

FACTORIZATION OF WEAKLY COMPACT OPERATORS BETWEEN BANACH SPACES AND FRÉCHET OR (LB)-SPACES

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Abstract. In this note we show that weakly compact operators from a Banach space X into a complete (LB)-space E need not factorize through a reflexive Banach space. If E is a Fréchet space, then weakly compact operators from a Banach space X into E factorize through a reflexive Banach space. The factorization of operators from a Fréchet or a complete (LB)-space into a Banach space mapping bounded sets into relatively weakly compact sets is also investigated.

1. Introduction and preliminaries

A linear operator $T \in L(X, Y)$ between Banach spaces is weakly compact if it maps the closed unit ball of X into a relatively weakly compact subset of Y . There are two possible extensions of this concept when the continuous linear operator $T \in L(F, E)$ is defined between locally convex spaces F and E . As in [5], we say that T is *reflexive* if it maps bounded sets into relatively weakly compact sets, and it is called *weakly compact* (as in [10, 42.2]) if there is a 0-neighborhood U in F such that $T(U)$ is relatively weakly compact in E . It can be easily seen that if $T \in L(F, E)$ is weakly compact, then T is reflexive. Although the converse is true if F is a Banach space, in general this is false, as the identity $T : E \rightarrow E$ on an infinite dimensional Fréchet Montel space E shows. One can take, for example, the space E of entire functions on the complex plane endowed with the compact open topology. On the other hand, van Dulst [24] showed that if F is a (DF)-space and E is a Fréchet space and T is reflexive, then T is weakly compact (see also [9, Corollary 6.3.8]). We refer the reader to [16] or [11] for (DF)-spaces. Grothendieck [6, Cor. 1 of Thm 11] and [7, IV,4.3, Cor. 1 of Thm 2] proved that if F is a quasinormable locally convex space (cf. [2] or [11]), E is a Banach space and T is reflexive, then T is weakly compact. This result can be seen e.g. in Junek [9, 6.3.4 and 6.3.5]. Extensions of these results for sets of operators can be seen in [19].

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Davis, Figiel, Johnson and Pelczyński [4] proved the following beautiful and important result: Every weakly compact operator between Banach spaces factorizes through a reflexive Banach space. J.C. Díaz and Domański [5] investigated the factorization of reflexive operators between Fréchet spaces through reflexive Fréchet spaces. In [3] we continued work by Brooks, Saitô and Wright, showing that weakly compact operators $T : A \rightarrow E$ from a C^* -algebra A into a complete locally convex space E constitute the natural non-commutative version of vector measures with values in E . See also [12, 13, 21, 27, 28]. A recent expository article on this topic is [26]. Davis, Figiel, Johnson and Pelczyński's result was used in [13] to prove that the so called Right topology can be described by a special system of seminorms defined by operators taking values into reflexive Banach spaces. We showed in [3, Remark 5.5] that this description of the Right topology does not hold for non-normable locally convex spaces. This led us naturally to the question of the factorization of weakly compact and reflexive operators between Banach spaces and Fréchet or complete (LB)-spaces considered in this note. The papers [14, 15] complete the basic knowledge on the Right topology and its variants and contain the first applications to linear and multilinear operators.

We use standard notation for functional analysis and locally convex spaces [7, 8, 10, 11, 16]. A Fréchet space is a complete metrizable locally convex space. We refer the reader to [2, 10, 11] for the theory of Fréchet and (DF)-spaces. For a locally convex space $E = (E, \tau)$, E' stands for the topological dual of E and we denote by $\sigma(E, E')$ the weak topology on E . The family of all absolutely convex 0-neighborhoods of a locally convex space E is denoted by $\mathcal{U}_0(E)$, the family of all absolutely convex bounded subsets of E by $\mathcal{B}(E)$, and the family of all continuous seminorms on E by $cs(E)$. If E is a locally convex space and $q \in cs(E)$, E_q is the Banach space which appears as the completion of $(E/\text{Ker } q, \hat{q})$, $\hat{q}(x + \text{Ker } q) = q(x)$, $x \in E$. We denote by $\pi_q : E \rightarrow E_q$, $\pi_q(x) = x + \text{Ker } q$ and by $\pi_{p,q} : E_q \rightarrow E_p$, $p \leq q$, the canonical maps. If $B \in \mathcal{B}(E)$, the normed space generated by B is $E_B := (\text{span} B, p_B)$, p_B being the Minkowski functional of B . If $B \in \mathcal{B}(E)$, then $E_B \hookrightarrow E$ continuously. If E is sequentially complete, then E_B is a Banach space for every $B \in \mathcal{B}(E)$ which is closed. If X is a Banach space, X_1 stands for the closed unit ball of X .

