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THE METRIC DIMENSION OF COMB PRODUCT GRAPHS

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Abstract. A set of vertices W *resolves* a graph G if every vertex is uniquely determined by its coordinate of distance to the vertices in W. The minimum cardinality of a resolving set of G is called the *metric dimension* of G. In this paper, we consider a graph which is obtained by the comb product between two connected graphs. Let o be a vertex of H. The *comb product* between G and H, denoted by $G \rhd_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identifying the i-th copy of H at the vertex o to the i-th vertex of G. We give an exact value of the metric dimension of $G \rhd_o H$ where H is not a path or H is a path where the vertex o is not a leaf. We also give the sharp general bounds of $\beta(G \rhd_o P_n)$ for $n \geq 2$ where the vertex o is a leaf of P_n.

1. Introduction

Throughout this paper all graphs G are finite, connected, and simple. We denote by V the vertex set of G and by E the edge set of G. The distance between two vertices $u, v \in V(G)$, denoted by $d_G(u, v)$, is the length of the shortest path from u to v in G. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered subset of $V(G)$. The *representation* of a vertex v of G with respect to W is defined as the k-tuple $r(v \,|\, W) = (d_G(v, w_1), d_G(v, w_2), \ldots, d_G(v, w_k))$. The set W is called a *resolving set* of G if every two distinct vertices $x, y \in V(G)$ satisfy $r(x \,|\, W) \neq r(y \,|\, W)$. A *basis* of G is a resolving set of G with the minimum cardinality, and the *metric dimension* of G refers to its cardinality and is denoted by $\beta(G)$.

The metric dimension problems were first studied by Harary and Melter [8], and independently by Slater [20]. Slater considered the minimum resolving set of a graph as the location of the placement of a minimum number of sonar/loran detecting devices in a network. So, the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Meanwhile Chartrand et al. [5] applied the metric dimension on the robot navigation problem. They considered the robot navigation in a graph space. A resolving set for a graph corresponds to the

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presence of distinctively labelled "landmark" nodes in the graph. It is assumed that a robot navigating a graph can sense the distance to each of the landmarks, and hence uniquely determine its location in the graph. The metric dimension problem is also applied in the chemical structures, coin weighing problems, and the Mastermind strategy (see [5, 11, 18]).

Determining the metric dimension of a general graph is an NP-complete problem [7]. There is no efficient algorithm to find the metric dimension of general graph. However, Chartrand et al. [5] have obtained some results as follows.

Theorem 1.1 ([5]). *Let G be a connected graph of order $n \geq 2$. Then*
1. $\beta(G) = 1$ *if and only if* $G = P_n$.
2. $\beta(G) = n - 1$ *if and only if* $G = K_n$.
3. $\beta(G) = n - 2$ *if and only if* G *is either* $K_{r,s}$ *for* $r, s \geq 1$, *or* $K_r + \overline{K_s}$ *for* $r \geq 1, s \geq 2$, *or* $K_r + (K_1 \cup K_s)$ *for* $r, s \geq 1$.

Some authors also have proven the metric dimension of certain classes of graphs. Interested readers are referred to a number of relevant literature that are mentioned in the bibliography section, including [2, 4–6, 9, 12–14, 16, 20].

There are also some results of the metric dimension problem for graphs resulting from operations on graphs. Some results on certain joint product graphs have been proved in [3, 4, 19]. Caceres et al. [4] have determined the metric dimension of graphs which are obtained by Cartesian product of two or more graphs. The metric dimension of some graphs which are constructed by corona product of two graphs have been studied in [10, 21]. Saputro et al. [17] have showed the metric dimension of lexicographic product of connected graph G and an arbitrary graph H. Meanwhile Rodríguez-Velázquez et al. [15] obtained closed formulae and tight bounds for the metric dimension of strong product graphs.

