ON THE CONTACT AND CO-CONTACT OF HIGHER ORDER

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Abstract. The unifying methodologies are based on the construction of 'bridges' connecting distinct mathematical theories with each other. The purpose of this paper is to study the relationship between the geometric and algebraic formulation of completely integrable systems of order k and dimension n over a differentiable manifold, in terms of contact $C^{k,n}M$ and co-contact $(C^{k,n}M)^0$ of higher order, as seen in [A. Morimoto, Prolongation of Geometric Structures, Math. Inst. Nagoya University, Nagoya, (1969)], to establish an equivalence between both formulations.

1. Introduction

In this article we establish equivalence between the notions of completely integrable systems of higher order, given in [6], in the geometric and algebraic formulation, respectively.

Moreover, we clarify the concept of co-contact and the ideal associated with a differential system $(W^k)^\circ$ on co-contact manifold and we show some examples.

Let M be a smooth m-dimensional manifold and $C^{k,n}M$, $n \leq m$, the manifold of contact elements of order k and dimension n over M [8]. Given an n-submanifold $S \subset M$, we denote by $C_x^k S$ a contact element of order k of S at $x \in S$ and by ρ_0^k the canonical projection over M. We put $C^{0,n}M = M$. There exists a natural immersion

$$i^{1,k} : C^{k+1,n}M \to C^{1,n}(C^{k,n}M), \quad C_x^{k+1}S \mapsto C_{C_x^k S}^1 C^k S$$

where $C^k S \subset C^{k,n}M$ denotes the n-submanifold in $C^{k,n}M$, defined by the immersion $i^k : x \in S \subset M \mapsto C_x^k S \in C^{k,n}M$.

Let $J_0^k(\mathbb{R}^p, M)$ be the tangent bundle of p^k-velocities, i.e., the k-jets bundles of curves mappings from $\mathbb{R}^p$ to M with source $0 \in \mathbb{R}^p$. If $p = k = 1$, we have the tangent bundle TM, and if $p = 1, k = 2$, we have the second tangent bundle T^2M, and if $p = 1, k > 2$, we have the k-tangent bundle $T^k M$.

Moreover, $J_0^k(\mathbb{R}^p, M)$ can be immersed in $T(T^{k-1}M)$ as the invariant set of the canonical involution α [3, 4] and thus $T^k M \equiv T(T^{k-1}M)$.

We recall [4] that two n-dimensional submanifolds $S_1, S_2 \subset M$ have co-contact of order k and codimension n in $x \in S_1 \cap S_2$ if the annihilator of the r-th order tangent vector at $x \in S_1$, denoted by $(T_x^r S_1)^0$, coincides with the annihilator of the r-th order tangent vector at $x \in S_2$, where r is an arbitrary integer with $0 \leq r \leq k$.

The equivalence class of co-contact elements of an n-dimensional submanifold S is called the co-contact element of order k and codimension n at x and denoted by $(C_x^k S)^0$. The set $(C^{k,n}M)^0$ of all co-contact elements of order k and codimension n on M has a differentiable manifold structure induced by the local bijections between coordinate neighborhoods of the manifold $C^{k,n}M$.

Denote by $(\rho_r^k)^0 : (C_x^k S)^0 \in (C^{k,n}M)^0 \mapsto (C_x^r S)^0 \in (C^{r,n}M)^0$, the canonical projection, with $0 \leq r \leq k$ and $(C^{0,n}M)^0 = M$. By *a differential system of order k and dimension n in $(C^{k,n}M)^0$* we mean an imbedded submanifold $(W^k)^0 \subset (C^{k,n}M)^0$. *The solution of a differential system $(W^k)^0$ in a point $(X_0^k)^0 \in (W^k)^0$* is an n-dimensional imbedded submanifold $S \subset M$, with $x_0 = (\rho_0^k)^0((X_0^k)^0) \in S$, such that $(C_x^k S)^0 \in (W^k)^0$, $x \in S$ and $(C_{x_0}^k S)^0 = (X_0^k)^0$.

The differential system $(W^k)^0$, of order k and dimension n in $(C^{k,n}M)^0$ is completely integrable in a given point $(X_0^k)^0 \in (W^k)^0$, if there exists a solution $S \subset M$ at $(X_0^k)^0$.

If $(\rho_{k-1}^k)^0$ is an immersion in a neighborhood $(U^k)^0$ of $(X^k)^0$, then it was proved in [5] that there exists an ideal defined in an open subset $(U^{k-1})^0$ of $(W^{k-1})^0 = (\rho_{k-1}^k)^0(W^k)^0$, such that this ideal is locally generated by independent 1-forms in $(U^{k-1})^0$. The ideal associated with the differential system $(W^k)^0$ is denoted by $\mathcal{Y}(W^k)^0$.

In this paper we clarify the concept of co-contact and the ideal associated to the differential system $(W^k)^0$ and we show some examples. Moreover, we show that necessaries conditions of integrability over $(W^k)^0$ and W^k are "equivalents". In Sections 2 and 3, we recall briefly, as seen in [4, 5], the geometric formulation (contact theory) and the algebraic formulation (co-contact theory) of differential systems completely integrable but we clarify the concept of co-contact and the ideal associated with the differential system $(W^k)^0$.