An (LB)-space $E := \text{ind}_n E_n$ is a Hausdorff countable inductive limit of Banach spaces. Every (LB)-space is a (DF)-space and every (DF)-space is quasinormable (see [16, 8.3.37]). An (LB) space E is called *regular* if every bounded subset in E is contained and bounded in a step E_m . An (LB)-space is complete if and only if it is quasicomplete. Every complete (LB)-space is regular.

2. Results

PROPOSITION 2.1. *Let $E := \text{ind}_n E_n$ be a complete (LB)-space. The following conditions are equivalent:*

- (1) *Every weakly compact operator T from an arbitrary Banach space X into E factorizes through a reflexive Banach space.*

- (2) Every weakly compact subset of E is contained and weakly compact in some step E_m .

Proof. We assume first that (1) is satisfied and fix a weakly compact subset K of E . Since E is complete, the closed absolutely convex hull B of K is also weakly compact by Krein's theorem [10, 24.5.(4')]. Accordingly, the canonical inclusion $T : E_B \rightarrow E$ is weakly compact. By condition (1), there are a reflexive Banach space Y and continuous linear operators $T_1 \in L(E_B, Y)$ and $T_2 \in L(Y, E)$ such that $T = T_2 \circ T_1$. The continuity of T_1 yields $\lambda > 0$ such that $T_1(B) \subset \lambda Y_1$. Since $T_2 : Y \rightarrow \text{ind}_n E_n$ is continuous, we can apply Grothendieck's factorization theorem [11, Theorem 24.33] to find m such that $T_2(Y) \subset E_m$ and $T_2 : Y \rightarrow E_m$ is continuous. The unit ball Y_1 of the reflexive Banach space Y is $\sigma(Y, Y')$ -compact, hence $T_2(Y_1)$ is $\sigma(E_m, E'_m)$ -compact in E_m . Now, $T(B) = B = T_2 T_1(B) \subset T_2(\lambda Y_1) = \lambda T_2(Y_1)$. Therefore B , and hence K , is $\sigma(E_m, E'_m)$ -compact in E_m and condition (2) is proved.

Conversely, we assume that condition (2) holds and take a weakly compact operator $T : X \rightarrow E$ from a Banach space X into E . The image $T(X_1)$ of the unit ball of X is $\sigma(E, E')$ -compact in E . We can apply condition (2) to find m such that the closure K of $T(X_1)$ is $\sigma(E_m, E'_m)$ -compact in E_m . This implies that $T(X) \subset E_m$ and that $T : X \rightarrow E_m$ is weakly compact between the Banach spaces X and E_m . By the theorem of Davis, Figiel, Johnson and Pelczyński [4], $T : X \rightarrow E_m$ factorizes through a reflexive Banach space Y . This implies that the operator $T : X \rightarrow E$ also factorizes through the reflexive Banach space Y . ■

THEOREM 2.2. *There are a Banach space X , a complete (LB)-space $E := \text{ind}_n E_n$ and a weakly compact operator $T \in L(X, E)$ which does not factorize through a reflexive Banach space.*

Proof. By Valdivia [22, Chapter 1, 9.4.(11)], a complete (LB)-space E satisfies condition (2) in Proposition 2.1 if and only if it satisfies Retakh's condition (M_0) : there exists an increasing sequence $\{U_n\}_{n=1}^\infty$ of absolutely convex 0-neighbourhoods U_n in E_n such that for each $n \in \mathbf{N}$ there exists $m(n) > n$ with the property that the topologies $\sigma(E, E')$ and $\sigma(E_{m(n)}, E'_{m(n)})$ coincide on U_n . See also [17, 18, 25]. Bierstedt and Bonet [1] showed that there exist complete co-echelon spaces $E = \text{ind}_n \ell^\infty(v_n)$ of order infinity which do not satisfy condition (M_0) . In fact it is enough to take E as the strong dual of a distinguished non quasinormable Köthe echelon space $\lambda_1(A)$ of order 1; see e.g. [2]. We can apply Proposition 2.1 to find a Banach space X and a weakly compact operator $T : X \rightarrow E$ which does not factorize through a reflexive Banach space. ■

The following result is well-known to specialists. For the convenience of the reader, we give a brief proof.

