,\ldots,m-1$), the collocation approximations u,v converge to the solutions y,z respectively. If $c_m < 1$, the collocation approximations u,v converge to the solutions y,z for any $m \ge 1$ if and only if*

$$-1 \le R(\infty) = (-1)^m \prod_{i=1}^{m}\frac{1-c_i}{c_i} \le 1.$$

Moreover, the following error estimates hold

$$\left\|y-u\right\|_\infty = \begin{cases} o(h^m), \textit{ if } c_m = 1 \\ o(h^m), \textit{ if } c_m < 1 \textit{ and } -1 \le R(\infty) < 1 \\ o(h^{m-1}), \textit{ if } c_m < 1 \textit{ and } R(\infty) = 1, \end{cases} \tag{3.1}$$

$$\left\|z-v\right\|_\infty = \begin{cases} o(h^m), \textit{ if } c_m = 1 \\ o(h^m), \textit{ if } c_m < 1 \textit{ and } -1 \le R(\infty) < 1 \\ o(h^{m-1}), \textit{ if } c_m < 1 \textit{ and } R(\infty) = 1. \end{cases} \tag{3.2}$$

Proof. The errors $e = y-u$ and $\epsilon = z-v$ satisfy the system

$$\begin{aligned} e_n\left(t_n\right)-a(t_n)\epsilon(t_n) &= \int_0^{t_{n,j}}\left(K_{11}\left(t_{nj},s\right)e\left(s\right)+K_{12}\left(t_{n,j},s\right)\epsilon\left(s\right)\right)ds \\ 0 &= \int_0^{t_{n,j}}\left(K_{21}\left(t_{n,j},s\right)e\left(s\right)+K_{22}\left(t_{n,j},s\right)\epsilon\left(s\right)\right)ds, \end{aligned}$$

where $e_n = e|_{\sigma_n}$ and $\epsilon_n = \epsilon|_{\sigma_n}$. Then similar to the proof of Theorem 3.1 of [6], the following cases have to be considered:

Case I. If $c_m = 1$, then the following linear system can be derived:

$$\begin{pmatrix} I - hK_{11}^{(n,n)} & -A_n - hK_{12}^{(n,n)} \\ K_{21}^{(n,n)} & K_{22}^{(n,n)} \end{pmatrix} \begin{pmatrix} E_n \\ \varepsilon_n \end{pmatrix} = h\sum_{i=0}^{n-1} \begin{pmatrix} K_{11}^{(n,i)} & K_{12}^{(n,i)} \\ \widetilde{K}_{21}^{(n,i)} & \widetilde{K}_{22}^{(n,i)} \end{pmatrix} \begin{pmatrix} E_i \\ \varepsilon_i \end{pmatrix} + O(h^m),$$

where $E_n = (e_n(t_{n,1}), \dots, e_n(t_{n,m}))^T$, $\varepsilon_n = (\epsilon_n(t_{n,1}), \dots, \epsilon_n(t_{n,m}))^T$, $n = 0, \dots, N-1$, and the matrices

$$\widetilde{K}_{lr}^{(n,i)} = \left(\int_0^1 \partial_1 K_{lr}\left(\xi_j, t_i + \tau h\right) L_k\left(\tau\right)\, d\tau \right)_{j,k=1,\dots,m}$$

$$K_{lr}^{(n,i)} = \left(\int_0^{c_j} K_{lr}\left(t_{n,j}, t_i + \tau h\right) L_k\left(\tau\right)\, d\tau \right)_{j,k=1,\dots,m}$$

with $l, r = 1, 2$, $\partial_1 K_{lr}(t,s) = \frac{\partial K_{lr}}{\partial t}(t,s)$, and ξ_j between $t_{n-1,m}$ and $t_{n,j}$. By (2.6) the inverse of the matrix on the left-hand side exists and is bounded if h is sufficiently small. Then the result follows from Gronwall's inequality [2].

