A NOTE ON GENERALIZED WHITNEY MAPS

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Abstract. For metrizable continua, there exists the well-known notion of a Whitney map. Garcıa-Velazquez extends the definition of Whitney map for $C(X)$, where X is an arbitrary continuum (not necessarily metrizable). He also shows that the examples he considers do not admit such generalized Whitney maps. In this paper we shall investigate the properties of continua which admit such generalized Whitney maps.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$. The cardinality of a set A is denoted by $\text{card}(A)$.

Let X be a space. We define its hyperspaces as the following sets:

$$2^X = \{F \subseteq X : F \text{ is closed and non-empty}\},$$
$$C(X) = \{F \in 2^X : F \text{ is connected}\},$$
$$X(n) = \{F \in 2^X : F \text{ has at most } n \text{ points}\}, \ n \in \mathbb{N}.$$

For any finitely many subsets $S_1, \ldots, S_n$, let

$$\langle S_1, \ldots, S_n \rangle = \left\{ F \in 2^X : F \subset \bigcup_{i=1}^{n} S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i \right\}.$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{\langle U_1, \ldots, U_n \rangle : U_i \text{ is an open subset of } X \text{ for each } i \text{ and each } n < \infty\}$, and $C(X)$ is a subspace of 2^X.

Let X and Y be the spaces and let $f : X \to Y$ be a mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = \{f(x) : x \in F\}$ for $F \in 2^X$. By [12, Theorem 5.10, p. 170], 2^f is continuous and $2^f(C(X)) \subset C(Y)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

The concept of Whitney maps is a very powerful tool in hyperspace theory of metric compact spaces. In the 1930's, Whitney constructed special types of

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functions on spaces of sets for the purpose of studying families of curves ([18] and [19]). In 1942, Kelley made significant use of Whitney's functions in studying hyperspaces of metric continua [7]. In 1978, Whitney's functions are called Whitney maps (see Chapter XIV of [3]). Whitney maps and Whitney levels are widely used in the theory of metrizable continua, for details see the book [5].

In [1], first examples are presented of non-metrizable continua X which admit and ones which do not admit a map $\mu : C(X) \to [0, 1]$. Stone [16] gives another example of non-metrizable continua X which admits a Whitney map $\mu : C(X) \to [0, 1]$. See also [15].

Garcıa-Velazquez [3] extends the definition of Whitney level (for $C(X)$) so that it does not depend on Whitney maps and can be given on arbitrary continua (that is, continua that are not necessarily metrizable). In the paper [4], an example is given of a non-metrizable continuum admitting a generalized Whitney map. The author of that paper gives an example of a non-metrizable continuum admitting a generalized Whitney map. Moreover, the author gives restrictions on what types of Whitney maps are possible. In this paper the following definitions are given.

DEFINITION 1.1. A generalized arc is a continuum J with its topology given by a strict linear order $\rhd$. It is denoted by $\langle J, \rhd \rangle$.

DEFINITION 1.2. If X is a continuum, a generalized Whitney map for $C(X)$ is a map $\mu \colon C(X) \to \langle J, \rhd \rangle$ where $\langle J, \rhd \rangle$ is a generalized arc and the following conditions hold:

a) $\mu(\{x\}) = \min J$ for each $x \in X$,

b) $\mu(A) \rhd \mu(B)$ whenever $A, B \in C(X)$ and $A \subset B$, and

c) $\mu(X) = \max J$.

For some topological properties P, if X is a continuum with P that admits a Whitney map $p : C(X) \to J$ then the weight of J must be equal to the weight of X. Among these properties P we find local connectedness, rim-metrizability, semi-aposyndesis and arcwise-connectedness. The technical condition needed is one related to the representation of X as an inverse limit.

2. Factorizable inverse systems

We shall use the notion of inverse system as in [2, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

An element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [2, p. 135].

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a \colon \lim \mathbf{X} \to X_a$, for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ form inverse systems.

LEMMA 2.1. *Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$ *and* $C(X) = \lim C(\mathbf{X})$.

For a cardinal τ we say that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is τ-*directed* if for each $B \subseteq A$ with $\mathrm{card}(B) \leq \tau$ there is an $a \in A$ such that $a \geq b$ for each $b \in B$. Inverse system $\mathbf{X}$ is σ-*directed* if $\mathbf{X}$ is $\aleph_0$-directed. We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is λ-*system* if it is λ-directed.

THEOREM 2.2. *For each Tychonoff cube* I^τ, $\tau \geq \aleph_1$, *there exists* $\lambda < \tau$ *and an inverse* λ-*system* $\mathbf{I} = \{I^a, P_{ab}, A\}$ *of the cubes* I^a, $\mathrm{card}(a) = \lambda$, *such that* I^τ *is homeomorphic to* $\lim \mathbf{I}$.

