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HARMONIC STARLIKE FUNCTIONS OF COMPLEX ORDER INVOLVING HYPERGEOMETRIC FUNCTIONS

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Dedicated to my father Prof. P.M. Gangadharan (1938–2011)

Abstract. A family of harmonic starlike functions of complex order in the unit disc has been introduced and investigated by S.A. Halim and A. Janteng [Harmonic functions starlike of complex order, Proc. Int. Symp. on New Development of Geometric function Theory and its Applications, (2008), 132–140]. In this paper we consider a subclass consisting of harmonic parabolic starlike functions of complex order involving special functions and obtain coefficient conditions, extreme points and a growth result.

1. Introduction

Let $\mathcal{H}$ denote the family of harmonic functions $f = h + \overline{g}$ that are orientation preserving and univalent in the open disc $\triangle = \{z : |z| < 1\}$ with h and g given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \;\; g(z) = \sum_{n=1}^{\infty} b_n z^n, \;\; |b_1| < 1. \tag{1.1}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class $\mathcal{S}$ of normalized univalent functions if the co-analytic part of f is identically zero, i.e. $g \equiv 0$. Also, we denote by $\overline{\mathcal{H}}$ the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \;\; g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \;\; |b_1| < 1. \tag{1.2}$$

The seminal work of Clunie and Sheil-Small [2] on harmonic mappings gave rise to many studies of subclasses of complex-valued harmonic univalent functions. In particular, Silverman [18], Jahangiri [7] Rosy et al. [17], Halim and Janteng [6] and others (see [10,11,12]) have investigated properties of various subclasses of $\mathcal{H}$ related to harmonic starlike functions.

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The Hadamard product (or convolution) of two power series

$$\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \tag{1.3}$$

and

$$\psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \tag{1.4}$$

in S is defined (as usual) by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n. \tag{1.5}$$

For positive real values of $\alpha_1, \dots, \alpha_l$ and $\beta_1, \dots, \beta_m$ ($\beta_j \neq 0, -1, \dots; j = 1, 2, \dots, m$) the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$ {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_l)_n}{(\beta_1)_n \dots (\beta_m)_n} \frac{z^n}{n!} \tag{1.6}$$

$$(l \leq m+1;\ l, m \in N_0 := N \cup \{0\}; z \in \Delta),$$

where N denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \begin{cases} 1, & n = 0 \\ a(a+1)(a+2)\dots(a+n-1), & n \in N. \end{cases} \tag{1.7}$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel and Laguerre polynomial. Let

$$H[\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m] : \mathcal{S} \to \mathcal{S}$$

be a linear operator defined by

$$\begin{aligned} H[\alpha_1, \dots \alpha_l; \beta_1, \dots, \beta_m]\phi(z) &= H_m^l[\alpha_1]\phi(z) \\ &:= z\ {}_lF_m(\alpha_1, \alpha_2, \dots \alpha_l; \beta_1, \beta_2 \dots, \beta_m; z) * \phi(z) \\ &= z + \sum_{n=2}^{\infty} \omega_n(\alpha_1; l; m)\ \phi_n z^n, \end{aligned} \tag{1.8}$$

where

$$\omega_n(\alpha_1; l; m) = \frac{(\alpha_1)_{n-1} \dots (\alpha_l)_{n-1}}{(\beta_1)_{n-1} \dots (\beta_m)_{n-1}} \frac{1}{(n-1)!}. \tag{1.9}$$

