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ON THE FINE SPECTRUM OF GENERALIZED UPPER DOUBLE-BAND MATRICES Δ^{uv} OVER THE SEQUENCE SPACE ℓ_1

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Abstract. The main purpose of this paper is to determine the fine spectrum of generalized upper triangle double-band matrices Δ^{uv} over the sequence space ℓ_1.

1. Introduction

In functional analysis, the spectrum of an operator generalizes the notion of eigenvalues for matrices. The spectrum of an operator over a Banach space is partitioned into three parts, which are the point spectrum, the continuous spectrum and the residual spectrum. The calculation of three parts of the spectrum of an operator is called calculating the fine spectrum of the operator.

Several authors have studied the spectrum and fine spectrum of linear operators defined by some particular limitation matrices over some sequence spaces. We introduce knowledge in the existing literature concerning the spectrum and the fine spectrum. The fine spectrum of the Cesaro operator on the sequence space ℓ_p for $(1 < p < \infty)$ has been studied by Gonzalez [12]. Also, Wenger [20] examined the fine spectrum of the integer power of the Cesaro operator over c, and Rhoades [17] generalized this result to the weighted mean methods. Reade [16] worked the spectrum of the Cesaro operator over the sequence space c_0. Okutoyi [15] computed the spectrum of the Cesaro operator over the sequence space bv. The fine spectrum of the Rhally operators on the sequence spaces c_0 and c is studied by Yildirim [22]. The fine spectra of the Cesaro operator over the sequence spaces c_0 and bv_p have determined by Akhmedov and Basar [1, 4]. Akhmedov and Basar [2, 3] have studied the fine spectrum of the difference operator Δ over the sequence spaces ℓ_p, and bv_p, where $(1 \le p < \infty)$. The fine spectrum of the Zweier matrix as an operator over the sequence spaces ℓ_1 and bv_1 have been examined by Altay and Karakus [6]. Altay and Basar [5, 9] have determined the fine spectrum of the difference operator Δ over the sequence spaces c_0, c and ℓ_p, where $(0 < p < 1)$. The fine spectrum of the difference operator Δ over the sequence spaces ℓ_1 and bv is

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investigated by Kayaduman and Furkan [13]. Altun and Karakaya [7, 8] has been studied the fine spectra of Lacunary matrices and fine spectra of upper triangular double-band matrices. recently, Srivastava and Kumar [18] has been examined the fine spectrum of the generalized difference operator Δ_v over the sequence space c_0.

In this work, our purpose is to determine the fine spectra of the generalized upper double-band matrices Δ^{uv} as operators over the sequence space ℓ_1.

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. Let μ and ν be two sequence spaces and $A = (a_{n,k})$ be an infinite matrix operator of real or complex numbers $a_{n,k}$, where $n, k \in \mathbb{N} = \{0, 1, 2, \dots\}$. We say that A defines a matrix mapping from μ into ν and denote it by $A : \mu \longrightarrow \nu$, if for every sequence $x = \big(x_k\big) \in \mu$ the sequence $Ax = \big((Ax)_n\big)$, the A-transform of x, is in ν, where $(Ax)_n = \sum_{k=0}^{\infty} a_{n,k}x_k$.

Let X and Y be Banach spaces and let $T : X \longrightarrow Y$ be a bounded linear operator. By $R(T)$, we denote the range of T, i.e.,

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

By $B(X)$, we denote the set of all bounded linear operators of X into itself. If X is any Banach space and $T \in B(X)$ then the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^*\psi)(x) = \psi(Tx)$ for all $\psi \in X^*$ and $x \in X$ with $\|T\| = \|T^*\|$.

Let $X \neq \Theta$ be a complex normed space and let $T : \mathcal{D}(T) \longrightarrow X$ be a bounded linear operator with domain $\mathcal{D} \subseteq X$. With T, we associate the operator $T_\lambda = T - \lambda I$, where λ is a complex number and I is the identity operator on $\mathcal{D}(T)$, if T_λ has an inverse, which is linear, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1}$$

and call it the *resolvent* operator of T.