Proof. a) Let us recall that the *Tychonoff cube* I^τ is the Cartesian product $\prod\{I_s : s \in S\}$, $\mathrm{card}(S) = \tau$, $I_s = [0,1]$ [2, p. 114]. If $\mathrm{card}(S) = \aleph_0$, the Tychonoff cube I^τ is called the *Hilbert cube*. Let A be the set of all subsets of S of the cardinality λ ordered by inclusion. If $a \subseteq b$, then we write $a \leq b$. It is clear that A is λ-directed. For each $a \in A$ there exists a cube I^a. If $a, b \in A$ and $a \leq b$, then there exists a projection $P_{ab} : I^b \to I^a$. Finally, we have system $\mathbf{I} = \{I^a, P_{ab}, A\}$.

b) Let us prove that I^τ is homeomorphic to $\lim \mathbf{I}$. Let $x \in I^\tau$. It is clear that $P_a(x) = x_a$ is a point of I^a and that $P_{ab}(x_b) = x_a$ if $a \leq b$. This means that (x_a) is a thread in $\mathbf{I} = \{I^a, P_{ab}, A\}$. Set $H(x) = (x_a)$. We have the mapping $H : I^m \to \lim \mathbf{I}$. It is clear that H is continuous, 1-1 and onto. Hence, H is a homeomorphism. ∎

THEOREM 2.3. *Let* X *be a compact Hausdorff space such that* $\mathrm{w}(X) \geq \aleph_1$ *and let* $\aleph_0 \leq \lambda < \mathrm{w}(X)$. *Then there exists an inverse* λ-*system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *such that* $\mathrm{w}(X_a) \leq \lambda$ *and* X *is homeomorphic to* $\lim \mathbf{X}$.

Proof. By [2, Theorem 2.3.23.], the space X is embeddable in $I^{w(X)}$. From Theorem 2.2 it follows that $I^{w(X)}$ is a limit of $\mathbf{I} = \{I^a, P_{ab}, A\}$ where A is the set from the proof of a) of Therem 2.2. Now, X is homeomorphic to a closed subspace of $\lim \mathbf{I}$. For each $a \in A$ let $X_a = P_m(X)$, where $P_m : I^m \to I^a$ is a projection of the Tychonoff cube I^m onto the cube I^a. Let p_{ab} be the restriction of P_{ab} onto X_b. We have the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leq \lambda$. By virtue of [2, Corollary 2.5.7] X is homeomorphic to $\lim \mathbf{X}$. Moreover, $\mathbf{X}$ is an inverse λ-system since $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse λ-system. ∎

A *cover* of a set X is a family $\{A_s : s \in S\}$ of subsets of X such that $X = \bigcup\{A_s : s \in S\}$. $\mathrm{Cov}(X)$ is the set of all coverings of topological space X. We say that a cover $\mathcal{B}$ of space X is *refinement* of a cover $\mathcal{A}$ of the same space if for every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $B \subset A$. If $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} \prec \mathcal{U}$.

LEMMA 2.4. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be an inverse system of compact spaces with surjective bonding mappings and limit* X. *For every finite cover* $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ *there exists an* $a(\mathcal{U}) \in A$ *such that for each* $b \geq a(\mathcal{U})$ *there is a finite cover* $\mathcal{U}_b = \{U_{b1}, U_{b2}, \ldots, U_{bm}\}$ *of* X_b *such that* $p_b^{-1}(\mathcal{U}_b) =$

$\{p_b^{-1}(U_{b1}), p_b^{-1}(U_{b2}), \ldots, p_b^{-1}(U_{bm})\}$ *is refinement of* $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$, *i.e.,* $p_b^{-1}(\mathcal{U}_b) \prec \mathcal{U}$.

Proof. By virtue of the definition of a base in X, for each $U_i \in \mathcal{U}$ we have $U_i = \bigcup\{p_a^{-1}(U_a) : a \in K_i \subset A\}$. Now, $\bigcup\{p_a^{-1}(U_a) : a \in K_i \cup K_2 \cup \cdots \cup K_n\}$ is a cover oh X since $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ is cover of X. There is a finite subfamily $\{p_{a1}^{-1}(U_{a1}), \ldots, p_{am}^{-1}(U_{am})\}$ of $\bigcup\{p_a^{-1}(U_a) : a \in K_i \cup K_2 \cup \cdots \cup K_n\}$ which covers X. We infer that there exists $a(\mathcal{U}) \in A$ such that $a(\mathcal{U}) \geq a1, \ldots, am$. For each $b \geq a(\mathcal{U})$ we have a finite cover $\{U_{b1} = p_{a1b}^{-1}(U_{a1}), \ldots, U_{bm} = p_{amb}^{-1}(U_{am})\}$, i.e., a finite cover $\mathcal{U}_b = \{U_{b1}, U_{b2}, \ldots, U_{bm}\}$ of X_b such that $p_b^{-1}(\mathcal{U}_b) = \{p_b^{-1}(U_{b1}), p_b^{-1}(U_{b2}), \ldots, p_b^{-1}(U_{bm})\}$ is refinement of $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$. ∎

THEOREM 2.5. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* λ-*directed inverse system of compact spaces with surjective bonding mappings and limit* X. *Let* Y *be a compact space of weight* λ *For each surjective mapping* $f : X \to Y$ *there exists an* $a \in A$ *such that for each* $b \geq a$ *there exists a mapping* $g_b : X_b \to Y$ *such that* $f = g_b p_b$.