It follows from (1.8) that

$$H_0^1[1]\phi(z) = \phi(z), H_0^1[2]\phi(z) = z\phi'(z).$$

The linear operator $H_m^l[\alpha_1]$ is the Dziok-Srivastava operator (see [4]) which was subsequently extended by Dziok and Raina [3] by using the Wright generalized hypergeometric function. Recently Srivastava et al. [19] defined the linear operator $\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}$ as follows:

$$\mathcal{L}_{\lambda,\alpha_1}^{0}\phi(z) = \phi(z),$$

$$\mathcal{L}_{\lambda,l,m}^{1,\alpha_1}\phi(z) = (1-\lambda)H_m^l[\alpha_1]\phi(z) + \lambda z(H_m^l[\alpha_1]\phi(z))' = \mathcal{L}_{\lambda,l,m}^{\alpha_1}\phi(z),\ (\lambda \geq 0), \tag{1.10}$$

$$\mathcal{L}_{\lambda,l,m}^{2,\alpha_1}\phi(z) = \mathcal{L}_{\lambda,l,m}^{\alpha_1}(\mathcal{L}_{\lambda,l,m}^{1,\alpha_1}\phi(z)) \tag{1.11}$$

and in general,

$$\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}\phi(z) = \mathcal{L}_{\lambda,l,m}^{\alpha_1}(\mathcal{L}_{\lambda,l,m}^{\tau-1,\alpha_1}\phi(z)), (l \le m+1;\ l,m \in N_0 = N \cup \{0\}; z \in \Delta). \quad (1.12)$$

If the function $\phi(z)$ is given by (1.3), then we see from (1.8), (1.9), (1.10) and (1.12) that

$$\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}\phi(z) := z + \sum_{n=2}^{\infty} \omega_n^{\tau}(\alpha_1;\lambda;l;m)\ \phi_n\ z^n, \quad (1.13)$$

where

$$\omega_n^{\tau}(\alpha_1;\lambda;l;m) = \left(\frac{(\alpha_1)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_m)_{n-1}}\ \frac{[1+\lambda(n-1)]}{(n-1)!}\right)^{\tau}, (n \in N \setminus \{1\}, \tau \in N_0) \quad (1.14)$$

unless otherwise stated. We note that when $\tau = 1$ and $\lambda = 0$ the linear operator $\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}$ would reduce to the familiar Dziok-Srivastava linear operator [4], includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer [1], Owa [14] and Ruscheweyh [16].

In view of the relationship (1.14) and the linear operator (1.13) for the harmonic function $f = h + \overline{g}$ given by (1.1), Murugusundaramoorthy et al. [11,12] have defined the operator

$$\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}f(z) = \mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z) + \overline{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z)}, \quad (1.15)$$

and studied the subclass of $\mathcal{H}$ in terms of this operator.

Goodman [5] introduced two interesting subclasses of $\mathcal{S}$, namely uniformly convex functions ($\mathcal{UCV}$) and uniformly starlike functions ($\mathcal{UST}$), and Ronning [15] introduced a subclass of starlike functions $\mathcal{S}_p$ corresponding to the class $\mathcal{UCV}$. In order to consider extension of the class $\mathcal{S}_p$, we study in this note the class of harmonic starlike functions of complex order based on the earlier works of Nasr and Aouf [13] and Halim and Janteng [6].

For $0 \le \alpha < 1$, b, a non-zero complex number with $|b| < 1$, we let $\mathcal{HL}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$ be the subclass of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ where h and g are of the form (1.1), satisfying

$$\Re(w(z)) = \Re\Big(1 + \frac{1}{b}\Big((1+e^{i\gamma})\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z))' - \overline{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z))'}}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z) + \overline{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z)}} - e^{i\gamma} - 1\Big)\Big) > \alpha, \quad (1.16)$$

$z \in \triangle$, and for all real γ. We also let $\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha) = \mathcal{HL}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha) \cap \overline{\mathcal{H}}$.

Remark. With the above conditions, if we choose $\gamma = 0$, we can define the generalized class of harmonic starlike functions of complex order satisfying the condition

$$\Re(w(z)) = \Re\Big(1 + \frac{2}{b}\Big(\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z))' - \overline{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z))'}}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z) + \overline{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z)}} - 1\Big)\Big) > \alpha.$$

In this note we obtain sufficient coefficient conditions for harmonic functions $f = h + \overline{g}$ of the form (1.1) to be in $\mathcal{HL}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$. We also show that these conditions are necessary when $f \in \overline{\mathcal{H}}\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}(b,\gamma,\alpha)$. We also obtain extreme points and growth results.

