

A COMMON GENERALIZATION OF FUZZY PRIMES

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Abstract. Let R be a commutative ring with identity. Let $FI(R)$ be the set of all fuzzy ideals of R and $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$ be a function. We introduce the concept of fuzzy ϕ -prime ideals of R and study some of its properties. It will be shown that under additional conditions fuzzy ϕ -primeness implies fuzzy primeness. We also prove that in the decomposable rings fuzzy $\phi_{(1)}$ -primes and fuzzy primes coincide. The behavior of this concept with fuzzy localization and fuzzy quotient is also studied.

1. Introduction

Throughout the paper, R will be a commutative ring with identity. In some sense, fuzzy prime ideals play the same role in fuzzy commutative algebra as the prime ideals in (ordinary) commutative algebra. Of course, a non-constant fuzzy ideal ξ is called fuzzy prime, if for two fuzzy ideals ζ, ρ of R , $\zeta \cdot \rho \subseteq \xi$ gives that either $\zeta \subseteq \xi$ or $\rho \subseteq \xi$; or equivalently for each $a, b \in R$, $t, s \in I = [0, 1]$, $t_a \cdot s_b \in \xi$ imply that either $t_a \in \xi$ or $s_b \in \xi$. By restricting or enlarging where a and/or b lie or restricting or enlarging where t_a , s_b and/or $t_a \cdot s_b$ lie, we can generalize the notion of fuzzy prime ideals. One generalization is the notion of fuzzy primary ideal where we enlarge where t_a or s_b lies.

Recently, various generalizations of prime ideals have been studied by several authors (see for example [1–4]). In this paper we will introduce the concept of fuzzy ϕ -prime ideals and study some of their properties. This is a common generalization of fuzzy prime ideals. The connection of this notion with fuzzy prime ideals are studied. We show that under additional assumption fuzzy ϕ -primes are fuzzy prime (Theorem 3.1). Also it will be shown that fuzzy $\phi_{(1)}$ -primes are only of interest in the indecomposable rings (Theorem 4.2). The behavior of this concept with fuzzy localization and quotient will also be studied.

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2. Preliminaries

Let R be a non-zero commutative ring with identity. To ease in access, we recall some definitions, notations and known results which are needed for the development of the paper.

For any unexplained notation, we refer to the textbook [5]. As usual, a fuzzy subset of a non-empty set X , is a map $\lambda : X \rightarrow I$, ($I = [0, 1]$). Then for $t \in I$,

$$\lambda_t = \{x \in X \mid \lambda(x) \geq t\},$$

is the t -cut of λ . For two fuzzy subsets η, λ of X , $\eta \subseteq \lambda$ if for each $x \in X$, $\eta(x) \leq \lambda(x)$, and $\eta \subset \lambda$ if $\eta \subseteq \lambda$ and there exists $x \in X$ with $\eta(x) < \lambda(x)$.

Let Z be a subset of X and let $t \in I$. Then

$$t_Z(x) = \begin{cases} t & \text{if } x \in Z, \\ 0 & \text{otherwise,} \end{cases}$$

is a fuzzy subset of X . In particular 0_X and 1_X are fuzzy subsets of X . If $z \in X$, $t_z := t_{\{z\}}$ is a fuzzy point in X . Then

$$\lambda(z) \geq t \Leftrightarrow t_z \in \lambda.$$

DEFINITION 2.1. (1) A fuzzy subset ζ of R is called a fuzzy ideal of R if for each $a, b \in R$, $\zeta(a - b) \geq \zeta(a) \wedge \zeta(b)$, $\zeta(ab) \geq \zeta(a) \vee \zeta(b)$, and $\zeta(0) = 1$. Then

$$\zeta_* = \{r \in R \mid \zeta(r) = 1\},$$

is an ideal of R . We denote the set of all fuzzy ideals of R by $FI(R)$.

(2) For two fuzzy subsets ζ and ρ of R , the product of ζ and ρ , denoted $\zeta \cdot \rho$, is defined by

$$(\zeta \cdot \rho)(r) = \bigvee_{a,b \in R, r=ab} \{\zeta(a) \wedge \rho(b)\}$$

for all $r \in R$. In particular for $t, s \in I, a, r \in R$

$$(t_a \cdot \rho)(r) = \bigvee_{b \in R, r=ab} \{t \wedge \rho(b)\},$$

and $t_a \cdot s_r = (t \wedge s)_{ar}$. We note that whenever $\zeta, \rho \in FI(R)$, then $\zeta \cdot \rho, \zeta^n \in FI(R)$ for each $n \in \mathbf{N}$, where $\zeta^n = \zeta^{n-1} \cdot \zeta$.