Proof. Let B be a basis of Y with $\text{card}(B) = \lambda$ and let $\mathcal{V}$ be a collection of all finite subfamilies of B which cover Y. Clearly, $\text{card}(\mathcal{V}) = \lambda$. Consider an enumeration $\mathcal{V} = \{\mathcal{V}_v : v < \lambda\}$. For each $\mathcal{V}_v$ the family $f^{-1}(\mathcal{V}_v) = \{f^{-1}(U) : U \in \mathcal{V}_v\}$ is a covering of X. By virtue of Lemma 2.4 there exists an $a(v) \in A$ such that for each $b \geq a(v)$ there is a cover $\mathcal{V}_{vb}$ of X_b with $p_b^{-1}(\mathcal{V}_{vb}) \prec f^{-1}(\mathcal{V}_v)$. From the λ-directedness of A it follows that there is an $a \in A$ such that $a \geq a(v)$, for all $v < \lambda$. Let $b \geq a$. We claim that $f(p_b^{-1}(x_b))$ is degenerate. Suppose that there exists a pair u, v of distinct points of Y such that $u, v \in f(p_b^{-1}(x_b))$. Then there exists a pair x, y of distinct points of $p_b^{-1}(x_b)$ such that $f(x) = u$ and $f(y) = v$. Let U, V be a pair of disjoint open sets of Y such that $u \in U$ and $v \in V$. Consider the covering $\{U, V, Y \backslash \{u, v\}\}$. There exists a covering $\mathcal{V}_\nu \in \mathcal{V}$ such that $\mathcal{V}_\nu \prec \{U, V, X \backslash \{u, v\}\}$. We infer that there is a covering $\mathcal{V}_{vb}$ of X_b such that $p_b^{-1}(\mathcal{V}_{vb}) \prec f^{-1}(\mathcal{V}_v)$. It follows that $p_b(x) \neq p_b(y)$ since x and y lie in the disjoint members of the covering $f^{-1}(\mathcal{V}_v)$. This is impossible since $x, y \in p_b^{-1}(x_b)$. Thus, $f(p_b^{-1}(x_b))$ is degenerate. Now we define $g_b : X_b \to Y$ by $g_b(x_b) = f(p_b^{-1}(x_b))$. It is clear that $g_b p_b = f$. Let us prove that g_b is continuous. Let U be open in Y. Then $g_b^{-1}(U)$ is open since $p_b^{-1}(g_b^{-1}(U)) = f^{-1}(U)$ is open and p_b is quotient (as a closed mapping). ∎

3. Inverse systems whose limit X admits a generalized Whitney map for $C(X)$

In the sequel we shall use Definition 1.2 of generalized Whitney map and Definition 1.1 of generalized arc.

If X is a continuum, a λ-Whitney map for $C(X)$ is a map $\mu : C(X) \to J$, where J is some generalized arc of weight λ. If $J = [0, 1]$, then we say 'a Whitney map $\mu : C(X) \to [0, 1]$ instead of 'a λ-generalized Whitney map'.

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [13, (1.212.3), p. 204] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

A mapping $f : X \to Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [13, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

LEMMA 3.1. *Every hereditarily irreducible mapping is light.*

We say that a mapping $f : X \to Y$ is monotone if $f^{-1}(y)$ is connected for every $y \in Y$.

COROLLARY 3.2. *If $f : X \to Y$ is monotone and hereditarily irreducible, then f is one-to-one.*

LEMMA 3.3. [13, (1.212.3), p. 204] *A mapping $f : X \to Y$ is hereditarily irreducible if and only if the mapping $C(f) : C(X) \to C(Y)$ is light.*

Now we study inverse systems whose limit X admits a λ-generalized Whitney map for $C(X)$.

THEOREM 3.4. *Let X be a non-metric continuum of weight $\mathrm{w}(X) > \lambda$. Then X admits a λ-generalized Whitney map for $C(X)$ if and only if for each λ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of continua which admit Whitney maps for $C(X_a)$ and $X = \lim \mathbf{X}$ there exists an $a \in A$ such that for each $b \geq a$ the projection $p_b : \lim X \to X_b$ is hereditarily irreducible.*

Proof. Necessity. Consider the inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$ (Lemma 2.1). If $\mu \colon C(X) \to J$ is a λ-generalized Whitney map for $C(X)$, then, by Theorem 2.5, there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $\mu_b \colon C(p_b)(X) \to J$ with $\mu = \mu_b C(p_b)$. Suppose that p_b is not hereditarily irreducible. there exists a pair F, G of subcontinua of X with $F \subseteq G$, $F \neq G$, (i.e., F is a proper subcontinuum of G) such that $p_b(F) = p_b(G)$. It is clear that $C(p_b)(F) = C(p_b)(G)$. This means that $\mu_b C(p_b)(F) = \mu_b C(p_b)(G)$. From $\mu = \mu_b C(p_b)$ it follows that $\mu(F) = \mu(G)$. This is impossible since μ is a Whitney map for $C(X)$ and from $F \subseteq G, F \neq G$ it follows $\mu(F) < \mu(G)$. Hence, the projections $p_a, a \in A$, are hereditarily irreducible.