2. Main results

THEOREM 1. *Let $f = h + g$ be given by* (1.1). *If*

$$\sum_{n=2}^{\infty} \frac{[2n-2+(1-\alpha)|b|]}{(1-\alpha)|b|}|a_n|\omega_n^\tau(\alpha_1;\lambda;l;m) + \sum_{n=1}^{\infty} \frac{[2n+2-(1-\alpha)|b|]}{(1-\alpha)|b|}|b_n|\omega_n^\tau(\alpha_1;\lambda;l;m) \le 1 \quad (2.1)$$

where $a_1 = 1$, $0 \le \alpha < 1$ and b ($|b| \le 1$) is a non-zero complex number, then f is harmonic univalent and orientation-preserving in $\triangle$ and $f \in \mathcal{H}\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}(b,\gamma,\alpha)$.

Proof. First we establish that f is orientation preserving in $\triangle$. This is seen as follows, on using (2.1):

$$\begin{aligned}
|(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z))'| &\ge 1 - \sum_{n=2}^{\infty} n\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n|r^{n-1} \\
&> 1 - \sum_{n=2}^{\infty} n\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n| \\
&\ge 1 - \sum_{n=2}^{\infty} \left[\tfrac{2n-2+(1-\alpha)|b|}{(1-\alpha)|b|}\right]\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n| \\
&\ge \sum_{n=1}^{\infty} \left[\tfrac{2n+2-(1-\alpha)|b|}{(1-\alpha)|b|}\right]\omega_n^\tau(\alpha_1;\lambda;l;m)|b_n| \\
&\ge \sum_{n=1}^{\infty} n\omega_n^\tau(\alpha_1;\lambda;l;m)|b_n| \\
&\ge \sum_{n=1}^{\infty} n\omega_n^\tau(\alpha_1;\lambda;l;m)|b_n|r^{n-1} \ge |(\mathcal{L}^{\alpha_1}_{\lambda,l,m}g(z))'|.
\end{aligned}$$

To show that f is univalent in $\triangle$, we show that $f(z_1) \ne f(z_2)$ when $z_1 \ne z_2$. Suppose $z_1, z_2 \in \triangle$ so that $z_1 \ne z_2$. Since the unit disc $\triangle$ is simply connected and convex, we then have $z(t) = (1-t)z_1 + tz_2$ in D where $0 \le t \le 1$. Then we write

$$\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z_2) - \mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z_1) = \int_0^1 [(z_2-z_1)(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z(t))') + \overline{(z_2-z_1)(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z(t))')}]\, dt.$$

Since $z_2 - z_1 \ne 0$, dividing throughout by $z_2 - z_1$ and taking only the real parts we obtain

$$\begin{aligned}
\Re\Big(\frac{\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z_2) - \mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}f(z_1)}{z_2-z_1}\Big) &= \int_0^1 \Re[(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z(t)))' + \frac{\overline{(z_2-z_1)}}{z_2-z_1}\overline{(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z(t)))'}]\, dt \\
&> \int_0^1 [\Re(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z(t)))' - |(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z(t)))'|]\, dt. \quad (2.2)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Re(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z(t)))' - |(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z(t)))'| \\
&\qquad \geq \Re(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z(t)))' - \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n| \\
&\qquad \geq 1 - \sum_{n=2}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n| - \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n| \\
&\qquad \geq 1 - \sum_{n=2}^{\infty}\left[\tfrac{2n-2+(1-\alpha)|b|}{(1-\alpha)|b|}\right]\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n| \\
&\qquad\quad - \sum_{n=1}^{\infty}\left[\tfrac{2n+2-(1-\alpha)|b|}{(1-\alpha)|b|}\right]\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n| \\
&\qquad \geq 0 \ \text{ by } \quad (2.1).
\end{aligned}$$

Therefore this together with inequality (2.2) implies the univalence of f.