In Sections 4 and 5, some preliminary notions are introduced and studied and the main theorems (Theorem 4.3, Theorem 5.3) are formulated and proved in terms of differential systems and in terms of ideals respectively.

2. Contact theory and integrable systems

Let (M, N, ρ) be a fibred manifold with total space M, base M, projection ρ, $\dim M = m$, $\dim N = n$, $x_0 \in N$ and k an integer, $k \geq 0$. Two sections f, g whose domains contain x_0 are k-equivalent at x_0 if there is some fibred chart $x = (x_i, x^j)$ in x_0, with $1 \leq i \leq n$, $n+1 \leq j \leq m$, such that the r-partial derivatives, $0 \leq r \leq k$, of f and g coincide at x_0, i.e., $\partial^r f^j(x_0) = \partial^r g^j(x_0)$ where $f^j = x^j \circ f$ (resp. $g^j = x^j \circ g$) is the expression of f (resp. of g) in terms of (x_i, x^j).

The equivalence class of f, denoted by $j_{x_0}^k f$, is called a k-jet of f at x_0. We denote by $J^k M$ the differentiable manifold of all k-jets of sections of (M, N, ρ) and by $J_{x_0}^k M$ the set of k-jets at x_0. We shall define the jets of curves through x_0, by which we henceforth mean smooth functions $f : \mathbb{R} \to M$ such that $f(0) = x_0$.

Let f and g be a pair of curves through x_0. We will then say that f and g are equivalent to order k at x_0 if there is some neighborhood U of x_0, such that, for every smooth function $\varphi : U \to \mathbb{R}$, $j_{x_0}^k(\varphi \circ f) = j_{x_0}^k(\varphi \circ g)$. This equivalence relation is sometimes called the k-th order contact between curves at x_0.

We now define the k-jet of a curve f through x_0 to be the equivalence class of f. The k-th order jet space $J_0^k(\mathbb{R}, M)_{x_0}$ is then the set of k-jets of curves at x_0. As x_0 varies over M, $J_0^k(\mathbb{R}, M)_{x_0}$ forms a fibre bundle over M and the k-th order tangent bundle, often denoted in the literature by $T^k M$, the k-th order tangent vectors on M.

Two k-jets, $j_{x_0}^k f, j_{x_0}^k g \in J_{x_0}^k M$ are equivalent at x_0 if there exists an invertible k-jet $j_{x_0}^k h \in J_{x_0}^k N$, with $h : N \to N$, such that $j_{x_0}^k f = j_{x_0}^k g \circ j_{x_0}^k h = j_{x_0}^k g \circ h$. An equivalence class is called the contact element of order k and dimension n at $f(x_0) \in M$. The set of all contact elements of order k at x is denoted by $C_x^{k,n} M$, and $\underline{C}^{k,n} M$ denotes all the k-contact elements of sections of (M, N, ρ).

Let $S \subset M$ be an imbedded n-submanifold, then the tangent space $T_x S$ in $x \in S$ defines a local fibration (V, U, ρ) at $x \in M$ such that S is the images of a cross section f and the fibred chart $(V, x = (x_i, x^j))$ satisfies $x_i \circ f = \xi_i$, where ξ_i are the canonical coordinates in $\mathbb{R}^n$.

A coordinate neighborhood at $X^k = C_x^k S$ is given by

$$(\mathcal{V}_{X^k} = \underline{C}^k V, (x_i, x^j, p_{I_r}^j)), \quad p_{I_r}^j(C_x^k g(U)) = \frac{\partial^r}{\partial x_{i_1} \ldots \partial x_{i_r}} g^j(x),$$

where $I_r = (i_1, \ldots, i_r)$ is an ordered r-uple of integers $\{1, \ldots, n\}$. The set $\mathcal{V}_{X^k}$ of all k-contact transversal manifolds of (V, U, ρ) is called the chart associated with $X^k = C_x^k S$.

The manifold structure on $C^{k,n} M$ is given by all coordinate neighborhoods $(\mathcal{V}_{X^k}, (x_i, x^j, p_{I_r}^j))$, $X^k \in C^{k,n} M$. With this differential structure the natural projections,

$$\rho_0^k : C_x^k S \in C^{k,n} M \mapsto x \in M, \quad \rho_r^k : C_x^k S \mapsto C_x^r S \in C^{r,n} M$$

are submersions, and the natural injections

$$i^k : x \in S \mapsto C_x^k S, \quad i^{1,k} : C_x^{k+1} S \mapsto C_{C_x^k S}^1(C^k S) \in C^{1,n}(C^{k,n} M)$$

are immersions, moreover, the natural immersion $i^{1,k}$ makes it possible to identify

$$C_x^{k+1} S \simeq C_{C_x^k S}^1(C^k S) \simeq T_{C_x^k S}(C^k S).$$

By a *differential system of order k and dimension n* in $C^{k,n} M$, we mean an imbedded submanifold $W^k \subset C^{k,n} M$. A *solution of a differential system W^k at*

$X^k \in W^k$, is an n-dimensional imbedded submanifold $S \subset M$, with $x = \rho_0^k(X^k) \in S$, such that $C^k S \subset W^k$ and $C_x^k S = X^k$. A system W^k is *completely integrable* if for each $X^k \in W^k$ there exists a solution $S \subset M$ passing through X^k.