Case II. If $c_m < 1$, then we have the linear system

$$\begin{aligned} &\begin{pmatrix} I - hK_{11}^{(n,n)} & -A_n - hK_{12}^{(n,n)} \\ K_{21}^{(n,n)} & K_{22}^{(n,n)} \end{pmatrix} \begin{pmatrix} E_n \\ \varepsilon_n \end{pmatrix} \\ &\quad = \begin{pmatrix} hK_{11}^{(n,n-1)} & hK_{12}^{(n,n-1)} \\ K_{21}(t_n,t_n)M_1 + O(h) & K_{22}(t_n,t_n)M_1 + O(h) \end{pmatrix} \begin{pmatrix} E_{n-1} \\ \varepsilon_{n-1} \end{pmatrix} \\ &\qquad + h\sum_{i=0}^{n-2} \begin{pmatrix} K_{11}^{(n,i)} & hK_{12}^{(n,n-1)} \\ \widetilde{K}_{21}^{(n,i)} & \widetilde{K}_{22}^{(n,i)} \end{pmatrix} \begin{pmatrix} E_{n-1} \\ \varepsilon_{n-1} \end{pmatrix}) + O(h^m) \end{aligned} \tag{3.3}$$

where $M_1 = u_m e_m^T A - u_m b^T$, $u_m = (1, \dots, 1)^T$, $e_m = (0, \dots, 0, 1)^T$, and the vector $b = (b_0, \dots, b_m)^T$ with $b_j = \int_0^1 L_j\left(\tau\right)\, d\tau, j = 1, \dots, m$. The inverse of the matrix on the left-hand side has the form

$$\begin{pmatrix} I - hK_{11}^{(n,n)} & -A_n - hK_{12}^{(n,n)} \\ K_{21}^{(n,n)} & K_{22}^{(n,n)} \end{pmatrix}^{-1} = \begin{pmatrix} I - \frac{a_n A^{-1} K_{21}^{(n,n)}}{\alpha_n} + O(h) & \frac{a_n A^{-1}}{\alpha_n} + O(h) \\ -\frac{A^{-1} K_{21}^{(n,n)}}{\alpha_n} + O(h) & \frac{A^{-1}}{\alpha_n} + O(h) \end{pmatrix},$$

where $a_n = a(t_n)$ and $\alpha_n = K_{22}(t_n, t_n) + K_{21}(t_n, t_n)a_n$. Then

$$\begin{aligned} &\begin{pmatrix} I - hK_{11}^{(n,n)} & -A_n - hK_{12}^{(n,n)} \\ K_{21}^{(n,n)} & K_{22}^{(n,n)} \end{pmatrix}^{-1} \begin{pmatrix} hK_{11}^{(n,n-1)} & hK_{12}^{(n,n-1)} \\ K_{21}(t_n,t_n)M_1 + O(h) & K_{22}(t_n,t_n)M_1 + O(h) \end{pmatrix} \\ &\quad = \begin{pmatrix} K_{21}(t_n,t_n)\frac{a_n A^{-1}}{\alpha_n} M_1 + O(h) & K_{22}(t_n,t_n)\frac{a_n A^{-1}}{\alpha_n} M_1 + O(h) \\ K_{21}(t_n,t_n)\frac{A^{-1}}{\alpha_n} M_1 + O(h) & K_{22}(t_n,t_n)\frac{A^{-1}}{\alpha_n} M_1 + O(h) \end{pmatrix}. \end{aligned}$$

Thus (3.3) becomes

$$\begin{aligned} \begin{pmatrix} E_n \\ \varepsilon_n \end{pmatrix} &= \begin{pmatrix} K_{21}(t_n,t_n)\frac{a_n}{\alpha_n} M_0 & K_{22}(t_n,t_n)\frac{a_n}{\alpha_n} M_0 \\ K_{21}(t_n,t_n)\frac{1}{\alpha_n} M_0 & K_{22}(t_n,t_n)\frac{1}{\alpha_n} M_0 \end{pmatrix} \begin{pmatrix} E_{n-1} \\ \varepsilon_{n-1} \end{pmatrix} \\ &\quad + h\sum_{i=0}^{n-1} \begin{pmatrix} D_1^{(n,i)} & D_2^{(n,i)} \\ D_3^{(n,i)} & D_4^{(n,i)} \end{pmatrix} \begin{pmatrix} E_i \\ \varepsilon_i \end{pmatrix} + O(h^m) \end{aligned}$$

where $M_0 = A^{-1}M_1$ and $D_2^{(n,i)}$, $i = 2, \dots, 5$, denote bounded matrices. Then, the result follows as in [5] and [7] and by the help of the following lemmas.

LEMMA 3. [6] *Let* $M_0 = A^{-1}u_m(e_m^T A - b^T)$. *Then* M_0 *has rank one and its only nonzero eigenvalue is*

$$R(\infty) = (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i},$$

where $R(z) = 1 + zb^T(u - zA)^{-1}u_m$ *denotes the stability function of the Runge-Kutta method* (c, A, b), *moreover there exists a nonsingular matrix* P *such that* $M_0 = PDP^{-1}$ *with* $D = \operatorname{diag}(R(\infty), 0, \ldots, 0)$.