PROPOSITION 2.3. *Every weakly compact operator $T : X \rightarrow E$ from a Banach space X into a Fréchet space E factorizes through a reflexive Banach space.*

Proof. The closure B of $T(X_1)$ in E is a weakly compact set. By a result of Grothendieck (see e.g. [9, Corollary 6.4.5] or [19, Section 1 Lemma (e)]), there is

$C \in \mathcal{B}(E)$ such that B is weakly compact in E_C . The map $T : X \rightarrow E_C$ is well defined and weakly compact, hence it factorizes through a reflexive Banach space Y , by the theorem of Davis, Figiel, Johnson and Pelczyński [4]. Since the inclusion from E_C into E is continuous, the original map $T : X \rightarrow E$ factorizes through Y , too. ■

We now consider the factorization of reflexive maps from a locally convex space F into a Banach space X through a reflexive Banach space Y . Deep results concerning the factorization of reflexive maps between Fréchet spaces through a reflexive Fréchet space can be seen in Díaz, Domanski [5].

PROPOSITION 2.4. *The following conditions are equivalent for a reflexive operator T from a locally convex space F into a Banach space X :*

- (1) T factorizes through a reflexive Banach space.
- (2) T is weakly compact.

Proof. We assume that a reflexive operator $T \in L(F, X)$ satisfies condition (1). There are a reflexive Banach space Y and continuous linear operators $T_1 \in L(F, Y)$ and $T_2 \in L(Y, X)$ such that $T = T_2 \circ T_1$. Find $U \in \mathcal{U}_0(E)$ such that $T_1(U) \subset Y_1$. Since Y is reflexive, $T_2(Y_1)$ is relatively $\sigma(X, X')$ -compact in X . Therefore $T(U) \subset T_2(Y_1)$ is also relatively $\sigma(X, X')$ -compact in X , and T is weakly compact.

Conversely, suppose that $T \in L(F, X)$ is weakly compact and find $U \in \mathcal{U}_0(E)$ such that $T(U)$ is relatively $\sigma(X, X')$ -compact in X . Let q be the Minkowski functional of U . Then $T = S \circ \pi_q$, with $S : F_q \rightarrow X$ the unique continuous extension of $S(x + \text{Ker } q) := T(x)$. The closed unit ball of F_q is $B_q = \overline{\pi_q(U)}$, the closure taken in F_q . Therefore $S(B_q) \subset \overline{T(U)}$, the closure taken in X . This implies that S is weakly compact between the Banach spaces. By the theorem of Davis, Figiel, Johnson and Pelczyński [4], S factorizes through a reflexive Banach space Y ; hence T also factorizes through Y . ■

COROLLARY 2.5. *Every reflexive operator from a quasinormable locally convex space into a Banach space factorizes through a reflexive Banach space. In particular, reflexive operators from a (DF)-space into a Banach space factorize through a reflexive Banach space.*

Proof. This is a consequence of Proposition 2.4 and Grothendieck [6, Cor. 1 of Thm 11] and [7, IV,4.3, Cor. 1 of Thm 2]. ■

There is another class of locally convex spaces F such that every Banach valued reflexive operator factorizes through a reflexive Banach space. A locally convex space F is called *infra-Schwartz* (cf. [9, 7.1.3]) if for every continuous seminorm p on F there is a continuous seminorm $q \geq p$ such that the canonical map $\pi_{p,q} : F_q \rightarrow F_p$ is weakly compact. Complete infra-Schwartz spaces are projective limits of spectra of reflexive Banach spaces [9, 7.5.3] and every reflexive quasinormable locally convex space is infra-Schwartz [9, 7.5.2]. There are infra-Schwartz Fréchet spaces which are not quasinormable and quasinormable Fréchet spaces which are not infra-Schwartz. It follows from the definition that *every reflexive operator from*

an infra-Schwartz space into a Banach space factorizes through a reflexive Banach space. Infra-Schwartz Fréchet spaces were investigated by Floret; see [16, Section 8.5]. Valdivia [23] proved that a Fréchet space F is infra-Schwartz if and only if it is totally reflexive, i.e. every separated quotient of F is reflexive. Reflexive non totally reflexive Fréchet spaces exist. They are used in our last result.