Next we show that $f \in \mathcal{HL}^{\tau,\alpha_1}_{\lambda,l,m}(b,\gamma,\alpha)$. To do so, we need to show that when (2.1) holds, then (1.16) also holds true. Using the fact that $\Re w(z) \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ for $0 \leq \alpha < 1$ it suffices to show that

$$\begin{aligned}
&|(2b-\alpha b - e^{i\gamma}-1)(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z)}) \\
&+(1+e^{i\gamma})(z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z))' - \overline{(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z))'})| - |(1+\alpha b+e^{i\gamma})(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z) + \overline{\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z)})| \\
&\qquad - (1+e^{i\gamma})(z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z))' - \overline{z(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z))'})| \geq 0
\end{aligned}$$

On substituting for $(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}h(z))$ and $(\mathcal{L}^{\tau,\alpha_1}_{\lambda,l,m}g(z))$ we obtain

$$\begin{aligned}
&|(2b-\alpha b-(1+e^{i\gamma}))\left[z + \sum_{n=2}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)a_n z^n + \sum_{n=1}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)\overline{b_n z^n}\right] \\
&\quad + (1+e^{i\gamma})\left[z + \sum_{n=2}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)a_n z^n - \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)\overline{b_n z^n}\right]| \\
&\quad - |(1+\alpha b+e^{i\gamma})\left[z + \sum_{n=2}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)a_n z^n + \sum_{n=1}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)\overline{b_n z^n}\right] \\
&\quad - (1+e^{i\gamma})\left[z + \sum_{n=2}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)a_n z^n - \sum_{n=1}^{\infty} n\omega_n^{\tau}(\alpha_1;\lambda;l;m)\overline{b_n z^n}\right]| \\
&\quad \geq (2-\alpha)|b||z| - \sum_{n=2}^{\infty}|(2-\alpha)b + (1+e^{i\gamma})(n-1)|\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n||z|^n \\
&\qquad - \sum_{n=1}^{\infty}|(1+e^{i\gamma})(n+1) - (2-\alpha)b|\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\ |z|^n \\
&\qquad - \alpha|b||z| - \sum_{n=2}^{\infty}|(n-1)(1+e^{i\gamma}) - \alpha b|\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n|\ |z|^n \\
&\qquad - \sum_{n=1}^{\infty}|(n+1)(1+e^{i\gamma}) + \alpha b|\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\ |z|^n \\
&\qquad \geq 2(1-\alpha)|b||z|\{1 - \sum_{n=2}^{\infty}\left[\tfrac{2[2n-2+(1-\alpha)|b|]}{2(1-\alpha)|b|}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n|\right]\}
\end{aligned}$$

$$-2(1-\alpha)|b||z|\sum_{n=1}^{\infty}\left[\tfrac{2[2n+2-(1-\alpha)|b|]}{2(1-\alpha)|b|}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\right]$$

$$\geq 0,\quad\text{by}\quad(2.1).\quad\blacksquare$$

The function

$$f(z)=z+\sum_{n=2}^{\infty}\left[\tfrac{(1-\alpha)|b|}{[2n-2+(1-\alpha)|b|]}\right]x_nz^n+\sum_{n=1}^{\infty}\left[\tfrac{(1-\alpha)|b|}{[2n+2-(1-\alpha)|b|]}\right]\overline{y}_n\overline{z}^n,$$

where $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$, shows that the coefficient bound given by (2.1) is sharp.

The next theorem shows that condition (2.1) is necessary for $f\in\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$.

Theorem 2. *Let $f=h+\overline{g}$ be given by (1.2). Then $f\in\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{[2n-2+(1-\alpha)|b|]}{(1-\alpha)|b|}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n| +\sum_{n=1}^{\infty}\frac{[2n+2-(1-\alpha)|b|]}{(1-\alpha)|b|}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\leq 2 \quad (2.3)$$

where $a_1=1$, $0\leq\alpha<1$, b is a non-zero complex number such that $|b|<1$.

Proof. Since $\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)\subset\mathcal{H}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$, the if part of the Theorem 2 follows from Theorem 1. To prove the *only if* part, we show that when (2.3) does not hold then f is not in $\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$.

