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SOME RESULTS ON TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study ϕ-conformally (resp. conharmonically, projectively) flat trans-Sasakian manifolds.

1. Preliminaries

In 1985 J.A. Oubina [9] introduced a new class of almost contact metric manifolds, called trans-Sasakian manifold, which includes Sasakian, Kenmotsu and Cosymplectic structures. The local classification of trans-Sasakian manifold is given by J.C. Marrero [8]. Blair and Oubina [3] also obtained some fundamental results on this structure. D-homothetic deformations on trans-Sasakian manifold is studied by Shaikh et al. [14]. Conformally flat trans-Sasakian manifold is classified by Shaikh and Matsuyama [12]. Weakly symmetric trans-Sasakian structure is studied by Shaikh and Hui [13].

The Riemannian Christoffel curvature tensor R, the Weyl conformal curvature tensor C [15], the coharmonic curvature tensor K [7] and projective curvature tensor P [15] of $(2n+1)$-dimensional manifold M^{2n+1} are defined by

$$
\begin{aligned}
R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,\\
C(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y\\
&\qquad + g(Y,Z)QX - g(X,Z)QY],\\
&\qquad + \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y], \qquad (1.1)\\
K(X,Y)Z &= R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y\\
&\qquad + g(Y,Z)QX - g(X,Z)QY],\\
P(X,Y)Z &= R(X,Y)Z - \frac{1}{2n}[g(Y,Z)QX - g(X,Z)QY], \qquad (1.2)
\end{aligned}
$$

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respectively, where ∇ is the Levi$-$Civta connection, Q is the Ricci operator defined by $S(X,Y) = g(QX,Y)$, S is the Ricci tensor, τ is the scalar curvature and $X, Y, Z \in \chi(M^{2n+1})$, $\chi(M^{2n+1})$ being the Lie algebra of vector fields of M^{2n+1}. The paper is organized as follows:

In Section 2, we define a trans-Sasakian manifold and review some formulae which will be used in the later sections. In Section 3, we give the main results of the paper.

2. Trans-Sasakian manifolds

A differentiable manifold M^{2n+1} of class C^∞ is said to be an almost contact metric manifold [4], if it admits a $(1,1)$ tensor fields ϕ, a contravariant vector field ξ, a 1-form η and a Riemannin metric g, which satisfy

$$\begin{gathered}\phi^2 X = -X + \eta(X)\xi,\ \phi(\xi) = 0,\ \eta(\phi X) = 0,\\ g(X, \phi Y) = -g(\phi X, Y), \qquad g(X,\xi) = \eta(X), \qquad \eta(\xi) = 1,\\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),\end{gathered} \tag{2.1}$$

for all vector fields X, Y on M^{2n+1}.

An almost contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g)$ is said to be a trans-Sasakian manifold [9] if $(M^{2n+1} \times \mathbb{R}, J, G)$ belong to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on $M^{2n+1} \times \mathbb{R}$ defined by [6]

$$J\left(Z, f\frac{d}{dt}\right) = \left(\phi Z - f\xi, \eta(Z)\frac{d}{dt}\right),$$

for any vector field Z on M^{2n+1} and smooth function f on $M^{2n+1} \times \mathbb{R}$ and G is Hermitian metric on the product $M^{2n+1} \times \mathbb{R}$. This may be expressed by the condition [9]

$$(\nabla_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.2}$$

for some smooth functions α and β on M^{2n+1}, and we say that trans-Sasakian structure is of type (α,β). From equation (2.2), it follows that

$$\begin{gathered}\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi),\\ (\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).\end{gathered}$$

Further, on such a trans-Sasakian manifold M^{2n+1} with structure(ϕ,ξ,η,g), the following relations hold [5]

$$\begin{aligned}S(X,\xi) &= (2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\phi X)\alpha,\\ Q\xi &= (2n(\alpha^2 - \beta^2) - \xi\beta)\xi - (2n-1)\operatorname{grad}\beta + \phi(\operatorname{grad}\alpha).\end{aligned} \tag{2.3}$$

If ξ is an eigenvector of Q [11] then, we have either
$-(2n-1)\operatorname{grad}\beta + \phi(\operatorname{grad}\alpha) = a_1\xi$, or

$$-(2n-1)\,\mathrm{grad}\,\beta + \phi(\mathrm{grad}\,\alpha) = 0,$$

where a_1 is to be determined by applying η to both sides. We get

$$\begin{aligned}-(2n-1)\eta(\mathrm{grad}\,\beta) &= a_1,\\ g(\xi, -(2n-1)\,\mathrm{grad}\,\beta) &= a_1,\\ -(2n-1)\xi\beta &= a_1,\end{aligned}$$

hence $-(2n-1)\,\mathrm{grad}\,\beta + \phi(\mathrm{grad}\,\alpha) = -(2n-1)(\xi\beta)\xi.$

$$\begin{aligned}S(X,\xi) &= 2n(\alpha^2-\beta^2-\xi\beta)\eta(X),\\ Q\xi &= 2n(\alpha^2-\beta^2-\xi\beta)\xi,\\ S(\phi X,\phi Y) &= S(X,Y) - 2n(\alpha^2-\beta^2-\xi\beta)\eta(X)\eta(Y).\end{aligned} \tag{2.4}$$