The name resolvent is appropriate, since T_λ^{-1} helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1}y$ provided T_λ^{-1} exists. More important, the investigation of properties of T_λ^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_λ and T_λ^{-1} depend on λ, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all λ in the complex plane such that T_λ^{-1} exists. Boundedness of T_λ^{-1} is another property that will be essential. We shall also ask for what λ the domain of T_λ^{-1} is dense in X, to name just a few aspects. For our investigation of T, T_λ and T_λ^{-1}, we shall need some basic concepts in spectral theory which are given as follows (see [13, pp. 370–371]):

Definition 1.1. Let $X \neq \Theta$ be a complex normed space and $T : \mathcal{D}(T) \longrightarrow X$, be a linear operator with domain $\mathcal{D} \subseteq X$. A *regular* value of T is a complex number λ such that
$(R1)$ T_λ^{-1} exists,
$(R2)$ T_λ^{-1} is bounded,
$(R3)$ T_λ^{-1} is defined on a set which is dense in X.

The *resolvent* set $\rho(T,X)$ of T is the set of all *regular* values λ of T. Its complement $\sigma(T,X)=\mathbb{C}-\rho(T,X)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into three disjoint sets as follows:

The *point spectrum* $\sigma_p(T,X)$ is the set of all $\lambda\in\mathbb{C}$ such that T_λ^{-1} does not exist. The element of $\sigma_p(T,X)$ is called *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set of all $\lambda\in\mathbb{C}$ such that T_λ^{-1} exists and satisfies $(R3)$ but not $(R2)$, that is, T_λ^{-1} is unbounded.

The *residual spectrum* $\sigma_r(T,X)$ is the set of all $\lambda\in\mathbb{C}$ such that T_λ^{-1} exists but do not satisfy $(R3)$, that is, the domain of T_λ^{-1} is not dense in X. The condition $(R2)$ may or may not holds good.

Goldberg's classification of operator $T_\lambda=(T-\lambda I)$ (see [11, pp. 58–71]): Let X be a Banach space and $T_\lambda=(T-\lambda I)\in B(X)$, where $\lambda\in\mathbb{C}$. Again let $R(T_\lambda)$ and T_λ^{-1} denote the range and inverse of the operator T_λ, respectively. Then following possibilities may occur:

(A) $R(T_\lambda)=X$,
(B) $R(T_\lambda)\neq\overline{R(T_\lambda)}=X$,
(C) $\overline{R(T_\lambda)}\neq X$,
and
(1) T_λ is injective and T_λ^{-1} is continuous,
(2) T_λ is injective and T_λ^{-1} is discontinuous,
(3) T_λ is not injective.
If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1,A_2,A_3,B_1,B_2,B_3,C_1,C_2$ and C_3. If λ is a complex number such that $T_\lambda\in A_1$ or $T_\lambda\in B_1$, then λ is in the resolvent set $\rho(T,X)$ of T on X. The other classifications give rise to the fine spectrum of T. We use $\lambda\in B_2\sigma(T,X)$ to denote that the operator $T_\lambda\in B_2$, i.e. $R(T_\lambda)\neq\overline{R(T_\lambda)}=X$ and T_λ is injective but T_λ^{-1} is discontinuous. Similarly for the others.

Lemma 1.2. [11, p. 59] *A linear operator T has a dense range if and only if the adjoint T^* is one to one.*

Lemma 1.3. [11, p. 60] *The adjoint operator T^* is onto if and and only if T has a bounded inverse.*

Lemma 1.4. *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(c_0)$ from c_0 to itself if and only if*
(1) *the rows of A in ℓ_1 and their ℓ_1 norms are bounded.*
(2) *the columns of A are in c_0.*

Note: The operator norm of T is the supremum of the ℓ_1 norms of rows.

Lemma 1.5. *The matrix $A=(a_{nk})$ gives rise to a bounded linear operator $T\in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

Corollary 1.6. [11, Corollary II.5.3] $\sigma_r(T,X) \subseteq \sigma_p(T^*,X^*) \subseteq \sigma_r(T,X) \cup \sigma_p(T,X)$.

In this paper, we introduce a class of generalized upper triangular double-band matrices Δ^{uv} over space ℓ_1.

Let (u_k) be a sequence of positive real numbers such that $u_k \neq 0$ for each $k \in \mathbb{N}$ with $u = \lim_{k\to\infty} u_k \neq 0$ and (v_k) is either constant or strictly decreasing sequence of positive real numbers with $v = \lim_{k\to\infty} v_k \neq 0$, and $\sup_k v_k < u + v$. We define the operator Δ^{uv} on sequence space ℓ_1 as follows:

$$\Delta^{uv}x = \Delta^{uv}(x_n) = (v_n x_n + u_{n+1}x_{n+1})_{n=0}^{\infty}.$$

It is easy to verify that the operator Δ^{uv} can be represented by the matrix,

$$\Delta^{uv} = \begin{bmatrix} v_0 & u_1 & 0 & 0 & 0 & \cdots \\ 0 & v_1 & u_2 & 0 & 0 & \cdots \\ 0 & 0 & v_2 & u_3 & 0 & \cdots \\ 0 & 0 & 0 & v_3 & u_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

2. Main results

In this section, we compute spectrum, the point spectrum, the continuous spectrum and the residual spectrum of the generalized upper double-band matrices Δ^{uv} over the sequence space ℓ_1.

