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RESULTS CONCERNING SYMMETRIC GENERALIZED BIDERIVATIONS OF PRIME AND SEMIPRIME RINGS

Asma Ali, V. De Filippis and Faiza Shujat

Abstract. The purpose of this paper is to prove some results concerning symmetric biderivations and symmetric generalized biderivations on prime and semiprime rings which partially extend some results contained in Vukman, J., Symmetric biderivations on prime and semiprime rings, Aequationes Math. 38 (1989), 245–254 and Vukman, J., Two results concerning symmetric biderivations on prime rings, Aequationes Math. 40 (1990), 181–189.

1. Introduction

Throughout this paper all rings will be associative. We shall denote by $Z(R)$ the center of a ring R and by C the extended centroid of R, which is the center of the two-sided Martindale quotients ring Q (we refer the reader to [2] for more details). Recall that a ring R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and it is semiprime if $aRa = (0)$ implies $a = 0$. We shall write $[x, y]$ for $xy - yx$.

An additive mapping $d\colon R \to R$ is said to be a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such that $d_a(x) = [a, x]$, for all $x \in R$. A mapping $D\colon R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$, for all $x, y \in R$. A mapping $f\colon R \to R$ defined by $f(x) = D(x, x)$, where $D\colon R \times R \to R$ is a symmetric mapping, is called the trace of D. It is obvious that in the case $D\colon R \times R \to R$ is a symmetric mapping which is also biadditive (i.e., additive in both arguments), the trace f of D satisfies the relation $f(x + y) = f(x) + f(y) + 2D(x, y)$, for all $x, y \in R$. A biadditive mapping $D\colon R \times R \to R$ is called a biderivation if $D(xy, z) = D(x, z)y + xD(y, z)$ for all $x, y, z \in R$. Obviously, in this case the relation $D(x, yz) = D(x, y)z + yD(x, z)$ is also satisfied for all $x, y, z \in R$.

Typical examples are mappings of the form $(x, y) \mapsto \lambda[x, y]$ where $\lambda \in C$. We shall call such maps inner biderivations. It was shown in [4] that every biderivation D of a noncommutative prime ring R is of the form $D(x, y) = \lambda[x, y]$ for some $\lambda \in C$. Moreover, in [3], Brešar extended this result to semiprime rings.

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Maksa introduced in [6] the concept of a symmetric biderivation (see also [7], where an example can be found). It was shown in [6] that symmetric biderivations are related to general solution of some functional equations. Some results on symmetric biderivations in prime and semiprime rings can be found in [8, 9, 10]. The notion of additive commuting mappings is closely connected with the notion of biderivations. Every commuting additive mapping $f\colon R \to R$ gives rise to a biderivation on R. Namely linearizing $[x, f(x)] = 0$, for all $x, y \in R$, $(x, y) \mapsto [f(x), y]$ is a biderivation (moreover, it is an inner derivation in each argument).

The notion of generalized biderivation was introduced by Argaç in [1].

Let $D\colon R \times R \to R$ be a biderivation. A biadditive mapping $\Delta\colon R \times R \to R$ is said to be a generalized biderivation if for every $x \in R$, the map $y \mapsto \Delta(x, y)$ is a generalized derivation of R associated with D as well as for every $y \in R$, the map $x \mapsto \Delta(x, y)$ is a generalized derivation of R associated with D, i.e., $\Delta(x, yz) = \Delta(x, y)z + yD(x, z)$ and $\Delta(xy, z) = \Delta(x, z)y + xD(y, z)$ for all $x, y, z \in R$.

Example 1.1. Let R be a ring. If D is any biderivation of R and $\alpha\colon R \times R \to R$ is a biadditive map such that $\alpha(x, yz) = \alpha(x, y)z$ and $\alpha(xy, z) = \alpha(x, z)y$ for all $x, y, z \in R$, then $D + \alpha$ is a generalized D-biderivation of R.

In this paper, we prove some theorems on symmetric generalized biderivations of a ring which extend Vukman's results contained in [9, Theorem 1] and [8, Theorem 4].

2. Preliminaries

Throughout the paper, we make extensive use of the basic commutator identities (i) $[x, yz] = [x, y]z + y[x, z]$ and (ii) $[xy, z] = [x, z]y + x[y, z]$. Here we state the following results which will be useful in the sequel.