In this paper, we study the metric dimension of *comb product* of connected graphs G and H. In chemistry [1], some classes of chemical graphs can be considered as the comb product graphs. Let G and H be two connected graphs. Let o be a vertex of H. The *comb product* between G and H, denoted by $G \rhd_o H$, is a graph obtained by taking one copy of G and $|V(G)|$ copies of H and identify the i-th copy of H at the vertex o with the i-th vertex of G. By the definition of comb product, we can say that $V(G \rhd_o H) = \{(a, v) \mid a \in V(G), v \in V(H)\}$ and $(a, v)(b, w) \in E(G \rhd_o H)$ whenever $a = b$ and $vw \in E(H)$, or $ab \in E(G)$ and $v = w = o$. We consider two different vertices $a, b \in V(G)$ and a vertex $o \in V(H)$. We define $H(a) = \{(a, v) \mid v \in V(H)\}$, $G(o) = \{(v, o) \mid v \in V(G)\}$, and $P_G(a, b)$ is the shortest path from a to b in G.

We obtain four main results. The first result is related to $G \rhd_o H$ when G is a connected graph and either H is not a path or H is a path where the vertex o is not a leaf.

Theorem 1.2. *Let G and H be connected graphs of order at least 2. Let H be not a path or H be a path where the vertex o is not a leaf. If $|V(G)| = m$, then*

$$\beta(G \rhd_o H) = \begin{cases} m \cdot (\beta(H) - 1), & \textit{if there exists a basis of } H \textit{ containing the vertex } o, \\ m \cdot \beta(H), & \textit{otherwise.} \end{cases}$$

Our next results are related to $G \rhd_o P_n$ for a connected graph G and P_n where the vertex o is a leaf. Note that, there exists a basis of P_n containing the vertex o. However, the metric dimension of $G \rhd_o P_n$ with degree of the vertex o is 1, is different than $\beta(G \rhd_o H)$ in Theorem 1.2. In the next theorem, we give a lower bound of $\beta(G \rhd_o P_n)$ where degree of the vertex o is 1. We also prove that the bound is sharp.

THEOREM 1.3. *Let G be a connected graph of order $m \geq 2$. If o is a vertex of P_n with degree 1, then $\beta(G \rhd_o P_n) \geq \beta(G)$. The lower bound is sharp.*

For some cases, the lower bound in Theorem 1.3 cannot be attained. In some theorems below, we give the existence of a connected graph G such that $\beta(G \rhd_o P_n)$ is not equal to the lower bound in Theorem 1.3, where the vertex o is a leaf of P_n. Let v be a vertex of G. A *branch* of G at v is defined as a maximal subgraph of G which is isomorphic to a tree and containing v as an end point. So, if degree of v is k, then v has at most k different branches. A branch of v which is isomorphic to a path is called a *path branch* of v. If v contains at least two path branches, then v is called a *stem* of G. We prove that if G is a connected graph containing $p \geq 1$ stems, then $\beta(G \rhd_o P_n) \geq \beta(G) + p$. We also show that this lower bound is sharp.

THEOREM 1.4. *Let G be a connected graph containing $p \geq 1$ stems. If the vertex o is a leaf of P_n, then $\beta(G \rhd_o P_n) \geq \beta(G) + p$. The lower bound is sharp.*

We also give the existence of a connected graph G which does not contain any stem vertex such that $\beta(G \rhd_o P_n)$ also is not equal to the lower bound in Theorem 1.3, where the vertex o is a leaf of P_n.

THEOREM 1.5. *Let o be a leaf of P_n. There exists a connected graph G which does not contain any stem vertex such that $\beta(G \rhd_o P_n) > \beta(G)$.*

2. Proof of Theorem 1.2

First, we need to prove the following two propositions.

PROPOSITION 2.1. *Let G and H be connected graphs of order at least 2. Let H satisfy one of two conditions below.*

1. *H is not a path; or*
2. *H is a path and the degree of the vertex o is 2.*

Then there exist two distinct vertices $x, y \in V(H) \backslash \{o\}$ such that $d_{G \rhd_o H}((a, x), (a, o)) = d_{G \rhd_o H}((a, y), (a, o))$ for every vertex $a \in V(G)$.