Theorem 2.1. *The operator $\Delta^{uv} : \ell_1 \longrightarrow \ell_1$ is a bounded linear operator and*

$$\|\Delta^{uv}\| = \sup_k(|v_k| + |u_{k+1}|).$$

Proof. It is elementary. ■

If $T : \ell_1 \longrightarrow \ell_1$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : \ell_1^* \longrightarrow \ell_1^*$ is defined by the transpose of the matrix A. The dual space of ℓ_1 is isomorphic to ℓ_∞, the space of all bounded sequences, with the norm $\|x\| = \sup_k |x_k|$.

We now give the theorem about the point spectrum of the dual operator $(\Delta^{uv})^*$ of Δ^{uv} over the space ℓ_1^*.

Theorem 2.2. *The point spectrum of the operator $(\Delta^{uv})^*$ over ℓ_1^* is*

$$\sigma_p((\Delta^{uv})^*, \ell_1^*) = \emptyset$$

Proof. The proof of this theorem is divided into two cases.

Case (i): Suppose (v_k) is a constant sequence, say $v_k = v$ for all k. Consider $(\Delta^{uv})^* f = \lambda f$, for $f \neq \mathbf{0} = (0,0,0,\dots)$ in $\ell_1^* \cong \ell_\infty$, where

$$(\Delta^{uv})^* = \begin{bmatrix} v_0 & 0 & 0 & 0 & 0 & \cdots \\ u_1 & v_1 & 0 & 0 & 0 & \cdots \\ 0 & u_2 & v_2 & 0 & 0 & \cdots \\ 0 & 0 & u_2 & v_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \end{bmatrix}$$

This gives

$$\begin{aligned} v_0 f_0 &= \lambda f_0 \\ u_1 f_0 + v_1 f_1 &= \lambda f_1 \\ u_2 f_1 + v_2 f_2 &= \lambda f_2 \\ &\vdots \\ u_k f_{k-1} + v_k f_k &= \lambda f_k \\ &\vdots \end{aligned}$$

Let f_m be the first non-zero entry of the sequence (f_n). So we get $u_m f_{m-1} + v f_m = \lambda f_m$ which implies $\lambda = v$ and from the equation $u_{m+1} f_m + v f_{m+1} = \lambda f_{m+1}$ we get $f_m = 0$, which is a contradiction to our assumption. Therefore,

$$\sigma_p((\Delta^{uv})^*, \ell_1^*) = \emptyset.$$

Case (ii): Suppose (v_k) is a strictly decreasing sequence. Consider $(\Delta^{uv})^* f = \lambda f$, for $f \neq \mathbf{0} = (0, 0, 0, \dots)$ in $\ell_1^* \cong \ell_\infty$, which gives above system of equations. Hence, for all $\lambda \notin \{v_0, v_1, v_2, \dots\}$, we have $f_k = 0$ for all k, which is a contradiction. So $\lambda \notin \sigma_p((\Delta^{uv})^*, \ell_1^*)$. This shows that

$$\sigma_p((\Delta^{uv})^*, \ell_1^*) \subseteq \{v_0, v_1, v_2, \dots\}.$$

Let $\lambda = v_m$ for some m. Then $f_0 = f_1 = \dots = f_{m-1} = 0$. Now if $f_m = 0$, then $f_k = 0$ for all k, which is a contradiction. Also if $f_m \neq 0$, then

$$f_{k+1} = \frac{u_{k+1}}{v_m - v_{k+1}} f_k, \text{ for all } k \geq m,$$

and hence

$$\lim_{k\to\infty} \left| \frac{f_{k+1}}{f_k} \right| = \lim_{k\to\infty} \left| \frac{u_{k+1}}{v_m - v_{k+1}} \right| = \left| \frac{u}{v_m - v} \right| > 1 \text{ for all } k \geq m,$$

since $v_m < v + u$. Then, $f \notin \ell_1^*$. Thus

$$\sigma_p((\Delta^{uv})^*, \ell_1^*) = \emptyset. \quad \blacksquare$$

Theorem 2.3. *For any $\lambda \in C$, $\Delta_\lambda^{uv} : \ell_1 \longrightarrow \ell_1$ has a dense range.*