Lemma 2.1. [5, page 6, Corollary 2] *If R is a semiprime ring and I is an ideal of R, then $I \cap Ann(I) = (0)$.*

Theorem 2.2. [9, Theorem 1] *Let R be a prime ring of characteristic not two and three. If D_1, D_2 are the symmetric biderivations of R with trace f_1, f_2, respectively, such that $f_1(x)f_2(x) = 0$ for all $x \in R$, then either $D_1 = 0$ or $D_2 = 0$.*

Theorem 2.3. [8, Theorem 4] *Let R be a 2-torsion free semiprime ring. Suppose that there exists a symmetric biderivation $D\colon R \times R \to R$ such that $D(f(x), x) = 0$ for all $x \in R$, where f denotes the trace of D. In this case we have $D = 0$.*

Lemma 2.4. *Let R be a prime ring of characteristic different from two and I be a nonzero ideal of R. If D is a symmetric biderivation such that $D(x, x) = 0$ for all $x \in I$, then either $D = 0$ or R is commutative.*

Proof. Let $D(x, x) = 0$ for all $x \in I$. Linearization yields that $2D(x, y) = 0$ for all $x, y \in I$. Since characteristic of R is different from two, we have $D(x, y) = 0$

for all $x, y \in I$. Replacing y by ry, we get $D(x,r)y = 0$ for all $x, y \in I$ and $r \in R$. Substitute sx for x, we obtain $D(s,r)xy = 0$ for all $x, y \in I$ and $r, s \in R$. This implies that $D(s,r)R[x,y] = 0$ for all $x, y \in I$ and $r, s \in R$. Primeness of R yields that either $[x,y] = 0$ or $D(r,s) = 0$ for all $x, y \in I$ and $r, s \in R$. Hence one has that either $[I,I] = (0)$ and R is commutative, or $D = 0$. ■

3. Some generalizations to symmetric generalized biderivations

In this section we partially generalize Theorem 2.3. More precisely, we consider the case when the ring R is prime and replace symmetric biderivations with symmetric generalized biderivations. In this sense we obtain the following:

THEOREM 3.1. *Let R be a prime ring of characteristic not two and I be a nonzero ideal of R. If Δ is a symmetric generalized biderivation with associated biderivation D of R with trace f such that $\Delta(f(x), x) = 0$ for all $x \in I$, then one of the following holds:*

(i) $\Delta = 0$,

(ii) *R is commutative,*

(iii) *f is commuting on I.*

Proof. Suppose that

$$\Delta(f(x), x) = 0 \text{ for all } x \in I. \tag{3.1}$$

Linearizing (3.1) and using (3.1), we get

$$\Delta(f(x), y) + \Delta(f(y), x) + 2\Delta(D(x,y), x) + 2\Delta(D(x,y), y) = 0 \text{ for all } x, y \in I. \tag{3.2}$$

Replacing y by $-y$ in (3.2), we get

$$-\Delta(f(x), y) + \Delta(f(y), x) - 2\Delta(D(x,y), x) + 2\Delta(D(x,y), y) = 0 \text{ for all } x, y \in I. \tag{3.3}$$

Adding up (3.2) and (3.3) and using the fact that characteristic of R is not two, we find

$$\Delta(f(y), x) + 2\Delta(D(x,y), y) = 0 \text{ for all } x, y \in I. \tag{3.4}$$

Substitute xz for x in (3.4) to get

$$\Delta(f(y), x)z + xD(f(y), z) + 2\Delta(xD(z,y), y) + 2\Delta(D(x,y)z, y) = 0 \text{ for all } x, y, z \in I.$$

By computation, we get

$$\begin{aligned} &\Delta(f(y), x)z + xD(f(y), z) + 2\Delta(x,y)D(z,y) + 2xD(D(z,y), y) \\ &\quad + 2\Delta(D(x,y), y)z + 2D(x,y)D(z,y) = 0 \text{ for all } x, y, z \in I. \end{aligned} \tag{3.5}$$

In view of (3.4), (3.5) yields that

$$\begin{aligned} &xD(f(y), z) + 2\Delta(x,y)D(z,y) + 2xD(D(z,y), y) \\ &\quad + 2D(x,y)D(z,y) = 0 \text{ for all } x, y, z \in I. \end{aligned} \tag{3.6}$$

Replacing x by ux in (3.6), we obtain

$$\begin{aligned} &uxD(f(y), z) + 2\Delta(u, y)xD(z, y) + 2uD(x, y)D(z, y) + 2uxD(D(z, y), y) \\ &\quad + 2D(u, y)xD(z, y) + 2uD(x, y)D(z, y) = 0 \text{ for all } x, y, z, u \in I. \end{aligned} \tag{3.7}$$