(3) $\xi \in FI(R)$ is called non-constant if $\xi \neq 1_R$. Then a non-constant fuzzy ideal ξ is called a fuzzy prime ideal if for $\zeta, \rho \in FI(R)$, $\zeta \cdot \rho \subseteq \xi$ gives that $\zeta \subseteq \xi$ or $\rho \subseteq \xi$. This is equivalent to say that $\forall t, s \in I, a, b \in R; t_a \cdot s_b \in \xi$ implies $t_a \in \xi$ or $s_b \in \xi$

(4) (Extension Principle) Let X, Y be two non-empty sets and $f : X \rightarrow Y$ be a map. Let λ, ν be fuzzy subsets of X and Y respectively. Then $f(\lambda)$ and $f^{-1}(\nu)$ as fuzzy subsets of Y and X are defined by

$$f(\lambda)(y) = \begin{cases} \bigvee \{\lambda(x) \mid x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $y \in Y$, and $f^{-1}(\nu)(x) = \nu(f(x))$, for all $x \in X$.

Suppose that S is another non-zero commutative ring with identity and $f : R \rightarrow S$ is an epimorphism, and let $\zeta \in FI(R)$, $v \in FI(S)$. Then $f(\zeta) \in FI(R)$ and $f^{-1}(v) \in FI(R)$.

DEFINITION 2.2. (1) (see [1]) Let $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ be a function, where $\mathcal{I}(R)$ is the set of all ideals of R . The ideal $\mathfrak{p} \neq R$ is said to be a ϕ -prime ideal of R if for each $a, b \in R$, $ab \in \mathfrak{p} \setminus \phi(\mathfrak{p})$ implies that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Since $\mathfrak{p} \setminus \phi(\mathfrak{p}) = \mathfrak{p} \setminus (\mathfrak{p} \cap \phi(\mathfrak{p}))$, we always may assume that $\phi(\mathfrak{a}) \subseteq \mathfrak{a}$ for all $\mathfrak{a} \in \mathcal{I}(R)$.

(2) Let $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$ be a function. Let ξ be a non-constant fuzzy ideal of R . We will say that ξ is a fuzzy ϕ -prime ideal of R , if for each $a, b \in R, t, s \in I, t_a \cdot s_b \in \xi, t_a \cdot s_b \notin \phi(\xi)$, give that either $t_a \in \xi$ or $s_b \in \xi$. Since for each $a \in R, t \in I, t_a \in \xi, t_a \notin \phi(\xi)$ if and only if $t_a \in \xi, t_a \notin \xi \cap \phi(\xi)$ we always assume in this paper that $\phi(\zeta) \subseteq \zeta$ for all $\zeta \in FI(R)$.

EXAMPLE AND REMARK 2.3. In the rest of the paper we will use the following functions $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$.

(1) $\phi_{(0)}(\zeta) = 0_R, \forall \zeta \in FI(R)$, then fuzzy $\phi_{(0)}$ -prime ideals are exactly fuzzy prime ideal.

(2) $\phi_{(1)}(\zeta) = 1_0, \forall \zeta \in FI(R)$.

(3) $\phi_{(n)}(\zeta) = \zeta^n, \forall n \geq 2, \forall \zeta \in FI(R)$.

(4) $\phi_{(t,s,\omega)}(\zeta) = (\bigcap_{i=1}^{\infty} t_{(\zeta_*)^i} \cdot s_{\zeta_*}) \cup 1_0, \forall t, s \in I, \zeta \in FI(R)$.

It is clear that any fuzzy prime ideal is fuzzy $\phi_{(1)}$ -prime and each fuzzy $\phi_{(n)}$ -prime is fuzzy $\phi_{(n-1)}$ -prime for all $n \geq 2$.

3. Fuzzy ϕ -primes and fuzzy primes

In the rest of the paper, without any confusion, for two ideals $\zeta, \nu \in FI(R), a \in R, t \in I$, by $t_a \in \zeta \setminus \nu$, we mean $t_a \in \zeta, t_a \notin \nu$. We begin with the following theorem which shows that with an additional assumption fuzzy ϕ -primes are indeed fuzzy prime.

THEOREM 3.1. *Let $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$ be a function. Let ξ be a fuzzy ϕ -prime ideal of R such that for each $t, s \in I_0 = (0, 1], t_{\xi_*} \cdot s_{\xi_*} \notin \phi(\xi)$. Then ξ is a fuzzy prime ideal of R .*

Proof. Let $a, b \in R, t, s \in I$ such that $t_a \cdot s_b \in \xi$. If $t_a \cdot s_b \notin \phi(\xi)$, then ξ is fuzzy ϕ -prime gives that $t_a \in \xi$ or $s_b \in \xi$.