Sufficiency. Suppose that there exists an $a \in A$ such that for each $b \geq a$ the projection $p_b : \lim X \to X_b$ is hereditarily irreducible. Consider the inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$ (Lemma 2.1). Let $\mu_b \colon C(X_b) \to J$ be a λ-generalized Whitney map for $C(X_b)$, where $b \geq a$ is fixed. We shall prove that $\mu = \mu_b C(p_b) : C(X) \to J$ is a λ-generalized Whitney map for $C(X)$. Let F, G be a pair of subcontinua of X with $F \subseteq G$, $F \neq G$. We must prove that $\mu(F) < \mu(G)$. Now, $p_b(F) \subset p_b(G)$ and $p_b(F) \neq p_b(G)$ since p_b is hereditarily irreducible. We infer that $\mu_b(p_b(F)) < \mu_b(p_b(G))$ since μ_b is a λ-generalized Whitney map for $C(X_b)$. Moreover, $p_b(F) = C(p_b)(F)$ and $p_b(G) = C(p_b)(G)$. From $\mu_b(p_b(F)) < \mu_b(p_b(G))$ we have $\mu_b(C(p_b)(F)) < \mu_b(C(p_b)(G))$, i.e., $\mu_b C(p_b)(F) < \mu_b C(p_b)(G)$.

Finally, $\mu(F) < \mu(G)$ since $\mu = \mu_b C(p_b)$. We infer that $\mu = \mu_b C(p_b) \colon C(X) \to J$ is a λ-generalized Whitney map for $C(X)$. ∎

Let us observe that from the necessity in Theorem 3.4 the following corollary follows.

COROLLARY 3.5. *Let X be a continuum of weight* $\mathrm{w}(X) > \lambda$. *If X admits a λ-generalized Whitney map for $C(X)$, then for each λ-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *of continua such that $X = \lim \mathbf{X}$ there exists an $a \in A$ such that for each $b \geq a$ the projection $p_b \colon \lim X \to X_b$ is hereditarily irreducible.*

Let us recall that a mapping $f \colon X \to Y$ is monotone if $f^{-1}(y)$ is connected for every $y \in Y$.

THEOREM 3.6. [2, p. 462, 6.3.16. (a)] *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system of continua such that the bonding mappings p_{ab} are monotone, then the projections $p_a \colon \lim \mathbf{X} \to X_a, a \in A$, are monotone.*

COROLLARY 3.7. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a λ-directed inverse system of continua with monotone bonding mappings. Let $X = \lim \mathbf{X}$ be a continuum of weight* $\mathrm{w}(X) > \lambda$. *Then X admits no λ-generalized Whitney map for $C(X)$.*

Now we shall prove that hereditarily irreducible bonding mappings induce hereditarily irreducible projections.

THEOREM 3.8. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system of continua such that the bonding mappings p_{ab} are hereditarily irreducible, then the projections $p_a \colon \lim \mathbf{X} \to X_a, a \in A$, are hereditarily irreducible,*

Proof. Let $a \in A$; we shall prove that the projection $p_a \colon \lim X \to X_a$ is hereditarily irreducible. Let C^* and D^* be the continua in $\lim \mathbf{X}$ such that $C^* \subset D^*$ and $C^* \neq D^*$. There exists a point $x \in D^* \setminus C^*$ and a basic open neighborhood $p_b^{-1}(U_b)$ of x such that $p_b^{-1}(U_b) \cap C^* = \emptyset$. It follows that $p_b(x)$ is in $p_b(D^*) \setminus p_b(C^*)$. This means that $p_c(C^*) \subset p_c(D^*)$ and $p_c(C^*) \neq p_c(D^*)$ for every $c \geq b$. There exists $d \geq a, b$ since A is directed. It follows that $p_d(C^*) \subset p_d(D^*)$ and $p_d(C^*) \neq p_d(D^*)$. Now from the fact that $p_{da} \colon X_d \to X_a$ is hereditarily irreducible it follows that $p_{da}(p_d(C^*)) \subset p_{da}(p_d(D^*))$ and $p_{da}(p_d(C^*)) \neq p_{da}(p_d(D^*))$, i.e., $p_a(C^*) \subset p_a(D^*)$ and $p_a(C^*) \neq p_a(D^*)$. Thus, the projection $p_a \colon \lim \mathbf{X} \to X_a$ is hereditarily irreducible. ∎

QUESTION. Are the bonding mappings p_{ab} hereditarily irreducible if the projections $p_a \colon \lim \mathbf{X} \to X_a, a \in A$, are hereditarily irreducible?

4. D-continua

A continuum X is called a *D-continuum* if for every pair C, D of disjoint non-degenerate subcontinua there exists a subcontinuum $E \subset X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.

A family $\mathcal{N} = \{M_s : s \in S\}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighborhood U of x there exists an $s \in S$ such that $x \in M_s \subset U$ [2, p. 170]. The *network weight* of a space X is defined as the smallest cardinal number of the form $\mathrm{card}(\mathcal{N})$, where $\mathcal{N}$ is a network for X; this cardinal number is denoted by $\mathrm{nw}(X)$.

THEOREM 4.1. [2, p. 171, Theorem 3.1.19] *For every compact space X we have* $\mathrm{nw}(X) = \mathrm{w}(X)$.

Let us recall that $X(n) = \{F \in 2^X : F \text{ has at most } n \text{ points}\}$, $n \in \mathbb{N}$.

THEOREM 4.2. *Let X be a D-continuum of the weight $\mathrm{w}(X) > \lambda$. If X admits a λ-generalized Whitney map for $C(X)$, then* $\mathrm{w}(C(X) \setminus X(1)) \leq \lambda$.