Comparing (3.6) with (3.7), we get

$$\begin{aligned} &2\Delta(u, y)xD(z, y) + 2uD(x, y)D(z, y) + 2D(u, y)xD(z, y) \\ &\quad - 2u\Delta(x, y)D(z, y) = 0 \text{ for all } x, y, z, u \in I. \end{aligned}$$

Since R is of characteristic not two and replacing u by x, we have

$$\begin{aligned} &\Delta(x, y)xD(z, y) + xD(x, y)D(z, y) + D(x, y)xD(z, y) \\ &\quad - x\Delta(x, y)D(z, y) = 0 \text{ for all } x, y, z \in I. \end{aligned}$$

This implies that

$$\{[\Delta(x, y), x] + (xD(x, y) + D(x, y)x)\}D(z, y) = 0 \text{ for all } x, y, z \in I.$$

Therefore we have

$$\{[\Delta(x, y), x] + D(x^2, y)\}D(z, y) = 0, \quad \forall x, y, z \in I.$$

Replacing z by zu, we obtain $\{[\Delta(x, y), x]+D(x^2, y)\}zD(u, y) = 0$ for all $x, y, z, u \in I$. Since R is prime, we get either $[\Delta(x, y), x] + D(x^2, y) = 0$ or $D(u, y) = 0$. In this last case and by Lemma 2.4, we have that either R is commutative or $D = 0$. Moreover, if $D = 0$, then by (3.1) we get $\Delta = 0$.

On the other hand, if $[\Delta(x, y), x] + D(x^2, y) = 0$ for all $x, y \in I$, then replacing y by yz we find $\Delta(x, y)[z, x]+[\Delta(x, y), x]z+y[D(x, z), x]+[y, x]D(x, z)+yD(x^2, z)+D(x^2, y)z = 0$ for all $x, y, z \in I$. This implies that $\Delta(x, y)[z, x] + y[D(x, z), x] + [y, x]D(x, z) + yD(x^2, z) = 0$ for all $x, y, z \in I$. In particular, if we take $x = z$, then we have $y[f(x), x] + [y, x]f(x) + yD(x^2, x) = 0$ for all $x, y \in I$. Again replace y by ry and use the last relation to get

$$[r, x]yf(x) = 0 \text{ for all } x, y \in I,\ r \in R. \tag{3.8}$$

Replacing y by yx, we have

$$[r, x]yxf(x) = 0 \text{ for all } x, y \in I,\ r \in R. \tag{3.9}$$

Right multiplying by x in (3.8), we get

$$[r, x]yf(x)x = 0 \text{ for all } x, y \in I,\ r \in R. \tag{3.10}$$

Subtracting (3.9) and (3.10), we find

$$[r, x]y[f(x), x] = 0 \text{ for all } x, y \in I,\ r \in R. \tag{3.11}$$

Substituting $f(x)$ for r in (3.11), we get $[f(x), x]y[f(x), x] = 0$ for all $x, y \in I$. This implies that $y[f(x), x]Ry[f(x), x] = 0$ for all $x, y \in I$. Since R is prime, we have $y[f(x),x]=0$ for all $x,y\in I$, i.e., $[f(x),x]\in Ann(I)$. This implies that $[f(x),x]\in Ann(I)\cap I=(0)$, for all $x\in I$ by Lemma Lemma 2.1. Hence f is commuting on I. ■

Now we extend the result for the noncommutative case.

THEOREM 3.2. *Let R be a noncommutative prime ring of characteristic not two and I be an ideal of R. If Δ is a symmetric generalized biderivation with associated biderivation D of R with trace f such that $\Delta(f(x),y)=0$ for all $x,y\in I$, then $D=0$ and $\Delta=0$.*

Proof. Suppose that

$$\Delta(f(x),y)=0 \text{ for all } x,y\in I. \tag{3.12}$$

Replacing y by yz in (3.12), we have

$$\Delta(f(x),y)z+yD(f(x),z)=0 \text{ for all } x,y,z\in I.$$

Using (3.12), we get

$$D(f(x),z)=0 \text{ for all } x,z\in I. \tag{3.13}$$

Substitute $x+y$ for x in (3.13) to get

$$D(f(x),z)+D(f(y),z)+2D(D(x,y),z)=0 \text{ for all } x,y,z\in I.$$

Using (3.13) and the fact that R is not of characteristic two, we obtain

$$D(D(x,y),z)=0 \text{ for all } x,y,z\in I. \tag{3.14}$$

Replacing y by yu in (3.14), we find

$$\begin{aligned} &yD(D(x,u),z)+D(y,z)D(x,u)+D(x,y)D(u,z)\\ &\quad +D(D(x,y),z)u=0 \text{ for all } x,y,z,u\in I. \end{aligned}$$