In a trans-Sasakian manifold [6], we also have

$$\begin{aligned}R(X,Y)\xi &= (\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y]\\ &\quad + (Y\alpha)\phi X - (X\alpha)\phi Y - (Y\alpha)\phi X + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y,\\ R(\xi,X)\xi &= (\alpha^2-\beta^2-\xi\beta)[\eta(X)\xi - X],\end{aligned}$$

and

$$2\alpha\beta + \xi\alpha = 0.$$

A trans-Sasakian manifold M^{2n+1} is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

for any vector fields X and Y, where a, b are smooth functions on M^{2n+1}.

3. Main results

LEMMA 3.1. *In a trans-Sasakian manifold M^{2n+1}, the following statements are equivalent,*

(a) *ξ is an eigenvector of Ricci-operator;*

(b) *$\phi(\mathrm{grad}\,\alpha) - (2n-1)\,\mathrm{grad}\,\beta$ is parallel to ξ or zero.*

Proof. From equation (2.9) it is clear that statement (a) implies (b). It is also clear from equation (2.3) that (b) implies (a). ■

LEMMA 3.2. *In a trans-Sasakian manifold M^{2n+1} if $Q\phi = \phi Q$, then $\phi(\mathrm{grad}\,\alpha) - (2n-1)\,\mathrm{grad}\,\beta$ is parallel to ξ or zero.*

Proof. If $Q\phi = \phi Q$, then from equation (2.3), we have $Q\phi\xi = \phi Q\xi$, $\phi\{\phi(\mathrm{grad}\,\alpha) - (2n-1)\,\mathrm{grad}\,\beta\} = 0$l this implies that $\phi(\mathrm{grad}\,\alpha) - (2n-1)\,\mathrm{grad}\,\beta$ is parallel to ξ or zero. ■

LEMMA 3.3. *If ξ is an eigenvector of Ricci-operator Q and $(\alpha^2-\beta^2-\xi\beta) \neq 0$, then trans-Sasakian manifold cannot be flat.*

Lemma 3.4. *If $\phi(\operatorname{grad}\alpha) = (2n-1)\operatorname{grad}\beta$ and $(\alpha^2 - \beta^2 - \xi\beta) \neq 0$, then trans-Sasakian manifold cannot be flat.*

Due to these reasons, we have studied ϕ-conformally flat, ϕ-conharmonically flat and ϕ-projectively flat trans-Sasakian manifolds.

Definition 3.5. [10] A differentiable manifold M^{2n+1}, $(n > 1)$, satisfying the condition

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0, \tag{3.1}$$

is called ϕ-conformally flat.

In [1], the authors studied (k,μ)-contact metric manifolds satisfying equation (3.1). Now our aim is to find the characterization of trans-Sasakian manifolds satisfying the condition (3.1).

Definition 3.6. A differentiable manifold M^{2n+1}, $(n > 1)$, satisfying the condition

$$\phi^2 K(\phi X, \phi Y)\phi Z = 0, \tag{3.2}$$

is called ϕ-conharmonically flat.

In [2], the authors considered (k,μ)-contact manifolds satisfying condition (3.2). Now we will study the condition (3.2) on trans-Sasakian manifold.

Theorem 3.7. *Let M^{2n+1}, $(n > 1)$, be a ϕ-conformally flat trans-Sasakian manifold. Then M^{2n+1} is an η-Einstein manifold if ξ is an eigenvector of Q.*

Proof. Suppose that M^{2n+1} is a ϕ-conformally flat trans-Sasakian manifold. Then it is easy to see that $\phi^2 C(\phi X, \phi Y)\phi Z = 0$ holds if and only if $g(C(\phi X, \phi Y)\phi Z, \phi W) = 0$ for any $X, Y, Z \in \chi(M^{2n+1})$. So by the use of equation (1.2), ϕ-conformally flat means

$$\begin{aligned}
&g(R(\phi X, \phi Y)\phi Z, \phi W)\\
&\quad= \frac{1}{2n-1}[g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W)\\
&\qquad+ g(\phi X, \phi W)S(\phi Y, \phi Z) - g(\phi Y, \phi W)S(\phi X, \phi Z)]\\
&\qquad- \frac{r}{2n(2n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)].
\end{aligned} \tag{3.3}$$