Proof. Let X admit a λ-generalized Whitney map for $C(X)$. From Theorem 2.3 it follows that there exists a λ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of continua and surjective bonding mappings such that X is homeomorphic to $\lim \mathbf{X}$ and $\mathrm{w}(X_a) \leq \lambda$. Consider the inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ whose limit is $C(X)$. From Theorem 3.4 it follows that the projections p_b are hereditarily irreducible and $C(p_b)$ are light for some cofinal subset B of A. If for each $b \in B$, $C(p_b)$ is one-to-one, then we have a homeomorphism $C(p_b)$ of $C(X)$ onto $C(p_b)(X)$. Since $\mathrm{w}(C(p_b)(X)) \leq \lambda$, we have that $\mathrm{w}(C(X) \setminus X(1)) \leq \lambda$. Suppose that the restriction $C(p_b)|(C(X) \setminus X(1))$ is not one-to-one. Then there exists a nondegenerate continuum C_b in X_b and two nondegenerate continua C^*, D^* in X such that $p_b(C^*) = p_b(D^*) = C_b$. It is impossible that $C^* \subsetneq D^*$ or $D^* \subsetneq C^*$ since p_b is hereditarily irreducible. If $C^* \cap D^* \neq \emptyset$, then for a continuum $Y = C^* \cup D^*$ we have that C^* and D^* are subcontinua of Y and $p_b(Y) = p_b(C^*) = p_b(D^*) = C_b$ which is impossible since p_b is hereditarily irreducible. We infer that $C^* \cap D^* = \emptyset$. There exists a subcontinuum E such that $C^* \subset E, D^* \neq D^* \cap E \neq \emptyset$ since X is a D-continuum. Now $p_b(E \cup D^*) = p_b(E)$ which is impossible since p_b is hereditarily irreducible. Furthermore, $C(p_a)^{-1}(X_a(1)) = X(1)$ since from the hereditarily irreducibility of p_a it follows that no non-degenerate subcontinuum of X maps under p_a onto a point. Let $Y_a = C(p_a)(C(X))$. We infer that $C(p_a)^{-1}[Y_a \setminus X_a(1)] = C(X) \setminus X(1)$. It follows that the restriction $P_a = C(p_a)|(C(X) \setminus X(1))$ is one-to-one and closed [2, Proposition 2.1.4]. Hence, P_a is a homeomorphism and $\mathrm{w}(C(X) \setminus X(1)) \leq \lambda$. ∎

Now we shall prove the main theorem of this section.

THEOREM 4.3. *If a D-continuum X admits a λ-Whitney map $\mu \colon C(X) \to J$, then* $\mathrm{w}(X) = \lambda$.

Proof. By Theorem 4.2 we have that $\mathrm{w}(C(X) \setminus X(1)) \leq \lambda$. This means that there exists a base $\mathcal{B} = \{B_v : v < \lambda\}$ of $C(X) \setminus X(1)$. For each B_v let $C_v = \{x \in X : \exists\, B \in B_v \ (x \in B)\}$.

Claim 1. The family $\{C_v : v < \lambda\}$ is a network of X.

Let x be a point of X and let U be an open subset of X such that $x \in U$. There exists an open set V such that $x \in V \subset \mathrm{Cl}\, V \subset U$. Let K be a component of $\mathrm{Cl}\, V$ containing x. By the Boundary Bumping Theorem [14, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C(X) \setminus X(1)$. Now, $\langle U \rangle \cap (C(X) \setminus X(1))$ is a neighborhood of K in $C(X) \setminus X(1)$. It follows that there exists a $B_v \in \mathcal{B}$ such that $K \in B_v \subset \langle U \rangle \cap (C(X) \setminus X(1))$. It is clear that $C_v \subset U$ and $x \in C_v$ since $x \in K$. Hence, the family $\{C_v : v < \lambda\}$ is a network of X.

Claim 2. $\mathrm{nw}(X) = \lambda$.

Apply Claim 1 and the fact that the cardinality card $\mathcal{B} \le \lambda$.

Claim 3. $\mathrm{w}(X) = \lambda$.

By Claim 2 we have $\mathrm{nw}(X) = \lambda$. Moreover, by Theorem 4.1 $\mathrm{w}(X) = \lambda$. ∎

Now we have the following corollary.

COROLLARY 4.4. [8] *A D-continuum X admits a Whitney map $\mu : C(X) \to [0,1]$ if and only if it is metrizable.*

5. Applications

5.1. Generalized Whitney maps for rim-metrizable or locally connected continuum

A space X is said to be *rim-metrizable* if it has a basis $\mathcal{B}$ such that $\mathrm{Bd}(U)$ is metrizable for each $U \in \mathcal{B}$. Equivalently, a compact space X is rim-metrizable if and only if for each pair F, G of disjoint closed subsets of X there exists a metrizable closed subset of X which separates F and G.

Rim-metrizable spaces are a generalization of metrizable spaces.

Let us observe that every continuous image of an ordered compact space is rim-metrizable [11, p. 566, Theorem 5].

The properties of rim-metrizable spaces which are essential for the our purpose are established in Lemmas 5.1 and 5.3.