The first prolongation of a submanifold $W^k \subset C^{k,n}M$ is defined by Olver [7] as $PW^k = C^{1,n}W^k \cap C^{k+1,n}M$, where $C^{k+1,n}M$ is identified with its image by $i^{1,k}$ in $C^{1,n}(C^{k,n}M)$.

THEOREM 2.1. [4] *Let $W^k \subset C^{k,n}M$ be an imbedded submanifold and $X^k \in W^k$ such that the following conditions are satisfied:*

1) $\rho_{k-1}^k : W^k \to C^{k-1,n}M$ *is a local immersion in a neighborhood of $X^k \in W^k$;*

2) $\rho_k^{k+1} : PW^k \to W^k$ *is a local submersion in a neighborhood of X^k.*

Then W^k is completely integrable, that is, there exists a solution $S \subset M$ of the differential system W^k passing through X^k. Moreover, if $\tilde{S}$ is another submanifold of W^k passing through X^k, then there exists an open set $A \subset M$, $x = \rho_0^k(X^k) \in A$, such that $S \cap A = \tilde{S} \cap A$.

Let $\tilde{C}^{1,n}(C^{k,n}M)$ be the manifold of all contact elements of order 1 and dimension n, of n-submanifolds $\mathcal{S} \subset C^{k,n}M$, such that the natural projection $\rho_0^{1,k}$ restricted to $\mathcal{S}$ over M, has a maximal rank. The manifold $\tilde{C}^{1,n}(C^{k,n}M)$ is embedded in $C^{1,n}(C^{k,n}M)$, and the natural immersion

$$i^{1,k} : C^{k+1,n}M \to \tilde{C}^{1,n}(C^{k,n}M), \quad C_x^{k+1}S \mapsto C_{C_x^k S}^1 C^k S \qquad (2.1)$$

is an embedding.

3. Co-contact manifolds and higher order differential systems

Let $S \subset M$ be an embedded n-dimensional submanifold of M and $f : \mathbb{R}^n \to M$ be a local parametrization of S and a maximal rank. Given an n dimensional subspace $T_x S \subset T_x M$ we will denote by $(T_x S)^0$ the set of one-forms $\omega \in \Omega^1(C^{0,n}M)$ on the contact bundle $C^{0,n}M = M$ of order $k = 0$, which annihilate $T_x S$, i.e., $i^*\omega = 0$, where $i : S = im(f) \hookrightarrow C^{0,n}M$ is the natural inclusion map.

Now we will denote by $(T_x^2 S)^0$ the set of one-forms $\omega \in \Omega^1(C^{1,n}M)$ on the contact bundle $C^{1,n}M$ of order $k = 1$, which annihilate $T_x^2 S$, i.e., $i^*\omega = 0$, where the map $i : C^1 S \hookrightarrow C^{1,n}M$ is an embedding. At each point $x \in S$ the annihilator of tangent space of $(T_x^k S)$ denoted by $(T_x^k S)^0$ is the set of one-forms $\omega \in \Omega^1(C^{k-1,n}M)$ on the contact bundle $C^{k-1,n}M$ which annihilate $T_x^k S$, i.e., $i^*\omega = 0$, where $i : C^{k-1}S \hookrightarrow C^{k-1,n}M$ is the natural inclusion map.

We say that *two n-dimensional submanifolds S_1, S_2 have co-contact of order k and codimension n in $x \in S_1 \cap S_2$ if the annihilator of the r-th order tangent vector at $x \in S_1$ coincides with the annihilator of the r-th order tangent vector at $x \in S_2$*, i.e., $(T_x^r S_1)^0 = (T_x^r S_2)^0$ where r is an arbitrary integer with $0 \leq r \leq k$. The equivalence class of co-contact elements of an n-dimensional submanifold S is called the *co-contact element* of order k and codimension n at x and denoted by $(C_x^k S)^0$.

Let us also denote by $(C^{k,n}M)^0$ the set of all co-contact elements of order k and codimension n over M, and put $(C^{0,n}M)^0 = M$. Let $(X^k)^0 = (C_x^k S)^0$ be a co-contact element in $(C^{k,n}M)^0$, and consider a local coordinate system $(V, \varphi = (x_i, x^j))$, $1 \leq i \leq n$, $n+1 \leq j \leq m$, at $x \in M$, such that $\{dx_i\}$ generates $T_x^* S$.

Consider (V, U, ρ), a local fibration of V associated to S and $\mathcal{V}^k$, $(\mathcal{V}^k)^0$ open set defined as

$$\mathcal{V}^k = \underline{C}^k V, \qquad (\mathcal{V}^k)^0 = (\underline{C}^k V)^0.$$

Then the map

$$\Psi_V^k : \mathcal{V}^k \to (\mathcal{V}^k)^0, \qquad C_x^k g(U) \mapsto (C_x^k g(U))^0 \tag{3.1}$$

is a bijection.