Proof. If degree of the vertex o is at least 2, then we have nothing to prove. Suppose that degree of the vertex o is 1 which implies H is not a path. So, there exists a vertex z in H of degree at least 3. Let us consider two different vertices $p, q \in V(H)$ such that $pz, qz \in E(H)$ and $p, q \in V(H) \backslash V(P_H(o, z))$. Since $d_H(p, o) = d_H(p, z) + d_H(z, o) = 1 + d_H(z, o) = d_H(q, z) + d_H(z, o) = d_H(q, o)$, by the definition of comb product, we obtain that $d_{G \rhd_o H}((a, x), (a, o)) = d_{G \rhd_o H}((a, y), (a, o))$. $\square$

Proposition 2.2. *Let G and H be connected graphs of order at least 2. Let H be not a path or H be a path where the vertex o is not a leaf. For every distinct vertices $a, b \in V(G)$, if there exist distinct vertices $x, y, z \in V(H)\backslash\{o\}$ such that $d_{G \triangleright_o H}((a,x),(a,y)) = d_{G \triangleright_o H}((b,z),(a,y))$, then*

$$d_{G \triangleright_o H}((a,x),(a,o)) > d_{G \triangleright_o H}((b,z),(b,o)).$$

Proof. Note that we have

$$\begin{aligned} d_{G \triangleright_o H}((a,x),(a,y)) &\leq d_{G \triangleright_o H}((a,x),(a,o)) + d_{G \triangleright_o H}((a,o),(a,y)) \quad \text{and} \\ d_{G \triangleright_o H}((b,z),(a,y)) &= d_{G \triangleright_o H}((b,z),(b,o)) + d_{G \triangleright_o H}((b,o),(a,o)) \\ &\quad + d_{G \triangleright_o H}((a,o),(a,y)). \end{aligned}$$

Therefore, we obtain that

$$d_{G \triangleright_o H}((b,z),(b,o)) + d_{G \triangleright_o H}((b,o),(a,o)) \leq d_{G \triangleright_o H}((a,x),(a,o)).$$

Since $d_{G \triangleright_o H}((b,o),(a,o)) \geq 1$, we have $d_{G \triangleright_o H}((b,z),(b,o)) < d_{G \triangleright_o H}((a,x),(a,o))$. □

According to Proposition 2.1, in order to determine a resolving set of $G \triangleright_o H$, we must find a subset $S(a) \subseteq H(a)$ for every $a \in V(G)$.

Lemma 2.3. *Let G and H be connected graphs of order at least 2. Let H be not a path or H be a path where the vertex o is not a leaf. Let W be a basis of $G \triangleright_o H$. For any vertex $a \in V(G)$, if $S(a) = W \cap H(a)$, then $S(a) \neq \emptyset$. Moreover, if B is a basis of H, then $|S(a)| \leq |B|$.*

Proof. Suppose that there exists a vertex $a \in V(G)$ such that $S(a) = \emptyset$. Let $u \in W$ and $u \in V(G \triangleright_o H) \setminus H(a)$. By Proposition 2.1, there exist two different vertices (a,x) and (a,y) such that $d_{G \triangleright_o H}((a,x),(a,o)) = d_{G \triangleright_o H}((a,y),(a,o))$. It follows that $d_{G \triangleright_o H}(u,(a,x)) = d_{G \triangleright_o H}(u,(a,y))$, which implies $r((a,x) \,|\, W) = r((a,y) \,|\, W)$, a contradiction.

Now, let us consider $S(a) = \{(a,v) \,|\, v \in B\}$. Since B resolves every two distinct vertices of H, we obtain that $S(a)$ also resolves every two different vertices of $H(a)$. □

Although by Lemma 2.3, a basis of $G \triangleright_o H$ contains vertices of $H(a)$ for every $a \in V(G)$, we show that there exists a basis W of $G \triangleright_o H$ such that $(a,o) \notin W$ for every $a \in V(G)$. On the other hand, $W \cap G(o) = \emptyset$.

Lemma 2.4. *Let G and H be connected graphs of order at least 2. If H is not a path or H is a path where the vertex o is not a leaf, then there exists a basis W of $G \triangleright_o H$ such that $W \cap G(o) = \emptyset$.*

Proof. Suppose that there exists a basis W of $G \triangleright_o H$ containing (a,o) for some $a \in V(G)$. We define $W' = W \backslash \{(a,o)\}$. Let us consider two distinct vertices $x, y \in V(G \triangleright_o H) \backslash W'$. We distinguish two cases.