LEMMA 5.1 [17, Theorem 1.2] *Let X be a nondegenerate rim-metrizable continuum and let Y be a continuous image of X under a light mapping. Then $\mathrm{w}(X) = \mathrm{w}(Y)$.*

LEMMA 5.2 [17, Theorem 1.4] *Let $f : X \to Y$ be a light mapping of a nondegenerate continuum X onto a space Y. If X admits a basis of open sets whose boundaries have weight $\le \mathrm{w}(Y)$, then $\mathrm{w}(Y) = \mathrm{w}(X)$.*

LEMMA 5.3 [17, Theorem 3.2] *The monotone image of any rim-metrizable continuum is also rim-metrizable.*

THEOREM 5.4. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces and surjective bonding mappings p_{ab}. Then:*

1) *There exists an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact spaces such that m_{ab} are monotone surjections and $\lim \mathbf{X} = \lim M(\mathbf{X})$,*

2) *If $\mathbf{X}$ is λ-directed, then $M(\mathbf{X})$ is λ-directed.*

3) *If $w(X_a) \leq \tau$ and $\lim \mathbf{X}$ is either locally connected or rim-metrizable continuum, then $w(M_a) \leq \tau$ for every $a \in A$.*

Proof. 1) The proof of 1) is broken into several steps. We give the partial proof. The complete prof is in [9, pp. 110–111].

a) Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system with limit X and the projections $p_a : X \to X_a, a \in A$. For every mapping $p_a : X \to X_a$ there exists a monotone-light factorization $p_a = \ell_a m_a$, where $m_a : X \to M_a$ is monotone and $\ell_a : M_a \to X_a$ is light [2, p. 451, Theorem 6.2.22]. We have a collection of spaces $M_a, a \in A$.

b) For every bonding mapping $p_{ab} : X_b \to X_a, b \geq a$, we define $m_{ab} : M_b \to M_a$ as follows. Let x be a point of $M_b, x_b = \ell_b(x)$ and $x_a = p_{ab}(x_b)$. Then $m_b^{-1}(x)$ is a component in $p_b^{-1}(x_b)$. This means that there exists a unique point $y \in M_a$ such that the component $m_a^{-1}(x)$ of $p_a^{-1}(x_a)$ contains $m_b^{-1}(x)$ since $p_b^{-1}(x_b) \subset p_a^{-1}(x_a)$. Set $m_{ab}(x) = y \in M_a$. The mapping $m_{ab} : M_b \to M_a$ is defined. From the definition of m_{ab} it follows

$$p_a = \ell_a m_a,$$
$$p_{ab}\ell_b = \ell_a m_{ab},$$
$$m_{ab}m_b = m_a.$$

As in [9, p. 110, Theorem 3.7.] we infer that $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ is an inverse system. Moreover, $\lim \mathbf{X}$ and $\lim M(\mathbf{X})$ are homeomorphic.

2) Obvious since $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ and $\mathbf{X} = \{X_a, p_{ab}, A\}$ are defined over same set A.

3) If $\lim \mathbf{X}$ is rim-metrizable, then by Lemma 5.3 every space M_a is rim-metrizable and by Lemma 5.1 $w(X_a) = w(M_a)$. If X is locally connected, then apply [10, Theorem 1]. ∎

The main theorems of this subsection are the following ones.

THEOREM 5.5. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a λ-directed inverse system of compact spaces and surjective bonding mappings p_{ab}. If $\lim \mathbf{X}$ is a locally connected continuum which admits a generalized Whitney map for $C(\lim \mathbf{X})$, then there exists an $a \in A$ such that for each $b \geq a$ the projection $m_b : \lim M(\mathbf{X}) \to M_a$ is a homeomorphism.*

Proof. Consider the inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ (Theorem 5.4) whose limit $\lim M(\mathbf{X})$ is homeomorphic to $\lim \mathbf{X}$. By Theorem 3.5 there exists an $a \in A$ such that for each $b \geq a$ the projection $m_b : \lim M(X) \to M_a$ is hereditary irreducible. This means that m_b is light. Light and monotone mappings are 1-1 and, consequently, a homeomorphism. See also 3.2. The proof is completed. ∎

THEOREM 5.6. *If X is a locally connected continuum such that $\mathrm{w}(X) > \lambda$, then X admits no λ-generalized Whitney map $\mu_\lambda : C(X) \to J$.*

Proof. By Theorem 2.3 we infer that there exists an inverse λ-system $\mathbf{X} = \{X_a, p_{ab}, A\}, \lambda < \mathrm{w}(X)$, such that X is homeomorphic to $\lim \mathbf{X}$. Now, inverse system of Theorem 5.4 $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ is an inverse system of spaces M_a with $\mathrm{w}(M_a) \leq \lambda$. If we assume X does admit a λ-generalized Whitney map, then by Theorem 5.5 there exists an $a \in A$ such that the projection $m_b : \lim M(\mathbf{X}) \to M_b$ is a homeomorphism, for every $b \geq a$. It follows that $\mathrm{w}(\lim M(\mathbf{X})) = \mathrm{w}(M_b) \leq \lambda$. This is impossible since $\lim M(\mathbf{X})$ is homeomorphic to X and $\mathrm{w}(X) > \lambda$. ∎

THEOREM 5.7. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a λ-directed inverse system of compact spaces and surjective bonding mappings p_{ab}. If $\lim \mathbf{X}$ is a rim-metrizable continuum which admits a generalized Whitney map for $C(\lim \mathbf{X})$, then there exists an $a \in A$ such that for each $b \geq a$ the projection $m_b : \lim M(X) \to M_a$ is hereditary irreducible.*