Moreover, if $Y^k = C_x^k g(U)$ is defined in coordinates as $Y^k = (x_i, x^j, p_{I_r}^j)$ with $1 \leq r \leq k$, then $(Y^k)^0$ is generated by one-forms defined as follows

$$\omega_{I_r}^j = dp_{I_r}^j - \sum_{i=1}^n p_{I_r,i}^j dx_i, \quad n+1 \leq j \leq m, \quad 0 \leq r \leq k-1,$$

where $dp_{I_0}^j = dx^j$, and I_r, i denotes the ordered $(r+1)$-uple of integers $\{1, \ldots, n\}$ given by the set $\{i_1, \ldots, i_r, i\}$. These one forms, are contact forms on $(Y^k)^0$, i.e., $i^* \omega = 0$ and the map Ψ_V^k allows us to define a differential structure on $(C^{k,n}M)^0$.

A coordinate neighborhood at $(X^k)^0 = (C_x^k S)^0$ is given by $((\mathcal{V}^k)^0, x_i, x^j, (p_{I_r}^j)^0)$, where $(p_{I_r}^j)^0 (Y^k)^0 = \omega_{I_r}^j ((Y^{k-1})(v_{Y^{k-1}}))$ with $(v_{Y^{k-1}})$ basis of $T_{Y^{k-1}}(C^{k-1}M)$. With this differential structure the natural injection, $(i^k)^0 : S \hookrightarrow (C^{k,n}M)^0$ is given in coordinates as

$$(i^k)^0 (x_i, f^j(x_i)) = (x_i, f^j(x_i), (p_{I_r}^j)^0 = \omega_{I_r}^j = dp_{I_r}^j - \sum_{i=1}^n \frac{\partial^{r+1} f^j}{\partial x_{I_r,i}} dx_i)$$

where $0 \leq r \leq k-1$ and $f(U) = S$. Moreover, $(i^k)^0$ is an imbedding and $(i^k)^0(S)$, denoted by $(C^k S)^0$, is a regular submanifold of dimension n. The natural projection $(\rho_r^k)^0 : (C^{k,n}M)^0 \to (C^{r,n}M)^0$ with $0 \leq r < k$ is a submersion.

Let $(\tilde{C}^{1,n}(C^{k,n}M)^0)^0$ be the manifold of all co-contact elements of order 1 and codimension n, of n-submanifolds $\mathcal{S}^0 \subset (C^{k,n}M)^0$, such that the natural projection $(\rho_0^{1,k})^0$ restricted to $\mathcal{S}^0$ over M, has a maximal rank.

REMARKS 3.3. 1. Let $(X^{k+1})^0 = (C_x^{k+1} S)^0 \in (C^{k+1,n}M)^0$. Then we have the natural identification

$$T_{(C_x^k S)^0}^0 (C^k S)^0 \simeq (i^{1,k})^0 (C_x^{k+1} S)^0 \in (\tilde{C}_{(C_x^k S)^0}^{1,n} (C^{k,n}M)^0)^0.$$

2. Note that generally $p_{\{I_{k-1}, i_1\}, i}^j \neq p_{\{I_{k-1}, i\}, i_1}^j$.

3. Let $i^{1,k}$ be the natural immersion given in (2.1) and $\Psi_V^{1,k}$ be the map defined

as $C^1_{X^k} \mathcal{S}^k \mapsto (C^1_{(X^k)^0}(\mathcal{S}^k)^0)^0$. Then the following diagram

$$
\begin{array}{ccc}
(C^{k+1,n}V)^0 & \xrightarrow{\;(i^{1,k})^0\;} & (C^{1,n}(C^{k,n}V)^0)^0 \\[2mm]
\Psi^{k+1}_V \uparrow & & \uparrow \Psi^{1,k}_V \\[2mm]
C^{k+1,n}V & \xrightarrow{\quad i^{1,k}\quad} & C^{1,n}C^{k,n}V
\end{array}
$$

is commutative.

4. Any section local $F : U \subset \mathbb{R} \to V \subset M$ can be prolonged to a smooth map $(C^k)^0 F : \{x \mapsto (C^k_{F(x)}S)^0 = dF^j_{I_r} - \sum_{i=1}^n \frac{\partial^{r+1}F^j}{\partial x_{I_{r,i}}} dx_i\}$ with $F^j_{I_r} = \frac{\partial^{r+1}F^j}{\partial x_{I_{r,i}}}$ and where $0 \leq r \leq k-1$, $F(U) = S$. Note that $(C^k)^0 F$ satisfies $(\rho^k_0)^0 o (C^k)^0 F = id_U$.