1. $x, y \in H(a)$
   If $d_{G \triangleright_o H}(x,(a,o)) \neq d_{G \triangleright_o H}(y,(a,o))$, then according to Lemma 2.3, x and y are resolved by a vertex in $W' \backslash H(a)$. Otherwise, also according to Lemma 2.3, there exists a vertex $z \in W' \cap H(a)$ and $z \neq (a,o)$ that resolves x and y.

2. $x \in H(a)$ and $y \in H(b)$
   If there exists $v \in W' \cap H(a)$ such that $d_{G \rhd_o H}(v, x) \neq d_{G \rhd_o H}(v, y)$, then x and y are resolved by v. Otherwise, we have $d_{G \rhd_o H}(v, x) = d_{G \rhd_o H}(v, y)$ for every vertex $v \in W' \cap H(a)$. By Proposition 2.2, we obtain that $d_{G \rhd_o H}(x, (a, o)) > d_{G \rhd_o H}(y, (b, o))$. Now, we apply Proposition 2.2 to vertices in $H(b)$. So, there is no vertex $z \in V(H) \backslash \{o\}$ such that $d_{G \rhd_o H}(y, (b, z)) = d_{G \rhd_o H}(x, (b, z))$. It follows that there exists a vertex in $W' \cap H(b)$ that resolves x and y.

From the two previous cases, we obtain that W' is a resolving set of $G \rhd_o H$, a contradiction. □

By Lemma 2.3, in order to determine a resolving set of $G \rhd_o H$, we must find a subset $S(a) \subseteq H(a)$ for every $a \in V(G)$. However, by Lemma 2.4, there exists a basis W of $G \rhd_o H$ such that W does not contain any vertex of $G(o)$. In the next lemma, we show that $S(a)$ must contain at least $\beta(H) - 1$ vertices of $H(a) \backslash \{(a, o)\}$.

Lemma 2.5. *Let G and H be connected graphs of order at least 2. Let H be not a path or H be a path where the vertex o is not a leaf. Let W be a basis of $G \rhd_o H$. For any vertex $a \in V(G)$, if $S(a) = W \cap H(a)$, then $|S(a)| \geq \beta(H) - 1$.*

Proof. Suppose that there exists $a \in V(G)$ such that $|S(a)| \leq \beta(H) - 2$. We define $X = S(a) \cup \{(a, o)\}$. Since $|X| < \beta(H) - 1$, there exist two distinct vertices $u, v \in H(a)$ such that $r(u \,|\, X) = r(v \,|\, X)$ which implies $r(u \,|\, S(a)) = r(v \,|\, S(a))$ and $d_{G \rhd_o H}(u, (a, o)) = d_{G \rhd_o H}(v, (a, o))$.

Let $z \in W \setminus H(a)$. According to Lemma 2.3, we obtain that

$$\begin{aligned} d_{G \rhd_o H}(u, z) &= d_{G \rhd_o H}(u, (a, o)) + d_{G \rhd_o H}((a, o), z) \\ &= d_{G \rhd_o H}(v, (a, o)) + d_{G \rhd_o H}((a, o), z) = d_{G \rhd_o H}(v, z). \end{aligned}$$

Therefore, we have $r(u \,|\, W) = r(v \,|\, W)$, a contradiction. □

Combining Lemmas 2.3 and 2.5 for every vertex $a \in V(G)$, we obtain the general bounds of metric dimension of $G \rhd_o H$ for any connected graph G and a connected graph H which is not a path or if it is a path, then the vertex o is not a leaf.

Lemma 2.6. *Let G and H be connected graphs of order at least 2. Let H be not a path or H be a path where the vertex o is not a leaf. If $|V(G)| = m$, then*

$$m \cdot (\beta(H) - 1) \leq \beta(G \rhd_o H) \leq m \cdot \beta(H)$$

Now, we will show that for any connected graph G and a connected graph H which is not a path or H is a path with degree of the vertex o is 2, the metric dimension of $G \rhd_o H$ is equal to either lower bound or upper bound of Lemma 2.6. In other words, we are ready to prove Theorem 1.2.