Hence, assume that $t_a \cdot s_b \in \phi(\xi)$. In this case, we may assume that $t_a \cdot s_{\xi_*} \subseteq \phi(\xi)$. For, otherwise, $\exists c \in \xi_*$ such that $t \wedge s > \phi(\xi)(ac)$, which by the choice of c gives that $t_a \cdot s_{b+c} \in \xi$ and $t_a \cdot s_{b+c} \notin \phi(\xi)$. Therefore ξ is fuzzy ϕ -prime gives that either $t_a \in \xi$ or $s_{b+c} \in \xi$. So $t_a \in \xi$ or $s_b \in \xi$. By a similar argument we may assume that $t_{\xi_*} \cdot s_b \subseteq \phi(\xi)$. Now since $t_{\xi_*} \cdot s_{\xi_*} \notin \phi(\xi)$, there exist $a_1, b_1 \in \xi_*$ such that $t \wedge s > \phi(\xi)(a_1 b_1)$. Then $t_{a+a_1} \cdot s_{b+b_1} \in \xi \setminus \phi(\xi)$. Thus ξ is fuzzy ϕ -prime gives

that $t_{a+a_1} \in \xi$ or $s_{b+b_1} \in \xi$. Hence $t_a \in \xi$ or $s_b \in \xi$ and hence ξ must be a fuzzy prime ideal of R . ■

COROLLARY 3.2. *Let ξ be a fuzzy $\phi_{(1)}$ -prime ideal of R . Assume that for each $t, s \in I_0$, $t_{\xi_*} \cdot s_{\xi_*} \not\subseteq 1_0$. Then ξ is a fuzzy prime ideal of R . In particular if ξ is a fuzzy $\phi_{(1)}$ -prime ideal of R such that $(\xi_*)^2 \neq 0$, then ξ is a fuzzy prime ideal of R .*

Proof. In the previous theorem we put $\phi = \phi_{(1)}$. ■

COROLLARY 3.3. *Let ξ be a fuzzy ϕ -prime ideal of R such that $\phi(\xi) \subseteq t_{(\xi_*)^2} \cdot s_{\xi_*}$ for all $t, s \in I_0$. Then there exists $p, q \in I_0$ such that ξ is fuzzy $\phi_{(p,q,\omega)}$ -prime.*

Proof. In the case that ξ is a fuzzy prime ideal of R the result is clear.

Hence, we assume that ξ is not a fuzzy prime ideal. Then, by Theorem 2.1, there exists $p, q \in I_0$ such that $p_{\xi_*} \cdot q_{\xi_*} \subseteq \phi(\xi)$. Hence, by our assumption, we have

$$p_{\xi_*} \cdot q_{\xi_*} \subseteq \phi(\xi) \subseteq t_{(\xi_*)^2} \cdot s_{\xi_*} \subseteq t_{\xi_*} \cdot s_{\xi_*},$$

for all $t, s \in I_0$. This gives that

$$\phi(\xi) = p_{\xi_*} \cdot s_{\xi_*} = p_{(\xi_*)^2} \cdot q_{\xi_*}.$$

Thus $\phi(\xi) = (\bigcap_{i=1}^{\infty} p_{(\xi_*)^i} \cdot q_{\xi_*}) \cup 1_0$, and the result follows. ■

Let $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$ be a function. For each ideal \mathfrak{a} of R we put $\phi_*(\mathfrak{a}) = \phi(1_{\mathfrak{a}})_*$. Then $\phi_* : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ is a function. Note that (by our assumption on ϕ), $\phi_*(\mathfrak{a}) \subseteq \mathfrak{a}$, for each $\mathfrak{a} \in \mathcal{I}(R)$. In the next theorem we show that if for some fuzzy ideal ξ of R , $\phi(\xi)_* \subseteq \phi_*(\xi_*)$ (which is always the case for $\phi_{(0)}, \phi_{(1)}, \phi_{(t,s,\omega)}, \forall t, s \in I_0$; and for $\phi_n, n \geq 2$ if $\forall \{\xi(a) | a \in R \setminus \xi_*\} < 1$), then ξ is fuzzy ϕ -prime implies that ξ_* is ϕ_* -prime.

THEOREM 3.4. *Let ϕ be a function and ξ be a fuzzy ideal of R such that $\phi(\xi)_* \subseteq \phi_*(\xi_*)$. If ξ is fuzzy ϕ -prime, then ξ_* is a ϕ_* -prime ideal of R .*

Proof. Let $a, b \in R$ with $ab \in \xi_* \setminus \phi_*(\xi_*)$. Then $ab \in \xi_* \setminus \phi(\xi)_*$ and so $1_a \cdot 1_b \in \xi$, $1_a \cdot 1_b \notin \phi(\xi)$. Since ξ is fuzzy ϕ -prime, this gives that $1_a \in \xi$ or $1_b \in \xi$. Thus $a \in \xi_*$ or $b \in \xi_*$ and so ξ_* is a ϕ_* -prime ideal of R . ■