DEFINITION 3.4. By a differential system of order k and dimension n in $(C^{k,n}M)^0$ we mean an imbedded submanifold $(W^k)^0 \subset (C^{k,n}M)^0$. A solution of a differential system $(W^k)^0$ at $(X^k)^0 \in (W^k)^0$ is an n-dimensional imbedded submanifold $S \subset M$, with $x = (\rho^k_0)^0((X^k)^0) \in S$, such that

$$(i^k)^0(S) = (C^k S)^0 \subset (W^k)^0 \quad \text{and} \quad (C^k_x S)^0 = (X^k)^0.$$

EXAMPLE 3.5. Let (M^m, N^n, ρ) be a fiber manifold with local coordinates at $x \in M$ (x_i, x^j) and $(C^{k,n}M)^0$ the k order co-contact manifold. If $((\mathcal{V}^k)^0, (\mathcal{V}^{k-2})^0, (\rho^k_{k-2})^0)$ is a local fibration and $\Phi : (\mathcal{V}^{k-2})^0 \to (\mathcal{V}^k)^0$ is a local section of the special form $(C^1)^0 F$ with $F : (\mathcal{V}^{k-2})^0 \to (\mathcal{V}^{k-1})^0$, then $\Phi(\mathcal{V}^{k-2})^0$ is a subspace of $T^*(C^{k-1,n}M)$ generated by one-forms

$$dF^j - \sum_{i=1}^n \frac{\partial F^j}{\partial x_i} dx_i.$$

DEFINITION 3.6. The first prolongation of an n-submanifold $(W^k)^0 \subset (C^{k,n}M)^0$ is defined as

$$P(W^k)^0 = (C^{1,n}(W^k)^0)^0 \cap (C^{k+1,n}M)^0,$$

where $(C^{k+1,n}M)^0$ is identified with its image by $(i^{1,k})^0$ in $(\tilde{C}^{1,n}(C^{k,n}M)^0)^0$.

PROPOSITION 3.7. [4] Let $(W^k)^0 \subset (C^{k,n}M)^0$ be an imbedded submanifold, and let $(X^k)^0 \in (W^k)^0$, (V, U, ρ) be a local fibration at $x = \rho^k_0(X^k)$ associated with $(X^k)^0$. Suppose that $(\rho \circ (\rho^k_0)^0)|_{(W^k)^0}$ is a local submersion on U. Then

$$
(C^{k+1}_x S)^0 \in P(W^k)^0 \iff (T_{(C^k_x S)^0}(C^k S)^0)^0 \in (T_{(C^k_x S)^0}(W^k)^0)^0,
$$
$$
(PW^k)^0 \cap (\underline{C}^{k+1}V)^0 = (\underline{C}^{k+1}V)^0 \cap (C^{1,n}(W^k)^0)^0.
$$

PROPOSITION 3.8. [4] *Let $W^k \subset C^{k,n}M$ be an imbedded submanifold passing through X^k and $(W^k)^0 \subset (C^{k,n}M)^0$ be its annihilator. Let (V, U, ρ) be a local fibration at $x = \rho_0^k(X^k)$ associated with $(X^k)^0$ and $\Psi_V^{1,k}$ the map defined as*

$$C_{X^k}^1 \mathcal{S}^k \in C^{1,n}C^{k,n}V \mapsto (C_{(X^k)^0}^1 (\mathcal{S}^k)^0)^0.$$

Then $Z \in PW^k$ if and only if $\Psi_V^{1,k}(Z) \in P(W^k)^0$.

4. On the equivalence problem and integrable systems

Let $W^k \subset C^{k,n}M$ be a differential system and $(X^k) \in W^k$ be a contact element, (V, U, ρ) be a fiber manifold associated and $(X^k)^0 = (\Psi_V^k)^{-1}((X^k))$, where Ψ_V^k is given in (3.1). Let $((\mathcal{V}^k)^0, (p_{I_s}^j)^0)$, $(\mathcal{V}^k, p_{I_s}^j)$ be coordinate systems at $(X^k)^0$ and X^k respectively and let $(F^1), \ldots, (F^s)$ be smooth functions defined in a neighborhood of X^k such that

$$(\mathcal{V}^k) \cap W^k = \{(Y^k) \in (\mathcal{V}^k) : (F^1)((Y^k)) = \cdots = (F^s)((Y^k)) = 0\}.$$

DEFINITION 4.1. The differential system $(W^k)^0$ generated by $(F^j)^0 = F^j \circ (\Psi_V^k)^{-1}$ in a neighborhood $(\mathcal{V}^k)^0$ of $(X^k)^0$ is called the associated differential system at W^k.

DEFINITION 4.2. Let $W^k \subset C^{k,n}M$ be a differential system. We say that the system W^k is completely integrable, if the following conditions are satisfied:

1) The map $\rho_{k-1}^k : W^k \longrightarrow C^{k-1,n}M$ is a local immersion in a neighborhood of $(X^k) \in W^k$.

2) The map $\rho_k^{k+1} : P(W^k) \longrightarrow W^k$ is a local submersion in a neighborhood of X^k.

THEOREM 4.3. *Let W^k be a differential system in $\mathcal{V}^k$ and $(W^k)^0$ be the associated differential system in $(\mathcal{V}^k)^0$. Then $(W^k)^0$ is completely integrable if and only if W^k is completely integrable.*

Proof. Let $(X^k) = (C_x^k S)$ be a co-contact element, and let (V, U, ρ) be a fiber manifold associated, $(X^k)^0 = (\Psi_V^k)((X^k))$, where Ψ_V^k is given in (3.1). Let $((\mathcal{V}^k)^0, (p_{I_s}^j)^0)$, $(\mathcal{V}^k, p_{I_s}^j)$ be coordinate systems at $(X^k)^0$ and X^k respectively, and $(F^1), \ldots, (F^s)$ be smooth functions defined in a neighborhood of (X^k) such that

$$(\mathcal{V}^k) \cap W^k = \{(Y^k) \in (\mathcal{V}^k) : (F^1)((Y^k)) = \cdots = (F^s)((Y^k)) = 0\}.$$

Let

$$\underline{C}^k V = \mathcal{V}^k, \quad F^j = (F^j)^0 \circ \Psi_V^k,$$

and the differential system $(W^k)^0$ of order k in $(\mathcal{V}^k)^0$ be generated by $(F^j)^0$. We shall verify that $(W^k)^0$ satisfies the conditions of Definition 4.2.