Apply (3.14) to obtain

$$D(y,z)D(x,u)+D(x,y)D(u,z)=0 \text{ for all } x,y,z,u\in I. \tag{3.15}$$

Substituting yw for y in (3.15), we get

$$D(y,z)wD(x,u)+D(x,y)wD(u,z)=0 \text{ for all } x,y,z,u,w\in I. \tag{3.16}$$

In particular, if we replace x by z in (3.16), then we obtain $D(y,z)wD(z,u)+D(z,y)wD(u,z)=0$ for all $y,z,u,w\in I$. Since D is symmetric and using the fact that R is not of characteristic two, we have $D(y,z)wD(z,u)=0$ for all $y,z,u,w\in I$. Primeness of I yields that $D(z,u)=0$ for all $z,u\in I$. Using Lemma 2.4, we have $D=0$ and hence $\Delta=0$. ■

4. Main results

We dedicate the final section to the generalization of Theorems 2.2 and 2.3 to symmetric biderivations acting on ideals of semiprime rings.

We start with the following

THEOREM 4.1. *Let R be a prime ring of characteristic not two and three and I be an ideal of R. If D_1, D_2 are the symmetric biderivations of R with trace f_1, f_2 respectively such that $f_1(x)f_2(x) = 0$ for all $x \in I$, then either $D_1 = 0$ or $D_2 = 0$ unless R is commutative.*

Proof. Assume that R is not commutative. The linearization of the relation

$$f_1(x)f_2(x) = 0 \text{ for all } x \in I$$

gives

$$f_1(y)f_2(x) + 2D_1(x,y)f_2(x) + f_1(x)f_2(y) + 2D_1(x,y)f_2(y) + 2f_1(x)D_2(x,y) \\ + 2f_1(y)D_2(x,y) + 4D_1(x,y)D_2(x,y) = 0 \text{ for all } x, y \in I. \tag{4.1}$$

Substitute $-y$ for y in (4.1) to get

$$f_1(y)f_2(x) - 2D_1(x,y)f_2(x) + f_1(x)f_2(y) - 2D_1(x,y)f_2(y) - 2f_1(x)D_2(x,y) \\ - 2f_1(y)D_2(x,y) + 4D_1(x,y)D_2(x,y) = 0 \text{ for all } x, y \in I. \tag{4.2}$$

Adding up (4.1) and (4.2) and using 2-torsion freeness of R, we obtain

$$f_1(y)f_2(x) + f_1(x)f_2(y) + 4D_1(x,y)D_2(x,y) = 0 \text{ for all } x, y \in I. \tag{4.3}$$

Replacing y by $y + z$ in (4.3) , we find

$$f_1(y)f_2(x) + f_1(z)f_2(x) + 2D_1(y,z)f_2(x) + f_1(x)f_2(y) \\ + f_1(x)f_2(z) + 2f_1(x)D_2(y,z) + 4D_1(x,y)D_2(x,y) + 4D_1(x,z)D_2(x,z) \\ + 4D_1(x,y)D_2(x,z) + 4D_1(x,z)D_2(x,y) = 0 \text{ for all } x, y, z \in I. \tag{4.4}$$

Using (4.3), (4.4) gives that

$$2D_1(y,z)f_2(x) + 2f_1(x)D_2(y,z) + 4D_1(x,y)D_2(x,z) \\ + 4D_1(x,z)D_2(x,y) = 0 \text{ for all } x, y, z \in I. \tag{4.5}$$

Substitute y for x in (4.5) to get

$$6D_1(y,z)f_2(y) + 6f_1(y)D_2(y,z) = 0 \text{ for all } y, z \in I. \tag{4.6}$$

Replace z by zu in (4.6) and use (4.6) to obtain

$$[f_1(y),z]D_2(y,u) + D_1(y,z)[u,f_2(y)] = 0 \text{ for all } y,z,u \in I. \tag{4.7}$$

Again replace z by $f_1(y)z$ in (4.7) to get

$$f_1(y)[f_1(y),z]D_2(y,u) + f_1(y)D_1(y,z)[u,f_2(y)] \\ + D_1(y,f_1(y))z[u,f_2(y)] = 0 \text{ for all } y,z,u \in I. \tag{4.8}$$