COROLLARY 3.5. *Let ξ be a fuzzy ϕ -prime ideal of R . Then either there exists $t \in I_0$ such that $\xi_* \subseteq \mathcal{R}(\phi(\xi))_t$ or $\mathcal{R}(\phi(\xi)) \subseteq \xi$, where $\mathcal{R}(\phi(\xi))$ is the fuzzy radical of $\phi(\xi)$ defined by $\mathcal{R}(\phi(\xi))(r) = \bigvee_{n \in \mathbf{N}} \phi(\xi)(r^n)$ for all $r \in R$.*

Proof. If ξ is not a fuzzy ϕ -prime ideal of R , then by Theorem 3.1 there exists $p, q \in I_0$ such that $p_{\xi_*} \cdot q_{\xi_*} \subseteq \phi(\xi)$. With $t = p \wedge q$, this gives that

$$t_{\xi_*} = (p \wedge q)_{\xi_*} \subseteq \mathcal{R}((p \wedge q)_{\xi_*}) = \mathcal{R}(p_{\xi_*} \cdot q_{\xi_*}) \subseteq \mathcal{R}(\phi(\xi)),$$

and hence $\xi_* \subseteq \mathcal{R}(\phi(\xi))_t$. If ξ is a fuzzy prime ideal of R , then we have $\mathcal{R}(\phi(\xi)) \subseteq \mathcal{R}(\xi) = \xi$. ■

Comparing Theorem 3.4 by the fuzzy prime case the following questions arise.

QUESTION 1. What is the image of a fuzzy ϕ -prime ideal?

QUESTION 2. Under what additional condition, ϕ_* -primness of ξ_* implies the fuzzy ϕ -primness of ξ ?

We are only able to answer these in the $\phi_{(1)}$ case as the following shows.

THEOREM 3.6 *The fuzzy ideal ξ is fuzzy $\phi_{(1)}$ -prime if and only if ξ_* is $(\phi_{(1)})_*$ -prime and $|\text{Im}(\xi)| = 2$.*

Proof. (\Rightarrow). By Theorem 3.4, ξ_* is ϕ_* -prime. We show that $|\text{Im}(\xi)| = 2$. Let $a, b \in R \setminus \xi_*$. Let $\xi(a) = s$. Then $1_a \cdot s_1 \in \xi \setminus 1_0$ and we must have $s_1 \in \xi$. Thus $\xi(1) \geq s$ and so $\xi(b) \geq s$. Similarly we can show that $\xi(b) \leq \xi(a)$ and the result follows.

(\Leftarrow) is clear. ■

4. Ring decomposition and fuzzy ϕ -primes

Let R_1 and R_2 be two commutative rings with identity. It is well know fact in elementary ring theory that any prime ideal of the ring $R = R_1 \times R_2$ is of the form $\mathfrak{p}_1 \times R_2$ or $R_1 \times \mathfrak{p}_2$ where \mathfrak{p}_1 is a prime ideal of R_1 and \mathfrak{p}_2 is a prime ideal of R_2 . This fact also holds true in the fuzzy case. In fact, let $\zeta \in FI(R_1 \times R_2)$. For each $a_1 \in R_1, a_2 \in R_2$ we put

$$\zeta_1(a_1) = \vee \{ \zeta(a_1, b_2) \mid b_2 \in R_2 \},$$

and

$$\zeta_2(a_2) = \vee \{ \zeta(b_1, a_2) \mid b_1 \in R_1 \}.$$

Then it is easy to see that $\zeta_1 \in FI(R_1)$, $\zeta_2 \in FI(R_2)$ and $\zeta = \zeta_1 \times \zeta_2$, where $\zeta_1 \times \zeta_2$ is the complete direct product of ζ_1 and ζ_2 .

LEMMA 4.1. *Let ζ be a fuzzy ideal of $R = R_1 \times R_2$. Then ζ is a fuzzy prime ideal of R if and only if $\zeta = \zeta_1 \times 1_{R_2}$ for some fuzzy prime ideal ζ_1 of R_1 or $\zeta = 1_{R_1} \times \zeta_2$ for some fuzzy prime ideal ζ_2 of R_2 .*

Proof. One direction is clear.