Consider the commutative diagram, given in Proposition 3.2.

$$
\begin{array}{ccc}
\mathcal{V}^{k+1} & \xrightarrow{\Psi_V^{k+1}} & (\mathcal{V}^{k+1})^0 \\
\rho_k^{k+1} \downarrow & & \downarrow (\rho_k^{k+1})^0 \\
\mathcal{V}^k & \xrightarrow[\Psi_V^k]{} & (\mathcal{V}^k)^0 \\
\rho_{k-1}^k \downarrow & & \downarrow (\rho_{k-1}^k)^0 \\
\mathcal{V}^{k-1} & \xrightarrow[\Psi_V^{k-1}]{} & (\mathcal{V}^{k-1})^0
\end{array}
$$

Now, as by hypothesis $(\rho_{k-1}^k) : W^k \longrightarrow (C^{k-1,n}M)$ is a local immersion, then

$$
(\rho_{k-1}^k)^0 : (W^k)^0 \cap (\mathcal{V}^k)^0 \longrightarrow (C^{k-1,n}M)^0
$$

is also a local immersion, and Condition 1 of Definition 4.2 is verified.

To verify that $(\rho_k^{k+1})^0 : P(W^k)^0 \longrightarrow (W^k)^0$ is a local submersion we consider the commutative diagram given in Remark 3 of 3.3,

$$
\begin{array}{ccc}
(C^{k+1,n}V)^0 & \xrightarrow{(i^{1,k})^0} & (C^{1,n}(C^{k,n}V)^0)^0 \\
\Psi_V^{k+1} \simeq \uparrow & & \uparrow \simeq \Psi_V^{1,k} \\
C^{k+1,n}V & \xrightarrow[i^{1,k}]{} & C^{1,n}C^{k,n}V
\end{array}
$$

where

$$
(i^{1,k}) : C_x^{k+1}S \in C^{k+1}V \longmapsto C_{C_x^k S}^1(C^k S) \in C^1(C^k V)
$$

and $(i^{1,k})^0$ are immersions and the map

$$
\psi_V^{1,k} : C_{X^k}^1 \mathcal{S}^k \mapsto (C_{(X^k)^0}^{1,n}(\mathcal{S}^k)^0))^0
$$

is as seen in (3.1). Using Proposition 3.8 we have

$$
Z \in PW^k \iff \Psi_V^{1,k}(Z) \in (C^{1,n}(W^k)^0)^0 \cap (C^{k+1,n}V)^0 = P((W^k)^0),
$$

and hence $\Psi_V^{1,k}(PW^k) = P((W^k)^0)$.

Now, by hypothesis, $(\rho_k^{k+1}) : P(W^k) \longrightarrow W^k$ is a local submersion and

$$
\Psi_V^k \circ \rho_k^{k+1} = (\rho_k^{k+1})^0 \circ \Psi_V^{k+1}.
$$

Consequently,

$$
(\rho_k^{k+1})^o : P(W^k)^0 \longrightarrow (W^k)^0
$$

is a local submersion, and Condition 2 of Definition 4.2 is satisfied. Hence the differential system W^k is completely integrable.

The proof of the converse is similar. ∎

5. On the equivalence problem in terms of ideals

If M is a smooth n-dimensional manifold then we denote by d_k the dimension of $(C^{k,n}M)^0$ and by $\mathcal{F}_k^0(M)$ the algebra of smooth functions on $(C^{k,n}M)^0$. The Lie algebra over $\mathbb{R}$ with respect to the commutator product $[X, Y]$ in the set of all vector fields on $(C^{k,n}M)^0$ is denoted by $\Xi^k(M)$.

An ideal $\mathcal{Y}$ in $E^*(M)$ is called a differential ideal if it is closed under exterior differentiation, that is $d\mathcal{Y} \subset \mathcal{Y}$.

Let $\Omega^i(C^{k,n}M)^0$ be the algebra of all smooth i-forms on $(C^{k,n}M)^0$. The set of all differential forms on $(C^{k,n}M)^0$ is denoted by $\Omega^\star(C^{k,n}M)^0$ and it has the structure of a module over the ring $\mathcal{F}_k^0$ with respect to the operation of wedge product.

Consider an n-dimensional distribution

$$\mathcal{T}^k : (C^{k,n}M)^0 \ni \theta^k \mapsto \mathcal{T}_{\theta^k}^k \subset T_{\theta^k}(C^{k,n}M)^0.$$

Denote by $\mathcal{T}\Xi^k(M) \subset \Xi^k(M)$ the submodule of vector fields lying in $\mathcal{T}^k$ and

$$\mathcal{T}_k^1 = \{\omega \in \Omega^1(C^{k,n}M)^0 \mid i_X\omega = 0, X \in \mathcal{T}\Xi^k(M)\},$$

where i_X is the inner product.