PROPOSITION 2.6. *There exist a Fréchet Montel space F and a continuous surjection $T : F \rightarrow X$ onto a Banach space X which does not factorize through a reflexive Banach space.*

Proof. Köthe and Grothendieck constructed Fréchet Montel spaces F which have a quotient isomorphic to ℓ_1 or c_0 ; see [10, 31.7] and [11, Example 27.21]. Denote by $T : F \rightarrow X$ the quotient map. Since every bounded set in F is relatively compact, T is reflexive. If there were a reflexive Banach space Y and continuous linear operators $T_1 \in L(F, Y)$ and $T_2 \in L(Y, X)$ such that $T = T_2 \circ T_1$, then T_2 would be a continuous surjection from Y into X . By the open mapping theorem, this would imply that the non-reflexive Banach space X is a quotient of the Banach space Y . A contradiction. ■

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REFERENCES

- [1] K.D. Bierstedt, J. Bonet, *A question of D. Vogt on (LF)-spaces*, Archiv Math. **61** (1993), 170–172.
- [2] K.D. Bierstedt, J. Bonet, *Some aspects of the modern theory of Fréchet spaces*, RACSAM Rev. R. Acad. Ci. Serie A Mat. **97** (2003), 159–188.
- [3] J. Bonet, J.D.M. Wright, *Non-commutative locally convex measures*, Quart. J. Math. **62** (2011), 21–38, doi: 10.1093/qmath/hap018.
- [4] W.J. Davis, T. Figiel, W.B. Johnson, A. Pelczyński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327.
- [5] J. Díaz, P. Domański, *Reflexive operators with domain in Köthe spaces*, Manuscripta Math. **97** (1998), 189–204.
- [6] A. Grothendieck, *Sur les espaces (F) et (DF)*, Summa Brasil. Math. **3** (1954), 57–122.
- [7] A. Grothendieck, *Topological Vector Spaces*, New York, 1973.
- [8] H. Jarchow, *Locally Convex Spaces*; Math. Leitfäden, B.G. Teubner, Stuttgart, 1981.
- [9] H. Junek, *Locally Convex Spaces and Operator Ideals*; Teubner, Leipzig, 1983.
- [10] G. Köthe, *Topological Vector Spaces I and II*, Springer Verlag, Berlin-Heidelberg-New York, 1969 and 1979.
- [11] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford, 1997.
- [12] A.M. Peralta, *Topological characterization of weakly compact operators revisited*, Extracta Math. **22** (2007), 215–223.
- [13] A.M. Peralta, I. Villanueva, J.D.M. Wright, K. Ylinen, *Topological characterization of weakly compact operators*, J. Math. Anal. Appl. **325** (2007), 968–974.
- [14] A.M. Peralta, I. Villanueva, J.D.M. Wright, K. Ylinen, *Quasi-completely continuous multi-linear operators*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), 635–649.

- [15] A.M. Peralta, I. Villanueva, J.D.M. Wright, K. Ylinen, *Weakly compact operators and the strong** topology for a Banach space*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), 1249–1267.
- [16] P. Pérez Carreras, J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland, Amsterdam, 1987.
- [17] J.-H. Qiu, *Retakh's conditions and regularity properties of (LF)-spaces*, Arch. Math. **67** (1996), 302–307.
- [18] J.-H. Qiu, *Regularity properties of (LM)-spaces of type (M_0)* , Math. Proc. Roy. Ir. Acad. **99A** (1999), 105–111.
- [19] W. Ruess, *(Weakly) compact operators and DF spaces*, Pacific J. Math. **98** (1982), 419–441.
- [20] R.A. Ryan, *Dunford-Pettis properties*, Bull. Polish Acad. Sci. Math. **27** (1979), 373–379.
- [21] K. Saitô, J.D.M. Wright, *Extending a result of Ryan on weakly compact operators*, Proc. Edinb. Math. Soc. **49** (2006), 427–433.
- [22] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland, Amsterdam, 1982.
- [23] M. Valdivia, *A characterization of totally reflexive Fréchet spaces*, Math. Z. **200** (1989), 327–346.
- [24] D. van Dulst, *(Weakly) compact mappings into (F) spaces*, Math. Ann. **224** (1976), 111–115.
- [25] J. Wengenroth, *Derived Functors in Functional Analysis*, Lecture Notes in Mathematics **1810**, Springer, Berlin, 2003.
- [26] J.D.M. Wright, *Right topology for Banach spaces and weak compactness*, Atti. Semin. Mat. Fis. Univ. Modena Reggio Emilia **55** (2007), 153–163.
- [27] J.D.M. Wright, K. Ylinen, *Multilinear maps on products of operator algebras*, J. Math. Anal. Appl. **292** (2004), 558–570.
- [28] K. Ylinen, *Weakly compact operators into sequence spaces: a counterexample*, Proc. Amer. Math. Soc. **133** (2005), 1423–1425.

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