So let $\zeta = \zeta_1 \times \zeta_2$ be a fuzzy prime ideal of R . If $\zeta_1 \neq 1_{R_1}$ and $\zeta_2 \neq 1_{R_2}$, then there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that $\zeta_1(r_1) < 1$ and $\zeta_2(r_2) < 1$. Hence $1_{(r_1,0)} \cdot 1_{(0,r_2)} = 1_{(0,0)} \in \zeta$, gives that $1_{(r_1,0)} \in \zeta$ or $1_{(0,r_2)} \in \zeta$. So $\zeta_1(r_1) \wedge \zeta_2(0) = 1$ or $\zeta_1(0) \wedge \zeta_2(r_2) = 1$, which are not the case. Thus we get $\zeta = 1_{R_1} \times \zeta_2$ or $\zeta = \zeta_1 \times 1_{R_2}$. Suppose that $\zeta = 1_{R_1} \times \zeta_2$ and let $a_2, b_2 \in R_2$ and $t, s \in I$ such that $t_{a_2} \cdot s_{b_2} \in \zeta_2$. Then

$$t_{(1,a_2)} \cdot s_{(1,b_2)} = (t \wedge s)_{(1,a_2b_2)} \in 1_{R_1} \times \zeta_2$$

and $1_{R_1} \times \zeta_2$ is prime, give that $t_{(1,a_2)} \in 1_{R_1} \times \zeta_2$ or $s_{(1,b_2)} \in 1_{R_1} \times \zeta_2$. Hence $t_{a_2} \in \zeta_2$ or $s_{b_2} \in \zeta_2$ and ζ_2 must be a fuzzy prime ideal of R_2 . Similar argument works in the case $\zeta = \zeta_1 \times 1_{R_2}$ to show that ζ_1 is a fuzzy prime ideal of R_1 . ■

In the next theorem we show that, contrary to fuzzy prime ideals, fuzzy $\phi_{(1)}$ -primes are really only of interest in indecomposable rings. In fact, if ζ_1 is a fuzzy

$\phi_{(1)}$ -prime ideal of R_1 , then $\zeta_1 \times 1_{R_2}$ need not be a fuzzy $\phi_{(1)}$ -prime ideal of R . Indeed $\zeta_1 \times 1_{R_2}$ is fuzzy $\phi_{(1)}$ -prime if and only if it is actually fuzzy prime.

THEOREM 4.2. *Let $R = R_1 \times R_2$ and $\phi_{(1)} : FI(R) \rightarrow FI(R) \cup \{0_R\}$ be as in 2.3(2). If $\zeta = \zeta_1 \times \zeta_2$ is a fuzzy $\phi_{(1)}$ -prime of R , then either $\zeta_* = \{0\}$ or ζ is fuzzy prime.*

Proof. We may assume that $\zeta_* \neq \{0\}$. Now by Theorem 3.6, $\zeta_* = \zeta_{1*} \times \zeta_{2*}$ is a $(\phi_{(1)})_*$ -prime ideal of R . Since for each ideal \mathfrak{a} of R , $(\phi_{(1)})_*(\mathfrak{a}) = 0$, thus by [2, Theorem 7], ζ_* must be a prime ideal of R . On the other hand, since by Theorem 3.6, $|\text{Im}(\zeta)| = 2$, ζ must be a fuzzy prime ideal of R . ■

However, if ζ_1 is a fuzzy $\phi_{(1)}$ -prime ideal of R_1 , then for each function $\psi : FI(R) \rightarrow FI(R) \cup \{0_R\}$, $\zeta_1 \times 1_{R_2}$ must be a fuzzy ψ -prime ideal of $R = R_1 \times R_2$, if $1_0 \times 1_{R_2} \leq \psi(\zeta_1 \times 1_{R_2})$. To see this, let $(a_1, a_2), (b_1, b_2) \in R, t, s \in I_0$ such that $t_{(a_1, a_2)} \cdot s_{(b_1, b_2)} \in \zeta_1 \times 1_{R_2}$ and $t_{(a_1, a_2)} \cdot s_{(b_1, b_2)} \notin \psi(\zeta_1 \times 1_{R_2})$. Then $t_{(a_1, a_2)} \cdot s_{(b_1, b_2)} \in \zeta_1 \times R_2$ and $t_{(a_1, a_2)} \cdot s_{(b_1, b_2)} \notin \{1_0\} \times 1_{R_2}$. These in turn give that $t_{a_1} \cdot s_{b_1} \in \zeta_1$ and $t_{a_1} \cdot s_{b_1} \notin 1_0$. Hence ζ_1 is fuzzy $\phi_{(1)}$ -prime implies that $t_{a_1} \in \zeta_1$ or $s_{b_1} \in \zeta_1$. Therefore $t_{(a_1, a_2)} \in \zeta_1 \times 1_{R_2}$ or $s_{(b_1, b_2)} \in \zeta_1 \times 1_{R_2}$.

COROLLARY 4.3. *Let $R = R_1 \times R_2$ and let $\psi : FI(R) \rightarrow FI(R) \cup \{0_R\}$ be a function. Suppose that ζ_1 is a fuzzy $\phi_{(1)}$ -prime ideal of R_1 and that $1_0 \times 1_{R_2} \subseteq \psi(\zeta_1 \times 1_{R_2})$. Then $\zeta_1 \times 1_{R_2}$ is a fuzzy ψ -prime ideal of R .*

The following theorem is, in a sense, a more general result in this direction in the case $\phi = \psi_1 \times \psi_2$.