Proof. By Theorem 2.2, $\sigma_p((\Delta^{uv})^*, \ell_1^*) = \emptyset$. Hence $(\Delta^{uv})^* - \lambda I$ is one to one for all λ. By applying Lemma 1.2, we get the result. $\blacksquare$

Corollary 2.4. *Residual spectrum $\sigma_r(\Delta^{uv}, \ell_1)$ of operator Δ^{uv} over ℓ_1 is*

$$\sigma_r(\Delta^{uv}, \ell_1) = \emptyset$$

We define the operator Δ_{uv} on sequence space c_0 as follows:

$$\Delta_{uv} x = \Delta_{uv}(x_n) = (u_{n-2} x_{n-1} + v_n x_n)_{n=0}^{\infty} \text{ with } u_{-2} = u_{-1} = u_0 = 0$$

It is easy to verify that the operator Δ_{uv} can be represented by the matrix,

$$\Delta_{uv} = \begin{bmatrix} v_0 & 0 & 0 & 0 & 0 & \cdots \\ u_1 & v_1 & 0 & 0 & 0 & \cdots \\ 0 & u_2 & v_2 & 0 & 0 & \cdots \\ 0 & 0 & u_2 & v_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

THEOREM 2.5. $\sigma_p(\Delta_{uv}, c_o) = \emptyset$

Proof. The proof may be obtained by proceeding as in proving Theorem 2.2 so, we omit the details. ■

If $T : c_0 \longrightarrow c_0$ is a bounded linear operator with matrix A, then it is known that the adjoint operator $T^* : c_0^* \longrightarrow c_0^*$ is defined by the transpose of the matrix A. The dual space of c_0 is isomorphic to ℓ_1.

THEOREM 2.6. $\sigma_r(\Delta_{uv}, c_0) = \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$.

Proof. We show that the operator $\Delta_{uv} - \lambda I$ has an inverse and $\overline{R(\Delta_{uv} - \lambda I)} \neq c_0$ for λ satisfying $|\lambda - v| < u$. If $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$, then the operator $\Delta_{uv} - \lambda I$ is a triangle except for $\lambda = v$ (when (v_k) is a constant sequence) and $\lambda = v_k$, for some $k \in \mathbb{N}$ and consequently the operator $\Delta_{uv} - \lambda I$ has an inverse. Further by Theorem 2.5, the operator $\Delta_{uv} - \lambda I$ is one to one for $\lambda = v$ (when (v_k) is a constant sequence) and $\lambda = v_k$, for some $k \in \mathbb{N}$ and hence has an inverse. Now, we show that if $\lambda \in \{\lambda \in \mathbb{C} : |\lambda - v| < u\}$, then the operator $\Delta_{uv}^* - \lambda I$ is not one to one.

Suppose $\Delta_{uv}^* y = \lambda y$, for $y \neq \mathbf{0} = (0, 0, 0, \dots)$ in ℓ_1, where $\Delta_{uv}^* = \Delta^{uv}$. This gives

$$\begin{aligned} v_0 y_0 + u_1 y_1 &= \lambda y_0 \\ v_1 y_1 + u_2 y_2 &= \lambda y_1 \\ v_2 y_2 + u_3 y_3 &= \lambda y_2 \\ &\vdots \\ v_k y_k + u_{k+1} y_{k+1} &= \lambda y_k \\ &\vdots \end{aligned}$$

If $y_0 = 0$, then $y_k = 0$ for all k. Hence $y_0 \neq 0$ and solving the equation above, we get

$$y_{k+1} = \left(\frac{\lambda - v_k}{u_{k+1}} \right) y_k \text{ for all } k \geq 0,$$

and consequently

$$\lim_{k \to \infty} \left| \frac{y_{k+1}}{y_k} \right| = \left| \frac{v - \lambda}{u} \right| < 1 \text{ provided } |v - \lambda| < u.$$

Hence, $|v-\lambda|<u \Rightarrow y=(y_k)\in \ell_1$, which shows that $\Delta_{uv}^{*}-\lambda I$ is not one to one. Now Lemma 1.2 yields the fact that the range of the operator $\Delta_{uv}-\lambda I$ is not dense in c_0 and this step completes the proof. ■

THEOREM 2.7. $\sigma_p(\Delta^{uv},\ell_1)=\{\lambda\in\mathbb{C} : |\lambda-v|<u\}$.