PROPOSITION 5.1. [5] *If $(\rho_{k-1}^k)^0$ is an immersion in a neighborhood $(U^k)^0$ of $(X^k)^0$ then there exists an ideal denoted by $\mathcal{Y}(W^k)^0$ and defined in an open subset $(U^{k-1})^0$ of $(W^{k-1})^0 = (\rho_{k-1}^k)^0(W^k)^0$, such that it is an ideal locally generated by independent 1-forms in $(U^{k-1})^0$.*

The ideal $\mathcal{Y}(W^k)^0$ is called the ideal associated with the differential system $(W^k)^0$.

Furthermore, we have the following theorem.

THEOREM 5.2. [5] *The projection $(\rho_k^{k+1})^0 : P(W^k)^0 \to (W^k)^0$ is a local submersion on a neighborhood of $(X^k)^0 \in (W^k)^0$ if and only if $\mathcal{Y}^k(W^k)^0$ is a differential ideal.*

Now, the problem of equivalence in terms of ideals is terated in the following

THEOREM 5.3. *Let W^k be a differential system in $\mathcal{V}^k$, $X^k \in W^k$ and $(W^k)^0$ be the associated differential system $(\mathcal{V}^k)^0$ with $(X^k)^0 = \Psi_{(\mathcal{V}^k)^0}(X^k) \in (W^k)^0$. Suppose $(W^k)^0$ is transverse to the fiber $\mathcal{F}_{X^k}$. Then $\mathcal{Y}^k(W^k)^0$ is a differential ideal if and only if $\mathcal{Y}^k(W^k)$ is a differential ideal.*

Proof. Let $(X^k)^0 \in (W^k)^0$. As $(W^k)^0$ is transverse to the fiber $\mathcal{F}_{X^k}$ there exist an open subset $(U^k)^0$ of $(X^k)^0$ and function $F_{I_r}^j$ defined on neighborhood $(U^{k-1})^0$ of $(\rho_{k-1}^k)^0(X^k)^0$ such that

$$(p_{I_r}^j)^0 = F_{I_r}^j(x_i, x^j, (p_{I_{r-1}}^j)^0).$$

Now if $\mathcal{Y}^k(W^k)^0$ is a differential ideal, then we have by Theorem 5.2 that the map $(\rho_k^{k+1})^0 : P(W^k)^0 \longrightarrow (W^k)^0$ is a local submersion in a neighborhood of $(X^k)^0$. Moreover, the ideal $\mathcal{Y}^k(W^k)^0$ is an ideal locally generated by one-forms

$$\omega_{I_r}^j = dp_{I_r}^j - \sum_{i=1}^{n} F_{I_{r-1},i}^j dx_i, \qquad 1 \leq r \leq k-1$$

and by its exterior derivative $d\omega_{I_r}^j$ on $(U^{k-1})^0 = (\rho_{k-1}^k)^0(U^k)^0$.

This one-form and its exterior derivative are annihilates by the n-distribution:

$$\mathcal{T}^{k-1} = \{L_{(X^{k-1})^0} \in T_{(X^{k-1})^0}(C^{k-1,n}M)^0 \mid (X^{k-1})^0 \in (U^{k-1})^0, \omega_{I_r}^j((X^{k-1})^0) = 0\}$$

where $dp_{I_0}^j = dx^j$, $I_{r,i}$ denotes the ordered $r+1$-uple of integers $\{1,\ldots,n\}$ given by the set $\{i_1,\ldots,i_r,i\}$, and $\sigma = \left((\rho_{k-1}^k)^0\right)^{-1}|_{(U^{k-1})^0}$ is the section of the fiber bundle $((U^k)^0, (U^{k-1})^0, (\rho_{k-1}^k)^0)$ defined by

$$\sigma((x_i, x^j, (p_{I_r}^j)^0)) = (x_i, x^j, (p_{I_r}^j)^0, F_{I_r}^j(x_i, x^j, (p_{I_r}^j)^0))$$

Thus, $(\rho_{k-1}^k)^0$ is an immersion on $(U^{k-1})^0$, and hence $(W^k)^0$ is completely integrable.

Now, using Theorem 4.3, Proposition 5.1 and Theorem 5.2, we have the proof of the theorem. ∎

REMARK 5.4. Let $(W^k)^0$ be a differential system in $(\mathcal{V}^k)^0$ and W^k the associated differential system in $\mathcal{V}^k$ determined by $\Psi_{\mathcal{V}}^k$ and $i : W_k \hookrightarrow (C^{k,n}M)$ the embedding.

Now, consider the pull-back $i^*(\omega_{I_r}^j)$, where $\omega_{I_r}^j$ are one-forms of contact on $C^{k,n}M$. Then the differential ideal $\mathcal{Y}^k(W^k)^0$, can be considered as the ideal locally generated by $i^*(\omega_{I_r}^j)$ with the diffeomorphism $\Psi_{\mathcal{V}}^k$, i.e., $(\omega_{I_r}^j) \equiv d\Psi_{\mathcal{V}}^k \circ (\omega_{I_r}^j)$.