THEOREM 4.4. *Let $\psi_i : FI(R_i) \rightarrow FI(R_i) \cup \{0_{R_i}\}$, ($i = 1, 2$), be two functions and put $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$ with $\phi(\zeta_1 \times \zeta_2) = \psi_1(\zeta_1) \times \psi_2(\zeta_2)$ for all $\zeta_i \in FI(R_i)$. Then each of the following types is a fuzzy ϕ -prime ideal of R .*

- (1) $\zeta_1 \times \zeta_2$, where ζ_i is a non-constant ideal of R_i with $\psi_i(\zeta_i) = \zeta_i$.
- (2) $\zeta_1 \times 1_{R_2}$, where ζ_1 is a fuzzy prime ideal of R_1 .
- (3) $\zeta_1 \times 1_{R_2}$, where ζ_1 is a fuzzy ψ_1 -prime ideal of R_1 and $\psi_2(\zeta_2) = 1_{R_2}$.
- (4) $1_{R_1} \times \zeta_2$, where ζ_2 is a fuzzy prime ideal of R_2 .
- (5) $1_{R_1} \times \zeta_2$, where ζ_2 is a fuzzy ψ_2 -prime ideal of R_2 and $\psi_1(\zeta_1) = 1_{R_1}$.

Proof. First we show that each of the mentioned three type fuzzy ideals is fuzzy ϕ -prime.

(1) Since $\zeta_1 \times \zeta_2 \setminus \phi(\zeta_1 \times \zeta_2) = 0_R$, this is clear.

(2) If ζ_1 is fuzzy prime, then by Lemma 4.1, $\zeta_1 \times 1_{R_2}$ is fuzzy prime.

(3) Assume that ζ_1 is fuzzy ψ_1 -prime and $\psi_2(R_2) = 1_{R_2}$. Let $a_1, b_1 \in R_1, a_2, b_2 \in R_2, t, s \in I$ such that

$$t_{(a_1, a_2)} \cdot s_{(b_1, b_2)} \in \zeta_1 \times 1_{R_2} \setminus \psi_1(\zeta_1) \times \psi_2(R_2) = \zeta_1 \times 1_{R_2} \setminus \psi_1(\zeta_1) \times 1_{R_2}.$$

This gives that $t_{a_1} \cdot s_{b_1} \in \zeta_1 \setminus \psi_1(\zeta_1)$ and so $t_{a_1} \in \zeta_1$ or $s_{b_1} \in \zeta_1$. Thus $t_{(a_1, a_2)} \in \zeta_1 \times 1_{R_2}$ or $s_{(b_1, b_2)} \in \zeta_1 \times 1_{R_2}$. Hence $\zeta_1 \times 1_{R_2}$ is a fuzzy ϕ -prime ideal of R .

The proof of the cases (4) and (5) are similar to that of (2) and (3) respectively. ■

5. Connection with fuzzy localization and fuzzy quotient

Assume that S is a multiplicatively closed subset of R . It is a known fact in commutative algebra that there is a one-to-one correspondence between the set of all prime ideals \mathfrak{p} of R with $\mathfrak{p} \cap S = \emptyset$ and the set of all prime ideals of $S^{-1}R$, given by $\mathfrak{p} \rightarrow S^{-1}\mathfrak{p}$.

Let $f : R \rightarrow S^{-1}R$ be the natural map $r \rightarrow r/1$ and let $\mathfrak{o}(S) = \{r \in R | \exists s \in S, sr = 0\}$ be the kernel of f . Set $\pi : R \rightarrow R/\mathfrak{o}(S)$ be the natural ring epimorphism $r \rightarrow r + \mathfrak{o}(S) = \bar{r}$. Then it is well known that for $\zeta \in FI(R)$, $S^{-1}\zeta : S^{-1}R \rightarrow I$ given by

$$(S^{-1}\zeta)(a/s) = \vee \{t \in I | \bar{a}/s \in S^{-1}(\pi(\zeta)_t)\}, \quad a \in R, s \in S$$

is a fuzzy ideal of $S^{-1}R$ such that for each $a \in R, s, u \in S$, $(S^{-1}\zeta)(a/1) = (S^{-1}\zeta)(a/s) = (S^{-1}\zeta)(ua/1)$. It is easy to see that each fuzzy ideal of $S^{-1}R$ is extended, i.e., for each fuzzy ideal ζ' of $S^{-1}R$, $\zeta' = S^{-1}\zeta$, where $\zeta = f^{-1}(\zeta')$. Furthermore there is a one to-one correspondence between the set of all fuzzy prime ideals ξ of R with $\xi_* \cap S = \emptyset$ and the set of all fuzzy primes of $S^{-1}R$. In the next theorem we want to examine the behavior of fuzzy ϕ -prime property with localization. We say that the fuzzy ideal ξ of R has sup property if for any subset T of R there exists $t_0 \in T$ such that $\xi(t_0) = \vee_{t \in T} \xi(t)$.

