

## A LIOUVILLE TYPE THEOREM FOR $p$ -HARMONIC FUNCTIONS ON MINIMAL SUBMANIFOLDS IN $R^{n+m}$

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**Abstract.** In this note, we prove that if an  $n$ -dimensional complete noncompact minimal submanifold  $M$  in  $R^{n+m}$  has sufficiently small total scalar curvature, and  $u$  is a  $p$ -harmonic function on  $M$  with  $|du|^{2p-2} \in L^1(M)$ , then  $u$  is constant.

### 1. Introduction

In [2], Cao, Shen and Zhu showed that a complete connected stable minimal hypersurface in Euclidean space must have exactly one end. Its strategy was to utilize a result of Schoen-Yau asserting that a complete stable minimal hypersurface in Euclidean space can not admit a non-constant harmonic function with finite integral [8]. Later Ni [7] proved that if  $n$ -dimensional complete minimal submanifold  $M$  in Euclidean space has sufficient small total scalar curvature (i.e.  $\int_M |A|^n < C_1 < \infty$ ) then  $M$  has only one end. In [9], Seo improved the upper bound  $C_1$ . Due to this connection with harmonic functions, this allows one to estimate the number of ends of the above submanifold by estimating the dimension of the space of bounded harmonic function with finite Dirichlet integral [3].

A  $C^1$ -function  $u : (M, \langle \cdot \rangle_M) \rightarrow R$  is said to be  $p$ -harmonic function, if it satisfies the non-linear system  $\operatorname{div}(|du|^{p-2} du) = -\delta(|du|^{p-2} du) = 0$  in [1]. In case  $p = 2$ , the non-linear factor  $|du|^{p-2}$  disappears and the 2-harmonic function is simply called harmonic function. So the  $p$ -harmonic function is the generality of the harmonic function.

In this paper, following the work due to Ni and Seo, we study the  $p$ -harmonic function on complete noncompact minimal submanifold in  $R^{n+m}$ . We obtain a Kato type inequality for  $p$ -harmonic functions and the following Liouville type result:

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**THEOREM 1.1.** *Let  $M^n$  be an  $n$ -dimensional complete minimal submanifold in  $R^{n+m}$ ,  $n \geq 3$ , and  $u : M^n \rightarrow R$  be a  $p$ -harmonic function ( $p \geq 2$ ). If*

$$\left( \int_M |A|^n dv \right)^{\frac{1}{n}} < \sqrt{\left( \frac{n}{n-1} - \frac{(p-2)^2}{(p-1)^2} \right) \frac{n}{n-1} C_s^{-1}}$$

and  $|du|^{2p-2} \in L^1(M)$ , then  $u$  is constant. (Here  $C_s$  is a Sobolev constant in [3].)

**2. Proof of the theorem**

We begin with the following useful facts.

**LEMMA 2.1** (Bochner-Weitzenböck formula) [6] *Let  $u : M^n \rightarrow R$  be any smooth function on Riemannian manifold  $M^n$ . Then*

$$\begin{aligned} \frac{1}{2} \Delta |du|^{2(p-1)} &= -\langle \Delta_H (|du|^{p-2} du), |du|^{p-2} du \rangle \\ &\quad + |D(|du|^{p-2} du)|^2 + |du|^{2(p-2)} Ric^M(\nabla u, \nabla u). \end{aligned}$$

where  $D$  and  $\Delta_H = d\bar{\delta} + \delta d$  are respectively the covariant derivative and the Hodge Laplacian of the vector bundle  $T^*M \otimes u^{-1}TR$ .

**LEMMA 2.2.** [3] *Let  $M^n$  be an  $n$ -dimensional complete immersed minimal submanifold in  $R^{n+m}$ ,  $n \geq 3$ . Then for any  $\phi \in W_0^{1,2}(M)$  we have*

$$\left( \int_M |\phi|^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq C_s \int_M |\nabla \phi|^2 dv, \tag{1}$$

where  $C_s$  depends only on  $n$ .

**LEMMA 2.3.** [3] *Let  $M$  be an  $n$ -dimensional complete immersed minimal submanifold in space form  $F^{n+m}(c)$ . Then the Ricci curvature of  $M$  satisfies*

$$Ric^M \geq (n-1)c - \frac{n-1}{n} |A|^2. \tag{2}$$

We give a Kato type inequality for  $p$ -harmonic function.

**LEMMA 2.4.** *Let  $u : M^n \rightarrow R$  be a  $p$ -harmonic function on Riemannian manifold  $M$ , then we have*

$$|D(|du|^{p-2} du)|^2 \geq \frac{n}{n-1} |\nabla |du|^{p-1}|^2. \tag{3}$$

where  $p \geq 2$ .