Proof. This follows by Corollary 1.6 and Theorems 2.5–2.6.

THEOREM 2.8. *The spectrum of* Δ^{uv} *on* ℓ_1 *is given by*

$$\sigma(\Delta^{uv},\ell_1)=\{\lambda\in C : |\lambda-v|\le u\}.$$

Proof. Let $f\in\ell_\infty$ and consider $(\Delta_\lambda^{uv})^{*}x=f$. Then we have the linear system of equations

$$\begin{aligned}(v_0-\lambda)x_0&=f_0\\ u_1x_0+(v_1-\lambda)x_1&=f_1\\ u_2x_1+(v_2-\lambda)x_2&=f_2\\ &\vdots\\ u_kf_{k-1}+(v_k-\lambda)x_k&=f_k\\ &\vdots\end{aligned}$$

Solving the equations, for $x=(x_k)$ in terms of f, we get

$$x_0=\frac{1}{v_0-\lambda},\ \text{and}\ x_k=\frac{1}{v_k-\lambda}\sum_{i=0}^{k}\prod_{j=i}^{k-1}\left(\frac{u_{j+1}}{\lambda-v_j}\right)f_i,\ \text{for}\ k\ge 1.$$

Then $|x_k|\le S_k\|f\|_\infty$, where

$$\begin{aligned}S_k=&\frac{1}{|v_k-\lambda|}+\frac{u_k}{|v_{k-1}-\lambda||v_k-\lambda|}+\frac{u_{k-1}u_k}{|v_k-\lambda||v_{k-1}-\lambda||v_{k-2}-\lambda|}\\ &+\cdots+\frac{u_1u_2\ldots u_k}{|v_0-\lambda||v_1-\lambda|\cdots|v_k-\lambda|}.\end{aligned}$$

Clearly each S_k is finite. Now we prove that $\sup_k S_k$ is finite. If $u<|\lambda-v|$, then $\lim_{n\to\infty}\left|\frac{u_k}{v_{k-1}-\lambda}\right|=\frac{u}{|v-\lambda|}=p<1$. Then there exists $k\in\mathbb{N}$ such that $\frac{u_n}{|v_{n-1}-\lambda|}<p_0<1$, for all $n\ge k+1$ and so we get

$$\begin{aligned}S_{n+k}\le\frac{1}{|v_{n+k}-\lambda|}\Bigg(&\frac{u_1u_2\cdots u_k}{|v_0-\lambda||v_1-\lambda|\cdots|v_{k-1}-\lambda|}p_0^n\\ &+\frac{u_2u_3\ldots u_k}{|v_1-\lambda|\ldots|v_{k-1}-\lambda|}p_0^{n-1}+\cdots+p_0+1\Bigg).\end{aligned}$$

If we put $M=\max\left\{\frac{u_ju_{j+1}\ldots u_k}{|v_{j-1}-\lambda||v_j-\lambda|\ldots|v_k-\lambda|} : 1\le j\le k\right\}$, then we have

$$S_{n+k}\le\frac{M}{|v_{n+k}-\lambda|}\left(1+p_0+p_0^2+\cdots+p_0^n\right)\le\frac{M}{|v_{n+k}-\lambda|}\left(1+p_0+p_0^2+\cdots\right).$$

But, for large n, we have $\frac{1}{|v_{n+k}-\lambda|} < d < \frac{1}{u}$ and so $S_{n+k} \leq \frac{Md}{1-p_0}$, for all $n \geq k+1$. Thus, $\sup_k S_k < \infty$. This shows that $\|x\|_\infty \leq \sup_k S_k \|f\|_\infty < \infty$. Therefore $x \in \ell_\infty$. Hence, for $u < |\lambda - v|$, $(\Delta_\lambda^{uv})^*$ is onto, and by Lemma 1.3, Δ_λ^{uv} has a bounded inverse. This means that

$$\sigma_c(\Delta^{uv}, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| \leq u\}.$$

Combining this with Theorem 2.2 and Corollary 2.5, we get

$$\{\lambda \in C : |\lambda - v| < u\} \subseteq \sigma(\Delta^{uv}, \ell_1) \subseteq \{\lambda \in C : |\lambda - v| \leq u\}.$$