Comparing (4.7) with (4.8), we arrive at

$$D_1(y, f_1(y))z[u, f_2(y)] = 0 \text{ for all } y, z, u \in I.$$

This implies that $D_1(y, f_1(y))Rz[u, f_2(y)] = 0$ for all $y, z, u \in I$. Primeness of R yields that either $D_1(y, f_1(y)) = 0$ or $z[u, f_2(y)] = 0$ for all $y, z, u \in I$. If $D_1(y, f_1(y)) = 0$ for all $y \in I$, then conclusion follows from Theorem 2.3. Now consider the case when $z[u, f_2(y)] = 0$ for all $y, z, u \in I$. Primeness of R yields that $[u, f_2(y)] = 0$ for all $y, u \in I$. By linearizing we get $[u, D_2(x, y)] = 0$ for all $x, y, u \in I$. Replacing x by xz, we have $[u, x]D_2(z, y) + D_2(x, y)[u, z] = 0$ for all $x, y, u, z \in I$. In particular, we get $[x, z]D_2(z, y) = 0$ for all $x, y, z \in I$. This implies that $[x, z]vD_2(z, y) = 0$ for all $x, y, z, v \in I$. On the other hand, since R is not commutative, we also have that $[I, I] \neq 0$. Hence, primeness of I yields that $D_2(z, y) = 0$ for all $z, y \in I$. Application of Lemma 2.4 gives that $D_2 = 0$. ■

We conclude our paper with the following result which extends Theorem 2.3.

Theorem 4.2. *Let R be a 2-torsion free semiprime ring and I be a nonzero ideal of R. Let D be a symmetric biderivation of R such that $D(I, I) \subseteq I$. If f is the trace of D such that $D(f(x), x) = 0$ for all $x \in I$, then $D = 0$ on I.*

Proof. Suppose that

$$D(f(x), x) = 0 \text{ for all } x \in I. \tag{4.9}$$

Linearization yields that

$$D(f(x), y) + D(f(x), x) + D(f(y), x) + D(f(y), y) + 2D(D(x, y), x) + 2D(D(x, y), y) = 0 \text{ for all } x, y \in I. \tag{4.10}$$

Comparing (4.9) with (4.10), we get

$$D(f(x), y) + D(f(y), x) + 2D(D(x, y), x) + 2D(D(x, y), y) = 0 \text{ for all } x, y \in I. \tag{4.11}$$

Substituting $-y$ for y in (4.11), we find

$$-D(f(x), y) + D(f(y), x) - 2D(D(x, y), x) + 2D(D(x, y), y) = 0 \text{ for all } x, y \in I. \tag{4.12}$$

Adding up (4.11) and (4.12) and using 2-torsion freeness of R, we get

$$D(f(y), x) + 2D(D(x, y), y) = 0 \text{ for all } x, y \in I. \tag{4.13}$$

Replace x by xz in (4.13), we obtain

$$xD(f(y), z) + D(f(y), x)z + 2xD(D(z, y), y) + 4D(x, y)D(z, y) + 2D(D(x, y), y)z = 0 \text{ for all } x, y, z \in I. \tag{4.14}$$

In view of (4.13), (4.14) reduces to

$$4D(x, y)D(z, y) = 0 \text{ for all } x, y, z \in I.$$

Since R is 2-torsion free, we have $D(x,y)D(z,y) = 0$ for all $x,y,z \in I$. Substituting zx for z we get $D(x,y)zD(x,y) = 0$ for all $x,y,z \in I$. Hence we get $D(x,y)I = 0$ and $ID(x,y) = 0$, i.e., $D(x,y) \in Ann(I)$ for all $x,y \in I$. Since $D(I,I) \subseteq I$, we have $D(x,y) \in I \cap Ann(I) = (0)$ for all $x,y \in I$ by Lemma 2.1. Hence we get $D = 0$ on I. ∎

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A. A., Department of Mathematics, Aligarh Muslim University, Aligarh, India

E-mail: asma_ali2@rediffmail.com

V. D. F., Department of Mathematics and Computer Science, University of Messina, viale S. D'Alcontres, 98166, Messina, Italy

E-mail: defilippis@unime.it

F. Sh., Department of Applied Mathematics, Faculty of Eng. & Techn., Aligarh Muslim University, Aligarh, India

E-mail: faiza.shujat@gmail.com