For $\zeta \in FI(R)$ and $r \in R$, set $\zeta[S](r) = \vee_{s \in S} \zeta(sr)$. Then it is easy to see $\zeta[S]$ is a fuzzy ideal of R with $\zeta \subseteq \zeta[S]$.

LEMMA 5.1. *Let the notation be as in above. Assume that $\xi \in FI(R)$ has the sup property (e.g, ξ is fuzzy prime or fuzzy primary). Then*

- (1) *If $\zeta \in FI(R)$ such that $\zeta_* \cap S \neq \emptyset$, then $(\zeta \cdot \xi)[S] = \xi[S]$.*
- (2) *$S^{-1}(\xi[S]) = S^{-1}\xi$.*
- (3) *$(S^{-1}\xi)(a/1) \geq t \Leftrightarrow \exists s \in S$ such that $\xi(sa) \geq t$.*

Proof. (1) Clearly $(\zeta \cdot \xi)[S] \subseteq \xi[S]$. Let $r \in R, t \in I$ such $t_r \in \xi[S]$. Then $\vee_{s \in S} \xi(sr) \geq t$. Since ξ has the sup property, $\exists s \in S$ such that $\xi(sr) \geq t$. Choose $u \in \zeta_* \cap S$. Then

$$(\zeta \cdot \xi)[S](r) = \vee_{s' \in S} (\zeta \cdot \xi)(s'r) \geq (\zeta \cdot \xi)(usr) \geq \zeta(u) \wedge \xi(sr) \geq t,$$

and hence $t_r \in (\zeta \cdot \xi)[S]$ as required.

(2) It is clear that $S^{-1}(\xi) \subseteq S^{-1}(\xi[S])$. To prove the other inclusion, let $a \in R, t \in I$. Then

$$\begin{aligned} \bar{a} \in \pi(\xi[S])_t &\iff \pi(\xi[S])(\bar{a}) \geq t \\ &\iff \vee_{e \in \mathfrak{o}(S)} \xi[S](a + e) \geq t \\ &\iff \vee_{e \in \mathfrak{o}(S)} \vee_{s \in S} \xi(sa + se) \geq t \\ &\implies \vee_{e \in \mathfrak{o}(S)} \vee_{s \in S} \vee_{s_e \in S} \xi(s_e sa + s_e se) \geq t, \end{aligned}$$

where $s_e \in S$ is such that $s_e e = 0$.

$$\implies \bigvee_{e \in \mathfrak{o}(S)} \bigvee_{s \in S} \bigvee_{s_e \in S} \xi(s_e s a) \geq t.$$

Since ξ has the sup property, this last in turn gives that $\exists u \in S$ such that $\xi(ua) \geq t$. Thus

$$\pi(\xi)(\bar{u}a) = \bigvee_{e \in \mathfrak{o}(S)} \xi(ua + e) \geq t,$$

and so $\bar{u}a \in \pi(\xi)_t$. Therefore for $r \in R, t \in I$, if

$$\bar{r}/1 \in S^{-1}((\pi(\xi[S]))_t),$$

then $\exists s \in S, a \in R$, such that $\bar{r}/1 = \bar{a}/s$ and $\bar{a} \in (\pi(\xi[S]))_t$. So by the above observation, $\exists u \in S$, such that $\bar{r}/1 = \bar{u}a/us$ and $\bar{u}a \in \pi(\xi)_t$. Thus $\bar{r}/1 \in S^{-1}(\pi(\xi)_t)$.

Now, it is easy to see from the definition that $S^{-1}(\xi[S]) \subseteq S^{-1}\xi$.

(3) See [6, Lemma 3.11]. ■

Given a function $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$, we define the function $(S^{-1}\phi) : FI(S^{-1}R) \rightarrow FI(S^{-1}R) \cup \{0_{S^{-1}R}\}$ by $(S^{-1}\phi)(S^{-1}\zeta) = S^{-1}(\phi(\zeta[S]))$ if $\phi(\zeta[S]) \neq 0_R$ and $0_{S^{-1}R}$ if $\phi(\zeta[S]) = 0_R$. Then by Lemma 5.1 and noting that $\phi(\zeta) \subseteq \zeta$, we have $(S^{-1}\phi)(S^{-1}\zeta) \subseteq S^{-1}\zeta$.