Proof. Consider the inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ from Theorem 5.4 whose limit $\lim M(\mathbf{X})$ is homeomorphic to $\lim \mathbf{X}$. By Corollary 3.5 there exists an $a \in A$ such that for each $b \geq a$ the projection $m_b : \lim M(X) \to M_a$ is hereditary irreducible. This means that m_b is light. Light and monotone mappings are 1-1 and, consequently, homeomorphism. The proof is completed. ∎

THEOREM 5.8. *If X is a locally connected or rim-metrizable continuum of weight $\mathrm{w}(X) > \lambda$, then X admits no λ-generalized Whitney map $\mu : C(X) \to J$.*

THEOREM 5.9. *If X admits a basis of open sets whose boundaries have weight $\leq \lambda$ and $\mathrm{w}(X) > \lambda$, then X admits no λ-generalized Whitney map $\mu : C(X) \to J$.*

Proof. By Theorem 2.3 we infer that there exists an inverse λ-system $\mathbf{X} = \{X_a, p_{ab}, A\}, \lambda < \mathrm{w}(X)$, such that X is homeomorphic to $\lim \mathbf{X}$. If X admits a λ-generalized Whitney map $\mu : C(X) \to J$, then there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim X \to X_b$ is hereditarily irreducible (Corollary 3.5). This means that the projection $p_b : \lim X \to X_b$ is light. From Lemma 5.2 it follows that $\mathrm{w}(\lim \mathbf{X}) = \mathrm{w}(X_a) \leq \lambda$. This is impossible since $\lim(\mathbf{X})$ is homeomorphic to X and $\mathrm{w}(X) > \lambda$. ∎

5.2. Generalized Whitney maps for $C(X)$ if X is a semi-aposyndetic continuum

The concept of aposyndesis was introduced by Jones in [6]. A continuum is said to be *semi-aposyndetic* [5, p. 238, Definition 29.1], if for every $p \neq q$ in X, there exists a subcontinuum M of X such that $\mathrm{Int}(M)$ contains one of the points p, q and $X \setminus M$ contains the other one. Each locally connected continuum is semi-aposyndetic.

In the sequel we shall use the following result [14, p. 226, Exercise 11.52].

LEMMA 5.10. *If X is a continuum and if A and B are mutually disjoint subcontinua of X, then there is a component K of $X \setminus (A \cup B)$ such that $\mathrm{Cl}\, K \cap A \neq \emptyset$ and $\mathrm{Cl}\, K \cap B \neq \emptyset$.*

LEMMA 5.11. *If X is semi-aposyndetic continuum, then X is a D-continuum.*

Proof. Let us prove that for every pair C^*, D^* of disjoint non-degenerate subcontinua of X there exists a non-degenerate subcontinuum $E \subset X$ such that $C^* \cap E \neq \emptyset \neq D^* \cap E$ and $(C^* \cup D^*) \setminus E \neq \emptyset$. We shall consider two cases.

a) If either $\mathrm{Int}_X(C) \neq \emptyset$ or $\mathrm{Int}_X(D) \neq \emptyset$, then it suffices to apply Lemma 5.10 to the union $C^* \cup D^*$ and obtain a component K of $X \setminus (C^* \cup D^*)$ such that $\mathrm{Cl}\, K \cap C^* \neq \emptyset$ and $\mathrm{Cl}\, K \cap D^* \neq \emptyset$. Then $E = \mathrm{Cl}\, K$ is a continuum with properties $C^* \cap E \neq \emptyset \neq D^* \cap E$ and $(C^* \cup D^*) \setminus E \neq \emptyset$ since $\mathrm{Int}(C^*) \cap E = \emptyset$ or $\mathrm{Int}(D^*) \cap E = \emptyset$.

b) Assume that $\mathrm{Int}(C^*) = \emptyset$ and $\mathrm{Int}(D^*) = \emptyset$. There exist $x, y \in C^*$ such that $x \neq y$. Moreover, there exists a subcontinuum M of X such that $\mathrm{Int}(M)$ contains one of the points x, y and $X \setminus M$ contains the other one since X is semi-aposyndetic. Suppose that $x \in \mathrm{Int}(M)$ and $y \in X \setminus M$. If $M \cap D \neq \emptyset$, then we set $E = M$ and we have that $C^* \cap E \neq \emptyset \neq D^* \cap E$ and $(C^* \cup D^*) \setminus E \neq \emptyset$ since $y \in X \setminus M$. Suppose that $M \cap D^* = \emptyset$. Applying Lemma 5.10 to the union $C^* \cup D^* \cup M$ we obtain a component K of $X \setminus (C^* \cup D^* \cup M)$ such that $\mathrm{Cl}\, K \cap (C^* \cup M) \neq \emptyset$ and $\mathrm{Cl}\, K \cap D^* \neq \emptyset$. It is clear that $x \notin \mathrm{Cl}\, K$. If $\mathrm{Cl}\, K \cap C^* \neq \emptyset$, then we set $E = \mathrm{Cl}\, K$ and obtain a continuum E such that $C^* \cap E \neq \emptyset \neq D^* \cap E$ and $(C^* \cup D^*) \setminus E \neq \emptyset$ since $x \notin \mathrm{Cl}\, K$. If $\mathrm{Cl}\, K \cap C^* = \emptyset$, then $\mathrm{Cl}\, K \cap M \neq \emptyset$ and we set $E = \mathrm{Cl}\, K \cup M$. Now $y \notin E$, $C^* \cap E \neq \emptyset \neq D^* \cap E$ and $(C^* \cup D^*) \setminus E \neq \emptyset$. ∎