First, if $f\in\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$ then

$$\Re\left(1+\frac{1}{b}\left((1+e^{i\gamma})\frac{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z))'-\overline{z(\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z))'}}{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}h(z)+\overline{\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}g(z)}}-(e^{i\gamma}+1)\right)\right)-\alpha$$

$$=\Re\left(\frac{(1-\alpha)bz-\sum_{n=2}^{\infty}[(1-\alpha)b+(n-1)(1+e^{i\gamma})]\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n|z^n}{b\left(z-\sum_{n=2}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n|z^n+\sum_{n=1}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\overline{z}^n\right)}\right)$$

$$-\Re\left(\frac{\sum_{n=1}^{\infty}[(n+1)(1+e^{i\gamma})-(1-\alpha)b]\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\overline{z}^n}{b\left(z-\sum_{n=2}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n|z^n+\sum_{n=1}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\overline{z}^n\right)}\right)$$

$$=\Re\left(\frac{(1-\alpha)|b|^2-\sum_{n=2}^{\infty}[(1-\alpha)b+(n-1)(1+e^{i\gamma})]\overline{b}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n|z^{n-1}}{|b|^2\left(1-\sum_{n=2}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n|z^{n-1}+\frac{\overline{z}}{z}\sum_{n=1}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\overline{z}^{n-1}\right)}\right)$$

$$-\Re\left(\frac{+\frac{\overline{z}}{z}\sum_{n=1}^{\infty}[(n+1)(1+e^{i\gamma})-(1-\alpha)b]\overline{b}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\overline{z}^{n-1}}{|b|^2\big(1-\sum_{n=2}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|a_n|z^{n-1}+\frac{\overline{z}}{z}\sum_{n=1}^{\infty}\omega_n^{\tau}(\alpha_1;\lambda;l;m)|b_n|\overline{z}^{n-1}\big)}\right)\geq 0.$$

The above condition need hold for all values of γ, $|z| = r < 1$ and any b such that $0 < |b| < 1$. Choose $\gamma = 0$, b real and positive so that $|b| = b$ and $z = r < 1$ on positive real axis. Thus the above condition becomes

$$\frac{(1-\alpha)|b|^2 - \sum_{n=2}^{\infty}[(2n-2)+(1-\alpha)b]|b|\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n|r^{n-1}}{|b|^2\left(1-\sum_{n=2}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n|r^{n-1}+\sum_{n=1}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)|b_n|r^{n-1}\right)}$$
$$-\frac{\sum_{n=1}^{\infty}[(2n+2)-(1-\alpha)b]|b|\omega_n^\tau(\alpha_1;\lambda;l;m)|b_n|r^{n-1}}{|b|^2\left(1-\sum_{n=2}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)|a_n|r^{n-1}+\sum_{n=1}^{\infty}\omega_n^\tau(\alpha_1;\lambda;l;m)|b_n|r^{n-1}\right)} \geq 0. \quad (2.4)$$

We need to show that the numerator is positive since the denominator is positive. The numerator is

$$(1-\alpha)|b|^2 - |b|\left[\sum_{n=2}^{\infty}[(2n-2)+(1-\alpha)|b|]\,|a_n|\omega_n^\tau(\alpha_1;\lambda;l;m)r^{n-1}\right.$$
$$\left.-\sum_{n=1}^{\infty}[(2n+2)-(1-\alpha)|b|]\,|b_n|\omega_n^\tau(\alpha_1;\lambda;l;m)r^{n-1}\right]$$

which is negative if condition (2.3) does not hold. Thus, there exist some point $z_0 = r_0$ in $(0,1)$ and some real positive b for which the quotient in the above inequalities are negative, which contradicts the condition that $f \in \overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$. Hence the proof is complete. ■

Next, extreme points of the closed convex hull $\mathrm{clco}\,\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$ of $\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$ are determined.