EXAMPLE 5.5. Let $(W^1)^0 \subset (C^{1,2}M)^0$ be the differential system determined by the one form

$$\omega = dy - (y_x dx - \Phi dt) \in \Omega^1(M),$$

where Φ defines the differential system W^1 associated at $(W^1)^0$, i.e.,

$$W^1 = \{y_t = \Phi(x, t, y, y_x)\} \subset (C^{1,2}M).$$

The differential ideal $\mathcal{Y}^1(W^1)^0 \subset \Omega(W_1)$ is generated by the one form ω and its exterior derivative

$$d\omega = dy_x \wedge dx - \Phi_y dy \wedge dt - \Phi_{y_x} dy_x \wedge dt.$$

Suppose that $(\rho_0^1)^o : (C^{1,2}(W^1)^0)^0 \to ((W^1)^0)$ is a local submersion on a neighborhood of $(X^1)^o \in ((W^1)^0)$.

Consider $((X^1)^1)^0 \in (C^{1,2}(W^1)^0)^0$ such that $(\rho_0^1)^0((X^1)^1)^0 = (X^1)^0$ and $(X^1)^1 \in C^{1,2}(W^1)$, $X^1 \in W^1$ the associated contact of $((X^1)^1)^0$ and $(X^1)^0$, respectively.

Now, let $D_{X^1} \subset T_{X^1} W^1$ be a two-dimensional subspace, transversal to the fibre over $(X^1) \in W^1$, such that $\omega(X^1)(D_{X^1}) = 0$. This two-dimensional subspace is generated by two vectors w_1 and w_2 of the forms

$$T_{x^1} i(w_1) = \partial_x + y_x \partial_y + a_1 \partial_{y_x} + b_1 \partial_{y_t}$$

and

$$T_{x^1} i(w_2) = \partial_t + y_t \partial_y + a_2 \partial_{y_x} + b_2 \partial_{y_t}$$

since $D_{X^1} \subset T_{x^1} W^1$ is transversal to the fibre over $(X^1) \in W^1$. As the one-form ω annihilates D_{X^1}, then the tangency condition is given by

$$i^*(d\omega(X^1)(w_1, w_2)) = 0.$$

Moreover the components of fibre of this vectors are

$$a_1 = \Phi_{xx}, a_2 = \Phi_{tx}, b_1 = \Phi_{xt}, b_2 = \Phi_{tt},$$

thus we have the system of equations

$$\begin{cases} \omega(w_1) = 0 \\ \omega(w_2) = 0 \\ d\omega(w_1, w_2) = 0 \end{cases}$$

From this it follows that

$$\begin{cases} b_1 - \Phi_{u_x}(X^1)a_1 = -L_1^1(\Phi)(X^1) \\ b_2 - \Phi_{u_x}(X^1)a_2 = -L_2^1(\Phi)(X^1) \\ b_1 - a_2 = 0 \end{cases}$$

where L_i^k are the contact field defined by

$$L_i^k = \partial_{x_i} + \sum_{j=n+1}^m \sum_{I_r} \frac{\partial^r}{\partial p_{I_r}^j} p_{I_r,i}^j,$$

where $1 \leq r < k$.

Using the formal derivative

$$D_i = \partial_{x_i} + \sum_{j=n+1}^m \sum_{I_r} \frac{\partial^r}{\partial p_{I_r}^j} p_{I_r,i}^j,$$

where $1 \leq r \leq k$, we know that

$$\begin{cases} y_{xt} = D_x \Phi = L_1^1(\Phi) + y_{xx}\Phi_{y_x} \\ y_{tt} = D_t \Phi = L_2^1(\Phi) + y_{xt}\Phi_{y_x} \end{cases}$$

and hence y_{xt} and y_{tt} are determined for a value of y_{xx}.

Then, to find the solution space of this system, first identify $a_1 = y_{xx}$ and then use the above relationships. Thus, we have that $y_{tx} = a_2 = b_1 = y_{xt}$ and $a_1 = y_{xx}$

and consequently $\Phi_{tx} = \Phi_{xt}$ and from this it follows that the vectors w_1 and w_2 are the forms:

$$\begin{cases} T_{x^1} i(w_1) = \partial_x + y_x \partial y + y_{xx} \partial y_x + y_{xt} \partial y_t \\ T_{x^1} i(w_2) = \partial_t + y_t \partial y + y_{tx} \partial y_x + y_{tt} \partial y_t \end{cases}$$

The condition $i^*(d\omega(X^1)(w_1, w_2)) = 0$ shows that $\Phi_{tx} = \Phi_{xt}$ and that a local section $\gamma \in \Gamma_{loc}(\rho)$ exists such that $i^1\gamma(x) = X^1$. Hence $T_{x^1}(im(i^1\gamma))$ is generate by the vectors $T_{x^1} i(w_1)$ and $T_{x^1} i(w_2)$.

Consequently, $((X^1)^1)^0 \in P(W^1)^0$ and the map $(\rho_1^2)^0 : P(W^1)^0 \longrightarrow (W^1)^0$ is a local submersion in a neighborhood of $(X^1)^0$ and D_{X^1} is an integral element. $\blacksquare$

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