Since the spectrum of any bounded operator is closed, we have

$$\sigma(\Delta^{uv}, \ell_1) = \{\lambda \in C : |\lambda - v| \leq u\}. \quad \blacksquare$$

THEOREM 2.9. *Continuous spectrum $\sigma_c(\Delta^{uv}, \ell_1)$ of operator Δ^{uv} over ℓ_1 is*

$$\sigma_c(\Delta^{uv}, \ell_1) = \{\lambda \in C : |\lambda - v| = u\}.$$

Proof. Since $\sigma_r(\Delta^{uv}, \ell_1) = \emptyset$, $\sigma_p(\Delta^{uv}, \ell_1) = \{\lambda \in C : |\lambda - v| < u\}$ and $\sigma(\Delta^{uv}, \ell_1)$ is the disjoint union of the parts $\sigma_p(\Delta^{uv}, \ell_1)$, $\sigma_r(\Delta^{uv}, \ell_1)$ and $\sigma_c(\Delta^{uv}, \ell_1)$, we deduce that

$$\sigma_c(\Delta^{uv}, \ell_1) = \{\lambda \in C : |\lambda - v| = u\}. \quad \blacksquare$$

THEOREM 2.10. *If $|\lambda - v| < u$, then $\lambda \in A_3\sigma(\Delta^{uv}, \ell_1)$.*

Proof. Let $|\lambda - v| < u$. Then by Theorem 2.2, $\lambda \in (3)$; it remains to prove that Δ_λ^{uv} is surjective when $|\lambda - v| < u$. Let $y = (y_0, y_1, y_2, \dots) \in \ell_1$ and consider the equation $\Delta_\lambda^{uv} x = y$. Then we have the linear system of equations

$$\begin{aligned}
(v_0 - \lambda)x_0 + u_1 x_1 &= y_0 \\
(v_1 - \lambda)x_1 + u_2 x_2 &= y_1 \\
(v_2 - \lambda)x_2 + u_3 x_3 &= y_2 \\
&\vdots \\
(v_k - \lambda)x_k + u_{k+1} x_{k+1} &= y_k \\
&\vdots
\end{aligned}$$

Now, set $x_0 = 0$ and by solving these equations, we get $x_1 = \frac{1}{u_1} y_0$ and

$$x_k = \frac{1}{u_k}\left(\sum_{i=0}^{k-2}\left[\prod_{j=i+1}^{k-1}\frac{\lambda - v_j}{u_j}\right] y_i + y_{k-1}\right) \text{ for all } k \geq 2.$$

Then $\sum_k |x_k| \leq \sum_k S_k |y_k|$, where

$$S_k = \frac{1}{u_{k+1}} + \frac{1}{u_{k+2}}\frac{|v_{k+1} - \lambda|}{u_{k+1}} + \frac{1}{u_{k+3}}\frac{|v_{k+1} - \lambda|}{u_{k+1}}\frac{|v_{k+2} - \lambda|}{u_{k+2}} + \cdots \text{ for all } k.$$

Let

$$S_{n,k} = \frac{1}{u_{k+1}} + \frac{1}{u_{k+2}}\frac{|v_{k+1}-\lambda|}{u_{k+1}} + \frac{1}{u_{k+3}}\frac{|v_{k+1}-\lambda|}{u_{k+1}}\frac{|v_{k+2}-\lambda|}{u_{k+2}} + \cdots + \frac{1}{u_{k+n+1}}\frac{|v_{k+1}-\lambda|}{u_{k+1}}\frac{|v_{k+2}-\lambda|}{u_{k+2}}\cdots\frac{|v_{k+n}-\lambda|}{u_{k+n}} \text{ for all } k, n.$$

Then

$$S_n = \lim_{k\to\infty} S_{n,k} = \frac{1}{u} + \frac{|v-\lambda|}{u^2} + \frac{|v-\lambda|^2}{u^3} + \cdots + \frac{|v-\lambda|^n}{u^{n+1}}.$$

Now for $|\lambda - v| < u$, we can see that

$$S = \lim_{n\to\infty} S_n = \frac{1}{u} + \frac{|v-\lambda|}{u^2} + \frac{|v-\lambda|^2}{u^3} + \cdots < \infty,$$

hence (Sk) is a convergent sequence of positive real numbers with the limit S. Therefore, (S_k) is bounded and $\sup_k S_k < \infty$. Thus $\sum_k |x_k| \leq \sup_k S_k \sum_k |f_k| < \infty$. This shows that $x \in \ell_1$. ■

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