THEOREM 3. *$f \in \mathrm{clco}\,\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n) \quad (2.5)$$

where

$$h_1(z) = z,\ h_n(z) = z - \frac{(1-\alpha)|b|}{[2n-2+(1-\alpha)|b|]\omega_n^\tau(\alpha_1;\lambda;l;m)}z^n,\quad n = 2,3,\ldots;$$
$$g_n(z) = z + \frac{(1-\alpha)|b|}{[2n+2-(1-\alpha)|b|]\omega_n^\tau(\alpha_1;\lambda;l;m)}\overline{z}^n,\quad n = 1,2,\ldots;$$

$\sum_{n=1}^{\infty}(X_n+Y_n) = 1$, $X_n \geq 0$ and $Y_n \geq 0$. In particular, the extreme points of $\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. For functions f having the form (2.5), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(X_n h_n + Y_n g_n) \\ &= \sum_{n=1}^{\infty}(X_n+Y_n)\,z - \sum_{n=2}^{\infty}\frac{(1-\alpha)|b|}{[2n-2+(1-\alpha)|b|]\omega_n^\tau(\alpha_1;\lambda;l;m)}X_n z^n \\ &\quad + \sum_{n=1}^{\infty}\frac{(1-\alpha)|b|}{[2n+2-(1-\alpha)|b|]\omega_n^\tau(\alpha_1;\lambda;l;m)}Y_n\overline{z}^n. \end{aligned}$$

Thus

$$\sum_{n=2}^{\infty}\frac{[2n-2+(1-\alpha)|b|]\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{(1-\alpha)|b|}\Big(\frac{(1-\alpha)|b|}{[2n-2+(1-\alpha)|b|]\omega_n^{\tau}(\alpha_1;\lambda;l;m)}\Big)X_n$$
$$+\sum_{n=1}^{\infty}\frac{[2n+2-(1-\alpha)|b|]\omega_n^{\tau}(\alpha_1;\lambda;l;m)}{(1-\alpha)|b|}\Big(\frac{(1-\alpha)|b|}{[2n+2-(1-\alpha)|b|]\omega_n^{\tau}(\alpha_1;\lambda;l;m)}\Big)Y_n$$
$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\le 1.$$

Therefore, $f\in \operatorname{clco}\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$.

Conversely, suppose that $f\in \operatorname{clco}\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$. Set

$$X_n=\frac{2n-2+(1-\alpha)|b|}{(1-\alpha)|b|}|a_n|\omega_n^{\tau}(\alpha_1;\lambda;l;m),\ n=2,3,\ldots,$$

and

$$Y_n=\frac{2n+2-(1-\alpha)|b|}{(1-\alpha)|b|}|b_n|\omega_n^{\tau}(\alpha_1;\lambda;l;m),\ n=1,2,\ldots,$$

where $\sum_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$\begin{aligned}f(z)&=z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}b_n\overline{z}^n\\&=z-\sum_{n=2}^{\infty}\frac{(1-\alpha)|b|}{[2n-2+(1-\alpha)|b|]\omega_n^{\tau}(\alpha_1;\lambda;l;m)}X_nz^n\\&\quad+\sum_{n=1}^{\infty}\frac{(1-\alpha)|b|}{[2n+2-(1-\alpha)|b|]\omega_n^{\tau}(\alpha_1;\lambda;l;m)}Y_n\overline{z}^n\\&=z-\sum_{n=2}^{\infty}[X_n(h_n(z)-z)]+\sum_{n=1}^{\infty}[Y_n(g_n(z)-z)]\\&=\sum_{n=1}^{\infty}(X_nh_n+Y_ng_n).\end{aligned}$$

From Theorem 2, we can deduce that $0\le X_n\le 1$, $(n\ge 2)$ and $0\le Y_n\le 1$, $(n\ge 1)$. We define $X_1=1-\sum_{n=2}^{\infty}X_n-\sum_{n=1}^{\infty}Y_n$. Again from Theorem 2, $X_1\ge 0$. Therefore $\sum_{n=1}^{\infty}(X_nh_n+Y_ng_n)=f(z)$ as required in the theorem. ∎