LEMMA 4. *Let* $M_n = \begin{pmatrix} K_{21}(t_n,t_n)\frac{a_n}{\alpha_n}M_0 & K_{22}(t_n,t_n)\frac{a_n}{\alpha_n}M_0 \\ K_{21}(t_n,t_n)\frac{1}{\alpha_n}M_0 & K_{22}(t_n,t_n)\frac{1}{\alpha_n}M_0 \end{pmatrix}$. *Then* M_n *is diagonalizable, and its only nonzero eigenvalue is* $R(\infty)$.

Proof. It is clear that $\lambda = 0$ is an eigenvalue of M_n of multiplicity $2m-1$ and $\operatorname{trace}(M_n) = \operatorname{trace}(M_0) = R(\infty)$. ∎

4. Discretized collocation

In section 2 it is assumed that the integrals in (2.1) and (2.2) are evaluated analytically. But it is not always possible to compute these integrals, thus they have to be approximated by using the following appropriate quadrature formulas:

$$\int_0^{c_i} f(\tau)\, d\tau \approx \sum_{j=1}^{m} a_{ij} f(c_j),\ i = 1, \ldots, m,$$

$$\int_0^{1} f(\tau)\, d\tau \approx \sum_{j=1}^{m} b_j f(c_j),$$

where the coefficients a_{ij} and b_j are defined by

$$a_{ij} = \int_0^{c_i} L_j(\tau)\, d\tau,\ b_j = \int_0^1 L_j(\tau)\, d\tau,\ i,j = 1, \ldots, m.$$

The resulting approximations of y, z are denoted by $\widetilde{u}, \widetilde{v} \in S_{m-1}^{(-1)}(Z_N)$ and defined by

$$\begin{aligned}
&\widetilde{Y}_{n,j} - a(t_{n,j})\widetilde{Z}_{n,j} = \\
&\quad = f(t_{n,j}) + h\sum_{k=1}^{m} a_{jk}K_{11}(t_{n,j}, t_{n,k})\widetilde{Y}_{n,k} + h\sum_{k=1}^{m} a_{jk}K_{12}(t_{n,j}, t_{n,k})\widetilde{Z}_{n,k} \\
&\qquad + h\sum_{i=0}^{n-1}\sum_{k=1}^{m} b_k K_{11}(t_{n,j}, t_{i,k})\widetilde{Y}_{i,k} + h\sum_{i=0}^{n-1}\sum_{k=1}^{m} b_k K_{12}(t_{n,j}, t_{i,k})\widetilde{Z}_{i,k}
\end{aligned}$$

and

$$\begin{aligned}
0 = g(t_{n,j}) &+ h\sum_{k=1}^{m} a_{jk}K_{21}(t_{n,j}, t_{n,k})\widetilde{Y}_{n,k} + h\sum_{k=1}^{m} a_{j,k}K_{22}(t_{n,j}, t_{n,k})\widetilde{Z}_{n,k} \\
&+ h\sum_{i=0}^{n-1}\sum_{k=1}^{m} b_k K_{21}(t_{n,j}, t_{i,k})\widetilde{Y}_{i,k} + h\sum_{i=0}^{n-1}\sum_{k=1}^{m} b_k K_{22}(t_{n,j}, t_{i,k})\widetilde{Z}_{i,k},
\end{aligned}$$

where

$$\widetilde{u}_n\left(t_n+\tau h\right)=\sum_{j=1}^{m} L_j\left(\tau\right)\widetilde{Y}_{n,j}, \widetilde{v}_n\left(t_n+\tau h\right)=\sum_{j=1}^{m} L_j\left(\tau\right)\widetilde{Z}_{n,j}.$$

REMARK 1. Applying the same techniques as in the proof of Theorem 1, it can be shown that the error estimates (3.1) and (3.2) hold also for the discretized collocation method.

5. Numerical examples

In this section, we apply collocation and discretized collocation methods for equation (1.3). In Tables 1, 2, 5, 6, 9, and 10, the case $c_m = 1$ is considered. Then, in Tables 3, 4, 7, 8, 11, and 12, the cases $c_m < 1$ and $|R(\infty)| < 1$ are assumed, in all of these cases the methods are convergent. In Table 13, the orders of convergence of u and v are given. These results are well in line with the prediction of Theorem 1.