THEOREM 5.12. *If a semi-aposyndetic continuum X admits a Whitney map $\mu \colon C(X) \to J$, then $\mathrm{w}(X) = \lambda$.*

Proof. Apply Theorem 4.3. ∎

COROLLARY 5.13. *A semi-aposyndetic continuum X with $\mathrm{w}(X) > \lambda$ admits no Whitney map $\mu \colon C(X) \to J$.*

5.3. A C-continuum

A continuum X is said to be a *C-continuum* provided for each triple x, y, z of points of X, there exists a subcontinuum C of X which contains x and exactly one of the points y and z [20, p. 326].

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

LEMMA 5.14. *Each arcwise connected continuum is a C-continuum.*

Proof. Let x, y, z be a triple of points of an arcwise connected continuum X. There exists an arc $[x, y]$ with endpoints x and y. If $z \notin [x, y]$, then the proof is completed. If $z \in [x, y]$, then subarc $[x, z]$ contains x and z, but not y. The proof is completed. ∎

LEMMA 5.15. *The cartesian product $X \times Y$ of two non-degenerate continua X and Y is a C-continuum.*

Proof. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be a triple of points of the product $X \times Y$. Now we have $x_2 \neq x_3$ or $y_2 \neq y_3$. We will give the proof in the case $x_2 \neq x_3$ since the proof in the case $y_2 \neq y_3$ is similar. Now we have two disjoint continua $Y_2 = \{(x_2, y) : y \in Y\}$ and $Y_3 = \{(x_3, y) : y \in Y\}$. If $(x_1, y_1) \in Y_2$ or $(x_1, y_1) \in Y_3$, the proof is completed. Assume that $(x_1, y_1) \notin Y_2$ and $(x_1, y_1) \notin Y_3$. Consider the continua $X_2 = \{(x, y_2) : x \in X\}$ and $X_3 = \{(x, y_3) : x \in X\}$. The continuum $Y_1 = \{(x_1, y) : y \in Y\}$ contains a point (x_1, p) such that $(x_1, p) \notin X_2 \cup X_3$. Let $X_p = \{(x, p) : x \in X\}$. It is clear that a continuum $Y_1 \cup X_p \cup Y_2$ contains the points (x_1, y_1) and (x_2, y_2) but not (x_3, y_3). Similarly, a continuum $Y_1 \cup X_p \cup Y_3$ contains the points (x_1, y_1) and (x_3, y_3) but not (x_2, y_2). The proof is completed. ∎

LEMMA 5.16. [20, Theorem 1, p. 326] *If the continuum X is aposyndetic, then X is a C-continuum.*

REMARK 1. There exists a C-continuum which is not aposyndetic [20, p. 327].

REMARK 2. There exists a C-continuum which is not arcwise connected [20, p. 328].

A continuum X is said to be *colocally connected* provided that for each point $x \in X$ and each open $U \ni x$ there exists an open set V containing x such that $V \subset U$ and $X \setminus V$ is connected.

LEMMA 5.17. *Each colocally connected continuum X is a C-continuum.*

Proof. Let x, y, z be a triple of points of X. Now, $U = X \setminus \{x, y\}$ is an open set U such that $z \in U$. From the colocal connectedness of X it follows that there exists an open set V such that $z \in V \subset U$ and $X \setminus V$ is connected. Hence, X is a C-continuum since the continuum $X \setminus V$ contains the points x and y. ∎

Now we shall prove the main theorem of this subsection.

THEOREM 5.18. *If X is a C-continuum, then it is a D-continuum.*

Proof. Let us prove that for every pair C^*, D^* of disjoint non-degenerate subcontinua of X there exists a non-degenerate subcontinuum $E \subset X$ such that $C^* \cap E \neq \emptyset \neq D^* \cap E$ and $(C^* \cup D^*) \setminus E \neq \emptyset$. Let $x \in C^*$ and $y, z \in D^*$. There exists a continuum E such that either $x, y \in E, z \in X \setminus E$ or $x, z \in E, y \in X \setminus E$, since X is a C-continuum. We assume that $x, y \in E$, and $z \in X \setminus E$. It is clear that $C^* \cap E \neq \emptyset \neq D^* \cap E$ and $(C^* \cup D^*) \setminus E \neq \emptyset$ since $x \in C^* \cap E$, $y \in D^* \cap E$ and $z \in (C^* \cup D^*) \setminus E$. ∎

THEOREM 5.19. *A C-continuum X with $\mathrm{w}(X) > \lambda$ admits no Whitney map $\mu : C(X) \to J$.*

THEOREM 5.20. *A C-continuum X admits a Whitney map $\mu : C(X) \to [0, 1]$ if and only if it is metrizable.*

COROLLARY 5.21. *Let X be an (arcwise connected, aposyndetic, colocally connected or cartesian product $Y \times Z$ of continua) continuum such that $w(X) > \lambda$. Then X admits no λ-generalized Whitney map $\mu_\lambda : C(X) \to J$.*

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