2.1 Proof of Theorem 1.2

We distinguish two cases.

Case 1.2.1. There exists a basis of H containing the vertex o

By Lemma 2.6, we only need to show that $\beta(G \rhd_o H) \leq m \cdot (\beta(H) - 1)$. Let B be a basis of H containing the vertex o of H. For every vertex $a \in V(G)$, we define

$W(a) = \{(a, v) \mid v \in B \backslash \{o\}\}$ and $W = \bigcup_{a \in V(G)} W(a)$. Note that $|W| = m \cdot (\beta(H) - 1)$. We will show that W is a resolving set of $G \rhd_o H$.

Let us consider two different vertices (a, x) and (b, y) for $a, b \in V(G)$ and $x, y \in V(H)$.

1. Case $a = b$
   If x and y are resolved by a vertex in $B \backslash \{o\}$, then (a, x) and (b, y) are resolved by a vertex in $W(a)$, which implies $r((a, x) \mid W) \neq r((b, y) \mid W)$. Otherwise, x and y are resolved by the vertex o. It follows that (a, x) and (b, y) are resolved by (a, o). It means that $d_{G \rhd_o H}((a, x), (a, o)) \neq d_{G \rhd_o H}((b, y), (a, o))$. Note that for every $p \in V(G \rhd_o H) \backslash H(a)$ and $q \in H(a)$, $d_{G \rhd_o H}(p, q) = d_{G \rhd_o H}(p, (a, o)) + d_{G \rhd_o H}((a, o), q)$. According to Lemma 2.3, since there exists a vertex z in $V(G \rhd_o H) \backslash H(a)$ which is contained in W, we have that (a, o) and (b, y) are resolved by z.
2. Case $a \neq b$
   If there exists z in $W(a)$ such that $d_{G \rhd_o H}((a, x), z) \neq d_{G \rhd_o H}((b, y), z)$, then z resolves (a, x) and (b, y). Otherwise, for every vertex $z \in W(a)$, $d_{G \rhd_o H}((a, x), z) = d_{G \rhd_o H}((b, y), z)$. By Proposition 2.2, we obtain that $d_{G \rhd_o H}((a, x), (a, o)) > d_{G \rhd_o H}((b, y), (b, o))$. If we consider Proposition 2.2 to $W(b)$, then every vertex $w \in W(b)$ satisfies $d_{G \rhd_o H}((a, x), w) \neq d_{G \rhd_o H}((b, y), w)$ which implies, $r((a, x) \mid W) \neq r((b, y) \mid W)$.

From the two previous cases, it follows that W is a resolving set of $G \rhd_o H$.

Case 1.2.2. No basis of H contains the vertex o

By Lemma 2.6, we only need to show that $\beta(G \rhd_o H) \geq m \cdot \beta(H)$. Suppose that $\beta(G \rhd_o H) \leq m \cdot \beta(H) - 1$. Let W be a basis of $G \rhd_o H$. So, there exists a vertex $a \in V(G)$ such that $H(a)$ contributes $\beta(H) - 1$ vertices in W. Let $W(a) = W \cap H(a)$. Let us consider a subset S of $V(H)$ whose vertices are corresponded to vertices of $W(a)$. Let $A = S \cup \{o\}$. Since $|A| \leq \beta(H)$ and every basis of H does not contain the vertex o, there exist two different vertices $x, y \in V(H)$ such that $r(x \mid A) = r(y \mid A)$. Thus, $d_H(x, o) = d_H(y, o)$ and $r(x \mid S) = r(y \mid S)$, which implies $d_{G \rhd_o H}((a, x), (a, o)) = d_{G \rhd_o H}((a, y), (a, o))$ and $r((a, x) \mid W(a)) = r((a, y) \mid W(a))$. Therefore, we obtain that $r((a, x) \mid W) = r((a, y) \mid W)$, a contradiction. □

3. Proof of Theorem 1.3

Let $V(G) = \{g_1, g_2, \ldots, g_m\}$ and $V(P_n) = \{p_1, p_2, \ldots, p_n\}$ with $E(P_n) = \{p_i p_{i+1} \mid 1 \leq i \leq n - 1\}$.