EXAMPLE 1. Consider equation (1.3) with $K_{11}(t,s) = ts+1, K_{12}(t,s) = st^2$, $K_{21}(t,s) = st^2+1, K_{22}(t,s) = ts+5, a(t) = t+2$, and $f(t), g(t)$ are chosen so that the exact solutions are $y(t) = t+1$ and $z(t) = t+2$.

Here we found the following results.

Table 1: Collocation method for Example 1 using $N = 10$, $m = 3$, and $c_i = \frac{i}{m}, i = 1,\ldots,m$.

t	0.4	0.6	0.8	1
$\lvert y(t)-u(t)\rvert$	0.2×10^{-7}	0.47×10^{-7}	0.82×10^{-7}	0.22×10^{-6}
$\lvert z(t)-v(t)\rvert$	0.1×10^{-8}	0.2×10^{-7}	0.33×10^{-7}	0.75×10^{-7}

Table 2: Discretized collocation method for Example 1 using $N = 10$, $m = 3$, and $c_i = \frac{i}{m}, i = 1,\ldots,m$.

t	0.4	0.6	0.8	1
$\lvert y(t)-\widetilde{u}(t)\rvert$	0.18×10^{-7}	0.10×10^{-7}	0.17×10^{-7}	0.45×10^{-7}
$\lvert z(t)-\widetilde{v}(t)\rvert$	0.44×10^{-8}	0.32×10^{-8}	0.39×10^{-8}	0.15×10^{-7}

Table 3: Collocation method for Example 1 using $N = 5$, $m = 5$, and $c_i = \frac{i}{m+2}+\frac{1}{12}, i = 1,\ldots,m$, and $R(\infty) = -0.74$.

t	0.4	0.6	0.8	1
$\lvert y(t)-u(t)\rvert$	0.43×10^{-5}	0.50×10^{-5}	0.28×10^{-5}	0.21×10^{-5}
$\lvert z(t)-v(t)\rvert$	0.17×10^{-5}	0.18×10^{-5}	0.10×10^{-5}	0.64×10^{-6}

Table 4: Discretized collocation method for Example 1 using $N = 5$, $m = 5$, and $c_i = \frac{i}{m+2} + \frac{1}{12}, i = 1, \ldots, m$, and $R(\infty) = -0.74$.

t	0.4	0.6	0.8	1
$\lvert y(t) - \widetilde{u}(t)\rvert$	0.15×10^{-6}	0.58×10^{-5}	0.84×10^{-5}	0.15×10^{-4}
$\lvert z(t) - \widetilde{v}(t)\rvert$	0.4×10^{-8}	0.21×10^{-5}	0.28×10^{-5}	0.51×10^{-5}

Example 2. Consider equation (1.3) with $K_{11}(t,s) = te^s - 1, K_{12}(t,s) = se^t$, $K_{21}(t,s) = e^{t+s} + 2, K_{22}(t,s) = ts + 1, a(t) = e^{2t} + 1$, and $f(t), g(t)$ are chosen so that the exact solutions are $y(t) = 2t - 1$ and $z(t) = e^t$.

Here we found the following results.

Table 5: Collocation method for Example 2 using $N = 5$, $m = 3$, and $c_i = \frac{i}{m}, i = 1, \ldots, m$.

t	0.4	0.6	0.8	1
$\lvert y(t) - u(t)\rvert$	0.22×10^{-5}	0.26×10^{-5}	0.27×10^{-5}	0.72×10^{-5}
$\lvert z(t) - v(t)\rvert$	0.96×10^{-5}	0.12×10^{-4}	0.14×10^{-4}	0.17×10^{-4}

Table 6: Discretized collocation method for Example 2 using $N = 5$, $m = 6$, and $c_i = \frac{i}{m}, i = 1, \ldots, m$.

t	0.4	0.6	0.8	1
$\lvert y(t) - \widetilde{u}(t)\rvert$	0.12×10^{-6}	0.17×10^{-7}	0.30×10^{-6}	0.54×10^{-6}
$\lvert z(t) - \widetilde{v}(t)\rvert$	0.34×10^{-7}	0.34×10^{-7}	0.57×10^{-7}	0.64×10^{-7}

Table 7: Collocation method for Example 2 using $N = 5$, $m = 3$, and $c_i = \frac{i}{m+1} + \frac{1}{10}, i = 1, \ldots, m$, and $R(\infty) = -0.21$.

t	0.4	0.6	0.8	1
$\lvert y(t) - u(t)\rvert$	0.5×10^{-4}	0.57×10^{-4}	0.65×10^{-4}	0.39×10^{-4}
$\lvert z(t) - v(t)\rvert$	0.30×10^{-3}	0.45×10^{-3}	0.56×10^{-3}	0.14×10^{-3}