*Proof.* When  $p = 2$ ,  $u$  is a 2-harmonic function i.e. harmonic function, (3) is true. So we only need to prove the case for  $p \geq 3$ . Since  $u$  is  $p$ -harmonic function, we have

$$\delta(|du|^{p-2} du) = 0, \tag{4}$$

We choose a local frame field  $\{e_1, \dots, e_n\}$  of  $M^n$  near a fixed point  $q \in M$  such that  $\nabla_{e_i} e_j(q) = 0$ ,  $u_1(p) = du(e_1)(q) = |\nabla u|(q) = |du|(q) \neq 0$  and  $du(e_i)(q) = 0$  for  $i \geq 2$ . Then we have

$$\nabla_{e_j} |\nabla u| = \nabla_j |\nabla u| = \nabla_j \left( \sqrt{\sum u_i^2} \right) = \frac{\sum u_i u_{ij}}{|\nabla u|} = u_{1j}.$$

From (4), we have

$$\begin{aligned} 0 &= \delta(|du|^{p-2} du)(q) = - \sum_{i=1}^n D_{e_i}(|du|^{p-2} du)(e_i)(q) \\ &= -|du|^{p-2} \sum_{i=1}^n [(p-2)\nabla_i(\ln(|\nabla u|))u_i + u_{ii}](q), \end{aligned}$$

so we have

$$\sum_{i=1}^n [(p-2)\nabla_i(\ln(|\nabla u|))u_i + u_{ii}] = 0. \tag{5}$$

Now we compute,

$$\begin{aligned} |D(|du|^{p-2} du)|^2 - |\nabla |du|^{p-1}|^2 &= \sum_{ij} |du|^{2(p-2)} [(p-2)\nabla_i(\ln |\nabla u|)u_j + u_{ij}]^2 \\ &\quad - \sum_i |du|^{2(p-2)} [(p-2)\nabla_i(\ln |\nabla u|)|\nabla u| + \nabla_i |\nabla u|]^2 \\ &= \sum_{ij} |du|^{2(p-2)} [(p-2)\nabla_i(\ln |\nabla u|)u_j + u_{ij}]^2 \\ &\quad - \sum_i |du|^{2(p-2)} [(p-2)\nabla_i(\ln |\nabla u|)u_1 + u_{i1}]^2 \\ &\geq \sum_{i \neq 1} |du|^{2(p-2)} [(p-2)\nabla_i(\ln |\nabla u|)u_1 + u_{i1}]^2 \\ &\quad + \sum_{i \neq 1} |du|^{2(p-2)} [(p-2)\nabla_i(\ln |\nabla u|)u_i + u_{ii}]^2 \\ &\geq \sum_{i \neq 1} |du|^{2(p-2)} [(p-2)\nabla_i(\ln |\nabla u|)u_1 + u_{i1}]^2 \\ &\quad + |du|^{2(p-2)} \frac{1}{n-1} \left[ \sum_{i \neq 1} [(p-2)\nabla_i(\ln |\nabla u|)u_i + u_{ii}] \right]^2 \\ &\geq \frac{1}{n-1} \sum_i |du|^{2(p-2)} [(p-2)\nabla_i(\ln |\nabla u|)u_1 + u_{i1}]^2 \\ &= \frac{1}{n-1} |\nabla |du|^{p-1}|^2. \end{aligned} \tag{6}$$

where we use the equation (5) in the last inequality. From (6), we have

$$|D(|du|^{p-2} du)|^2 \geq \frac{n}{n-1} |\nabla |du|^{p-1}|^2. \quad \blacksquare$$

PROOF OF THEOREM 1.1. Since  $u$  is a  $p$ -harmonic function and (1), we have

$$\begin{aligned} |du|^{p-1} \Delta |du|^{p-1} &= -\langle \delta d(|du|^{p-2} du), |du|^{p-2} du \rangle \\ &\quad + |D(|du|^{p-2} du)|^2 - |\nabla |du|^{p-1}|^2 + |du|^{2(p-2)} Ric^M(\nabla u, \nabla u) \end{aligned}$$

From (2) and (3), we have

$$\begin{aligned} |du|^{p-1} \Delta |du|^{p-1} + \frac{n-1}{n} |A|^2 |du|^{2(p-1)} \\ \geq -\langle \delta d(|du|^{p-2} du), |du|^{p-2} du \rangle + \frac{1}{n-1} |\nabla |du|^{p-1}|^2 \end{aligned}$$

Let  $f = |du|^{p-1}$ . Then we have

$$f \Delta f + \frac{n-1}{n} |A|^2 f^2 \geq -\langle \delta d(|du|^{p-2} du), |du|^{p-2} du \rangle + \frac{1}{n-1} |\nabla f|^2 \quad (7)$$