Suppose that $\beta(G \rhd_o P_n) \leq \beta(G) - 1$. Let W be a basis of $G \rhd_o P_n$. Let $W' = \{(a, o) \mid (a, v) \in W, a \in V(G)\}$ and $B = \{a \in V(G) \mid (a, o) \in W'\}$. Note that $B \subseteq V(G)$. Since $|W'| \leq |W| \leq \beta(G) - 1$ and $|B| = |W'|$, we obtain that $|B| \leq \beta(G) - 1$. Thus, there exist two distinct vertices $x, y \in V(G)$ such that $r(x \mid B) = r(y \mid B)$. It follows that $r((x, o) \mid W') = r((y, o) \mid W')$ which implies $r((x, o) \mid W) = r((y, o) \mid W)$, a contradiction.

Now, we will show that the bound is sharp. Let G be a complete graph with $m \geq 3$ vertices. Note that $\beta(G) = m - 1$ [5]. Let $o = p_1$. We will show that $\beta(G \rhd_o P_n) = \beta(G)$. We only need to show that $\beta(G \rhd_o P_n) \leq \beta(G)$. We define $S = \{(g_i, p_n) \,|\, 1 \leq i \leq m-1\}$. Since $|S| = m - 1 = \beta(G)$, we will show that S is a resolving set of $G \rhd_o P_n$.

Let (g_i, p_r) and (g_j, p_t) be two different vertices in $V(G \rhd_o P_n) \setminus S$ where $i, j \in \{1, 2, \ldots, m\}$ and $r, t \in \{1, 2, \ldots, n\}$. We distinguish two cases below.

1. Case $i = j$
   Let $r < t$. If $(g_i, p_n) \in S$, then $d_{G\rhd_o P_n}((g_i, p_r), (g_i, p_n)) = d_{G\rhd_o P_n}((g_i, p_t), (g_i, p_n)) + t - r$. Otherwise, for $(g_k, p_n) \in S$ with $k \neq i$, we have $d_{G\rhd_o P_n}((g_i, p_t), (g_k, p_n)) = d_{G\rhd_o P_n}((g_i, p_r), (g_k, p_n)) + t - r$.
2. Case $i \neq j$
   So, $(g_i, p_n) \in S$ or $(g_j, p_n) \in S$. Without loss of generality, let $(g_i, p_n) \in S$. Then $d_{G\rhd_o P_n}((g_j, p_t), (g_i, p_n)) = d_{G\rhd_o P_n}((g_i, p_r), (g_i, p_n)) + d_{G\rhd_o P_n}((g_i, p_r), (g_j, p_t))$.

From the two previous cases, it follows that S is a resolving set of $G \rhd_o P_n$. □

4. Proof of Theorem 1.4

Let v be a vertex of G. A *branch* of G at v is defined as a maximal subset of G which is isomorphic to a tree and containing v as an end point. So, if degree of v is k, then v has at most k different branches. A branch of v which is isomorphic to a path is called a *path branch* of v. If v has at least 2 path branches, then v is called a *stem* of G. Let $\mathcal{A}(v)$ be the vertex set of all vertices in path branches of v.

LEMMA 4.1. *Let G be a connected graph and v be a stem of G having $k \geq 2$ path branches. If W is a resolving set of G, then $|\mathcal{A}(v) \cap W| \geq k - 1$.*

Proof. Suppose that $|\mathcal{A}(v) \cap W| \leq k - 2$. So, there exist two different path branches A and B of G at v such that $(V(A) \setminus \{v\}) \cap W = \emptyset$ and $(V(B) \setminus \{v\}) \cap W = \emptyset$. Let $a \in V(A)$ and $b \in V(B)$ such that $av, bv \in E(G)$. Note that for every vertex $x \in \{v\} \cup V(G) \setminus (V(A) \cup V(B))$, $d_G(x, a) = d_G(x, v) + d_G(v, a) = d_G(x, v) + d_G(v, b) = d_G(x, b)$. It follows that $r(a \,|\, W) = r(b \,|\, W)$, a contradiction. □

LEMMA 4.2. *Let G be a connected graph and v be a stem of G having $k \geq 2$ path branches. There exists a resolving set W of G such that $|\mathcal{A}(v) \cap W| = k - 1$.*

Proof. Let v be a stem of G having path branches of size $m_1, m_2, \ldots, m_k$. Let $X_j = \{x_{(i,m_j)} \mid 1 \leq i \leq m_j\}$ be a vertex set of j-th path branch of v with $j \in \{1, 2, \ldots, k\}$.