Let $\{e_{1,\ldots,}e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M^{2n+1}. Using that $\{\phi e_{1,\ldots,}\phi e_{2n}, \xi\}$ is a local orthonormal basis, if we put $X = W = e_i$ in equation

(3.3) and sum up with respect to i, then

$$\sum_{i=1}^{2n} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i)$$
$$= \frac{1}{2n-1}\sum_{i=1}^{2n}[g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)$$
$$+ g(\phi e_i, \phi e_i)S(\phi Y, \phi Z) - g(\phi Y, \phi e_i)S(\phi e_i, \phi Z)]$$
$$- \frac{r}{2n(2n-1)}\sum_{i=1}^{2n}[g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)]. \tag{3.4}$$

It can be easily verified that

$$\begin{aligned}
\sum_{i=1}^{2n} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) &= S(\phi Y, \phi Z) - (\alpha^2 - \beta^2 - \xi\beta)g(\phi Y, \phi Z),\\
\sum_{i=1}^{2n} S(\phi e_i, \phi e_i) &= r - 2n(\alpha^2 - \beta^2 - \xi\beta),\\
\sum_{i=1}^{2n} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) &= S(\phi Y, \phi Z),\\
\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) &= 2n,\\
\sum_{i=1}^{2n} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) &= g(\phi Y, \phi Z).
\end{aligned} \tag{3.5}$$

Using equations (3.5), equation (3.4) can be written as

$$S(\phi Y, \phi Z) = [\frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta)]g(\phi Y, \phi Z). \tag{3.6}$$

Replacing Y by ϕY and Z by ϕZ in equation (3.6), we have

$$\begin{aligned}
S(Y, Z) = {}&\left[\frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta)\right] g(Y, Z)\\
&- \left[\frac{r}{2n} - (2n+1)(\alpha^2 - \beta^2 - \xi\beta)\right] \eta(Y)\eta(Z).
\end{aligned}$$

Hence, M^{2n+1} is an η-Einstein manifold. This completes the proof of the theorem. ∎

Corollary 3.8. *Let M^{2n+1}, $(n > 1)$, be a ϕ-conharmonically flat trans-Sasakian manifold. Then M^{2n+1} is an η-Einstein manifold with zero scalar curvature if ξ is an eigen vector of Q.*

Definition 3.9. A differentiable manifold M^{2n+1}, satisfying the condition

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0,$$

is called ϕ-projectively flat.

Theorem 3.10. *Let M^{2n+1} be a ϕ-projectively flat trans-Sasakian manifold. Then M^{2n+1} is an η-Einstein manifold if ξ is an eigenvector of Q.*

Proof. Suppose that M^{2n+1} is a ϕ-projectively flat trans-Sasakian manifold. Then it is easy to see that $\phi^2 P(\phi X, \phi Y)\phi Z = 0$ holds if and only if $g(P(\phi X, \phi Y)\phi Z, \phi W) = 0$ for any $X, Y, Z \in \chi(M^{2n+1})$. So by the use of equation (1.2) ϕ-projectively flat means

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2n-1}[g(\phi Y, \phi Z)S(\phi X, \phi W) - g(\phi X, \phi Z)S(\phi Y, \phi W). \tag{3.7}$$

Let $\{e_{1,\dots,}e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M^{2n+1}. Using that $\{\phi e_{1,\dots,}\phi e_{2n}, \xi\}$ is a local orthonormal basis, if we put $X = W = e_i$ in equation (3.7) and sum up with respect to i, then

$$\sum_{i=1}^{2n} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2n-1}\sum_{i=1}^{2n}[g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)]. \tag{3.8}$$

Using equations (3.5), equation (3.8) can be written as

$$S(\phi Y, \phi Z) = \left[\frac{r}{2n} - \frac{(\alpha^2 - \beta^2 - \xi\beta)}{2n}\right] g(\phi Y, \phi Z). \tag{3.9}$$

Replacing Y by ϕY and Z by ϕZ in equation (3.9), we have

$$S(Y, Z) = \left[\frac{r}{2n} - \frac{(\alpha^2 - \beta^2 - \xi\beta)}{2n}\right] g(Y, Z) - \left[\frac{r}{2n} + \frac{4n^2-1}{2n}(\alpha^2 - \beta^2 - \xi\beta)\right] \eta(Y)\eta(Z).$$

Hence, M^{2n+1} is an η-Einstein manifold. This completes the proof of the theorem. ■

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