Fix a point  $x_0 \in M$  and take a cut-off function  $\eta$  satisfying  $0 \leq \eta \leq 1$ ,  $\eta = 1$  on  $B_{x_0}(r)$ ,  $\eta = 0$  on  $M - B_{x_0}(2r)$  and  $|d\eta| \leq \frac{1}{r}$ . Multiplying (7) by  $\eta^2$  and integrating over  $M$ , we get

$$\begin{aligned} \int_M \eta^2 f \Delta f + \int_M \eta^2 \frac{n-1}{n} |A|^2 f^2 \\ \geq - \int_M \eta^2 \langle \delta d(|du|^{p-2} du), |du|^{p-2} du \rangle + \frac{1}{n-1} \int_M \eta^2 |\nabla f|^2 \quad (8) \end{aligned}$$

Since for a function  $f$  on  $M$ ,  $|d(f d\phi)| \leq |df| |d\phi|$ . Hence we have

$$\begin{aligned} \left| \int_M \langle \eta^2 |du|^{p-2} du, \delta d(|du|^{p-2} du) \rangle \right| \\ = \left| \int_M \langle d(\eta^2 |du|^{p-2} du), d(|du|^{p-2} du) \rangle \right| \\ \leq \left| \int_M |d(\eta^2 |du|^{p-2})| |du| |d|du|^{p-2}| |du| \right| \\ \leq 2 \int_M \eta |d\eta| |du|^{p-2} |d|du|^{p-2}| |du|^2 + \int_M \eta^2 |d|du|^{p-2}|^2 |du|^2 \\ = 2 \frac{p-2}{p-1} \int_M \eta |d\eta| |du|^{p-1} |d|du|^{p-1}| + \left(\frac{p-2}{p-1}\right)^2 \int_M \eta^2 |d|du|^{p-1}|^2 \\ = 2 \frac{p-2}{p-1} \int_M \eta |d\eta| |f| |df| + \left(\frac{p-2}{p-1}\right)^2 \int_M \eta^2 |df|^2. \quad (9) \end{aligned}$$

Using integration by parts for (8) and using (9), we have

$$\begin{aligned} - \int_M \eta^2 |df|^2 - 2 \int_M \eta f \langle df, d\eta \rangle + \int_M \eta^2 \frac{n-1}{n} |A|^2 f^2 \\ \geq -2 \frac{p-2}{p-1} \int_M \eta |d\eta| |f| |df| - \left(\frac{p-2}{p-1}\right)^2 \int_M \eta^2 |df|^2 + \frac{1}{n-1} \int_M \eta^2 |df|^2 \quad (10) \end{aligned}$$

On the other hand, it follows from (2) and Hölder inequality that

$$\frac{n-1}{n} \int_M \eta^2 |A|^2 f^2 \leq \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} \left( \int_M (\eta f)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}}$$

$$\begin{aligned}
&\leq \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s \int_M |d(\eta f)|^2 \\
&\leq \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s \int_M (f^2 |d\eta|^2 + \eta^2 |df|^2) \\
&\quad + \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s \int_M 2\eta f \langle d\eta, df \rangle. \quad (11)
\end{aligned}$$

From (10), (11) and Schwarz inequality, we have

$$\begin{aligned}
&\left[ \frac{n}{n-1} - \left( \frac{p-2}{p-1} \right)^2 - \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s \right. \\
&\quad \left. - \varepsilon \left( \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s + \frac{p-2}{p-1} + 1 \right) \right] \int_M \eta^2 |df|^2 \\
&\leq \left[ \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s + \frac{1}{\varepsilon} \left( \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s + \frac{p-2}{p-1} + 1 \right) \right] \int_M f^2 |d\eta|^2 \\
&\leq \left[ \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s + \frac{1}{\varepsilon} \left( \frac{n-1}{n} \left( \int_M |A|^n \right)^{\frac{2}{n}} C_s + \frac{p-2}{p-1} + 1 \right) \right] \frac{1}{r^2} \int_M f^2
\end{aligned}$$

Since  $(\int_M |A|^n dv)^{\frac{1}{n}} < \sqrt{\left(\frac{n}{n-1} - \frac{(p-2)^2}{(p-1)^2}\right) \frac{n}{n-1} C_s^{-1}}$  and  $|du|^{2p-2} \in L^1(M)$ , i.e.  $|f|^2 \in L^1(M)$ , choosing  $\varepsilon > 0$  small and letting  $r \rightarrow \infty$ , we have  $|df| = 0$  on  $M$ , i.e.  $f$  is constant. Since  $|f|^2 \in L^1(M)$  and the volume of  $M$  is infinite, we have  $f = 0$ , then  $u$  is constant. This completes the proof of Theorem 1.1. ■

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