THEOREM 5.2. *Let S be a multiplicatively closed subset of R and let $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$ be a function. Assume that ξ be a fuzzy ϕ -prime ideal of R such that $\xi_* \cap S = \emptyset$ and that $S^{-1}(\phi(\xi)) \subseteq (S^{-1}\phi)(S^{-1}\xi)$. Then $S^{-1}\xi$ is a fuzzy $(S^{-1}\phi)$ -prime ideal of $S^{-1}R$.*

Proof. Let $a, b \in R, t, s \in I$ with $t_{a/1} \cdot s_{b/1} \in S^{-1}\xi$ and $t_{a/1} \cdot s_{b/1} \notin (S^{-1}\phi)(S^{-1}\xi)$. Then by our assumption, $(t \wedge s)_{ab/1} \in S^{-1}\xi$ and $(t \wedge s)_{ab/1} \notin S^{-1}(\phi(\xi))$. Therefore, by Lemma 5.1(3), $\exists w \in S$ such that $(t \wedge s)_{wab} \in \xi$ and $(t \wedge s)_{wab} \notin \phi(\xi)$. Since $\xi_* \cap S = \emptyset$, it follows that $t_a \wedge s_b \in \xi$ and $t_a \wedge s_b \notin \phi(\xi)$. Then, ξ is fuzzy ϕ -prime, implies that $t_a \in \xi$ or $s_b \in \xi$. Thus using once again Lemma 5.1(3), gives that $t_{a/1} \in S^{-1}\xi$ or $s_{b/1} \in S^{-1}\xi$ and the proof of this part is complete. ■

Next we examine the behavior of ϕ primes in the quotient rings.

For $\zeta \in FI(R)$, the set $R/\zeta = \{1_r + \zeta | r \in R\}$ is a commutative ring with nonzero identity $1_1 + \zeta$ for which the map $g : R \rightarrow R/\zeta$ is a ring epimorphism with $Kerg = \zeta_*$. Furthermore, there is a one-to-one correspondence between the set of all fuzzy ideals of R which are constant on ζ_* and the set of all fuzzy ideals of R/ζ , given by $\nu \rightarrow g(\nu)$, where for each $r \in R, g(\nu)(1_r + \zeta) = \bigvee \{\nu(a) | a \in R, \zeta(a-r) = 1\}$. Since ν is constant on ζ_* , it is fuzzy prime of R if and only if $g(\nu)$ is fuzzy prime of R/ζ .

Now, given $\zeta \in FI(R)$ and $\phi : FI(R) \rightarrow FI(R) \cup \{0_R\}$, we define $\phi^\zeta : FI(R/\zeta) \rightarrow FI(R/\zeta) \cup \{0_{R/\zeta}\}$ by $\phi^\zeta(g(\nu)) = g(\phi(\nu))$ if $\phi(\nu) \neq 0_R$ and $0_{R/\zeta}$ if $\phi(\nu) = 0_R$. We note that for each $r \in R$,

$$\begin{aligned} \phi^\zeta(g(\nu))(1_r + \zeta) &= g(\phi(\nu))(g(r)) \\ &\leq g(\nu)(g(r)) = g(\nu)(1_r + \zeta), \end{aligned}$$

and so $\phi^\zeta(g(\nu)) \subseteq g(\nu)$ for each $\nu \in FI(R)$ which is constant on ζ_* .

PROPOSITION 5.3. *Let the notation be as above and let ν be a fuzzy ideal of R which is constant on ζ_* . If ν is a fuzzy ϕ -prime ideal of R , then $g(\nu)$ is a fuzzy ϕ^ζ -prime ideal of R/ζ . Furthermore if $\phi(\nu)$ is constant on ζ_* , the converse holds.*

Proof. (\Rightarrow) Let $a, b \in R, t, s \in I$ such that $t_{g(a)} \cdot s_{g(b)} \in g(\nu) \setminus \phi^\zeta(g(\nu)) = g(\nu) \setminus g(\phi(\nu))$. This gives that $\nu(ab) = g(\nu)(g(ab)) \geq t \wedge s$ and that

$$\phi(\nu)(ab) \leq \vee \{ \phi(\nu)(r) \mid r \in R, g(r) = g(ab) \} = g(\phi(\nu))(g(ab)) < t \wedge s.$$

That is $t_a \cdot s_b \in \nu \setminus \phi(\nu)$ and ν is fuzzy ϕ -prime gives that $t_a \in \nu$ or $s_b \in \nu$. Therefore $t_{g(a)} \in g(\nu)$ or $s_{g(b)} \in g(\nu)$.

(\Leftarrow) Let $a, b \in R, t, s \in I$ such that $t_a \cdot s_b \in \nu \setminus \phi(\nu)$. This, by our assumption on $\phi(\nu)$, gives that $t_{g(a)} \cdot s_{g(b)} \in g(\nu) \setminus \phi^\zeta(\nu)$. Thus $t_{g(a)} \in f(\nu)$ or $s_{g(b)} \in g(\nu)$ by fuzzy ϕ^ζ -primeness of $g(\nu)$. Hence $t_a \in \nu$ or $s_b \in \nu$ and the result follows. ■

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