Table 8: Discretized collocation method for Example 2 using $N = 5$, $m = 5$, and $c_i = \frac{i}{m+2} + \frac{1}{6}, i = 1, \ldots, m$, and $R(\infty) = -0.08$.

t	0.4	0.6	0.8	1
$\lvert y(t) - \widetilde{u}(t)\rvert$	0.3×10^{-9}	0.12×10^{-5}	0.21×10^{-5}	0.91×10^{-5}
$\lvert z(t) - \widetilde{v}(t)\rvert$	0.2×10^{-7}	0.24×10^{-6}	0.69×10^{-7}	0.58×10^{-7}

Example 3. Consider equation (1.3) with $K_{11}(t,s) = se^t, K_{12}(t,s) = t\sin s$, $K_{21}(t,s) = st^2+1, K_{22}(t,s) = ts+5, a(t) = e^t+2$, and $f(t), g(t)$ are chosen so that the exact solutions are $y(t) = 2\cos t + 1$ and $z(t) = t\sin t$.

Here we found the following results.

Table 9: Collocation method for Example 3 using $N = 5$, $m = 3$, and $c_i = \frac{i}{m}, i = 1, \ldots, m$.

t	0.4	0.6	0.8	1
$\lvert y(t) - u(t)\rvert$	0.109×10^{-2}	0.143×10^{-2}	0.164×10^{-2}	0.407×10^{-3}
$\lvert z(t) - v(t)\rvert$	0.348×10^{-3}	0.497×10^{-3}	0.652×10^{-3}	0.342×10^{-4}

Table 10: Discretized collocation method for Example 3 using $N = 5$, $m = 3$, and $c_i = \frac{i}{m}, i = 1, \ldots, m$.

t	0.4	0.6	0.8	1
$\lvert y(t) - \widetilde{u}(t)\rvert$	0.895×10^{-3}	0.128×10^{-2}	0.157×10^{-2}	0.317×10^{-3}
$\lvert z(t) - \widetilde{v}(t)\rvert$	0.354×10^{-3}	0.467×10^{-3}	0.573×10^{-3}	0.124×10^{-3}

Table 11: Collocation method for Example 3 using $N = 5$, $m = 5$, and $c_i = \frac{i}{m+2} + \frac{1}{12}, i = 1, \ldots, m$, and $R(\infty) = -0.74$.

t	0.4	0.6	0.8	1
$\lvert y(t) - u(t)\rvert$	0.123×10^{-4}	0.8×10^{-6}	0.664×10^{-4}	0.418×10^{-4}
$\lvert z(t) - v(t)\rvert$	0.287×10^{-5}	0.101×10^{-6}	0.174×10^{-4}	0.945×10^{-5}

Table 12: Discretized collocation method for Example 3 using $N = 10$, $m = 5$, and $c_i = \frac{i}{m+2} + \frac{1}{12}, i = 1, \ldots, m$, and $R(\infty) = -0.74$.

t	0.4	0.6	0.8	1
$\lvert y(t) - \widetilde{u}(t)\rvert$	0.116×10^{-5}	0.242×10^{-5}	0.104×10^{-5}	0.503×10^{-5}
$\lvert z(t) - \widetilde{v}(t)\rvert$	0.483×10^{-6}	0.413×10^{-6}	0.44×10^{-6}	0.823×10^{-6}

Table 13: The orders of convergence for Example 3 using $c_i = \frac{i}{m}$, $i = 1, \ldots, m$.

n	$m = 2$	$m = 3$	$m = 2$	$m = 3$
5	u	u	v	v
10	1.536	2.980	2.183	2.977
15	1.626	2.9849	2.156	2.981
20	1.673	2.992	2.141	2.986
25	1.704	3.127	2.130	3.056

6. Conclusion

An efficient numerical scheme based on the collocation spline method was proposed for solving third kind integral-algebraic equations. Moreover, the discretized collocation method for approximating the solution of equation (1.3) was discussed. Error analysis was provided. The error is $o(h^m)$ if $c_m = 1$ or $c_m < 1$ and $-1 \leq R(\infty) < 1$ and $o(h^{m-1})$ if $c_m < 1$ and $R(\infty) = 1$. The results in this paper can be considered as a general case of the convergence theory that has been proved by Kauthen [6]. Three numerical examples were introduced showing that the methods are convergent with a good accuracy. All numerical results confirmed the theoretical estimates.

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