Theorem 4. *If $f\in\overline{\mathcal{H}}\mathcal{L}_{\lambda,l,m}^{\tau,\alpha_1}(b,\gamma,\alpha)$, then for $|z|=r<1$,*

$$|f(z)|\le(1+b_1)r+\Big(\frac{(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)}-\frac{4-(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)}|b_1|\Big)r^2$$

and

$$|f(z)|\ge(1-b_1)r-\Big(\frac{(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)}-\frac{4-(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)}|b_1|\Big)r^2$$

Proof. Let $f(z) \in \overline{\mathcal{H}\mathcal{L}}^{\tau,\alpha_1}_{\lambda,l,m}(b,\gamma,\alpha)$. On taking the absolute value of f, we have

$$\begin{aligned}
|f(z)| &\le (1+|b_1|)r + \sum_{n=2}^{\infty}[|a_n|+|b_n|]\ \omega_n^{\tau}(\alpha_1;\lambda;l;m))r^n \\
&\le (1+|b_1|)r + r^2\sum_{n=2}^{\infty}(|a_n|+|b_n|)\omega_n^{\tau}(\alpha_1;\lambda;l;m) \\
&= (1+|b_1|)r + \frac{(1-\alpha)|b|r^2}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)} \\
&\quad \times \sum_{n=2}^{\infty}\Big(\frac{2+(1-\alpha)|b|}{(1-\alpha)|b|}|a_n| + \frac{2+(1-\alpha)|b|}{(1-\alpha)|b|}|b_n|\Big)\omega_2^{\tau}(\alpha_1;\lambda;l;m) \\
&\le (1+|b_1|)r + \frac{(1-\alpha)|b|r^2}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)} \\
&\quad \times \sum_{n=2}^{\infty}\Big(\frac{2n-2+(1-\alpha)|b|}{(1-\alpha)|b|}|a_n| + \frac{2n+2-(1-\alpha)|b|}{(1-\alpha)|b|}|b_n|\Big)\omega_n^{\tau}(\alpha_1;\lambda;l;m) \\
&\le (1+|b_1|)r + \frac{(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)}\Big(1 - \frac{[4-(1-\alpha)|b|]}{(1-\alpha)|b|}|b_1|\Big)r^2 \\
&\le (1+|b_1|)r \\
&\quad + \Big(\frac{(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)} - \frac{4-(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)}|b_1|\Big)r^2
\end{aligned}$$

Similarly we can prove the other inequality. The result is sharp for the function

$$\begin{aligned}
f(z) = z + |b_1|\overline{z} + \Big(&\frac{(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)} \\
&- \frac{4-(1-\alpha)|b|}{[2+(1-\alpha)|b|]\omega_2^{\tau}(\alpha_1;\lambda;l;m)}|b_1|\Big)\overline{z}^2,\ |b_1| \le \frac{(1-\alpha)|b|}{4-(1-\alpha)|b|}. \quad \blacksquare
\end{aligned}$$

Concluding remarks. By choosing $\tau = 1$; $\lambda = 0$ and specializing the parameters α_1, l, m, the various results presented in this paper would provide interesting analogous results for the class of harmonic functions those considered earlier in [7–10,12,17,18]. In fact, by appropriately selecting these arbitrary sequences, the results presented in this paper would find further applications for the class of harmonic functions which would incorporate a generalized form of the Dziok-Srivastava linear operator [4] involving the Hadamard product (or convolution) of the function in (1.1) with the Fox-Wright generalization ${}_l\psi_m$ (see [3]) of the hypergeometric function ${}_lF_m$. Theorems 1 to 4 would thus eventually lead us further to new results for the class of functions (defined analogously to the class $f \in \overline{\mathcal{H}\mathcal{L}}^{\tau,\alpha_1}_{\lambda,l,m}(b,\gamma,\alpha)$), by associating instead the FoxWright generalized hypergeometric function ${}_l\psi_m$. Further, it is of interest to note that the results obtained in this paper yield various results studied in the literature by taking $\gamma = 0$ with $\tau = 1; \lambda = 0$. We choose to skip further details in this regard.

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