Let $G' = G \setminus \mathcal{A}(v)$ and S be a resolving set of G'. Note that for every $s \in V(G')$, we have $d_G(s, x_{(r,i)}) = d_G(s, x_{(t,i)})$ for $r, t \in \{1, 2, \ldots, k\}$ and $r \neq t$. So, no vertex in S can resolve $x_{(r,i)}$ and $x_{(t,i)}$. Choose vertex set $B = \{x_{(i,m_i)} \mid 1 \leq i \leq k-1\}$. We will show that every two distinct vertices a and b in $\mathcal{A}(v)$ are resolved by B.

1. $a, b \in X_i$ for $i \in \{1, 2, \ldots, k\}$

Let $a = x_{(i,r)}$ and $b = x_{(i,t)}$ where $1 \leq r < t \leq m_i$. If $i \neq k$, then $d_G(a, x_{(i,m_i)}) > d_G(b, x_{(i,m_i)})$, otherwise $d_G(a, x_{(1,m_1)}) < d_G(b, x_{(1,m_1)})$. Therefore $r(a \,|\, B) \neq r(b \,|\, B)$.

2. $a \in X_i$ and $b \in X_j$ for $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$
   Without loss of generality, let $i \in \{1, 2, \ldots, k-1\}$. Then $d_G(a, x_{(i,m_i)}) < d_G(b, x_{(i,m_i)})$. Therefore $r(a \,|\, B) \neq r(b \,|\, B)$.

From the two previous cases, we obtain that B is a resolving set of $\mathcal{A}(v)$.

Now, we define $W = S \cup B$. Since S and B are resolving sets of G' and $\mathcal{A}(v)$, respectively, we have that W is a resolving set of G. $\square$

Lemma 4.3. *Let G be a connected graph and v be a stem of G having $k \geq 2$ path branches. Let W be a resolving set of G such that $|\mathcal{A}(v) \cap W| = k - 1$. Then every two distinct vertices in $\mathcal{A}(v) \cap W$ are from different path branches of v.*

Proof. Suppose that there exist two distinct vertices in $\mathcal{A}(v) \cap W$ which are from a path branch of v. So, there exist two path branches A and B of v such that for $C = (V(A) \cup V(B)) \setminus \{v\}$, we have $W \cap C = \emptyset$. Let a and b be two vertices in A and B, respectively, which are adjacent to v. Note that for every $x \in V(G) \setminus C$, $d_G(a, x) = d_G(a, v) + d_G(v, x) = d_G(b, v) + d_G(v, x) = d_G(b, x)$. It follows that $r(a \,|\, W) = r(b \,|\, W)$, a contradiction. $\square$

Now, we are ready to prove Theorem 1.4.

4.1 Proof of Theorem 1.4

Let $V(P_n) = \{p_1, p_2, \ldots, p_n\}$ with $E(P_n) = \{p_i p_{i+1} \,|\, 1 \leq i \leq n-1\}$ and $o = p_1$. Let G be a connected graph with metric dimension $\beta(G)$. Let $v_1, v_2, \ldots, v_p$ be $p \geq 1$ stems of G. Suppose that $\beta(G \rhd_o P_n) \leq \beta(G) + p - 1$. Let B be a basis of $G \rhd_o P_n$.

For $i \in \{1, 2, \ldots, p\}$, let v_i has m_i path branches. Let $A_{(i,1)}, A_{(i,2)}, \ldots, A_{(i,m_i)}$ be m_i path branches of v_i. By Lemmas 4.2 and 4.3, $m_i - 1$ vertices from $m_i - 1$ different path branches of v_i are contributed in a basis W of G. We can say that $\beta(G) = r + \sum_{i=1}^{p}(m_i - 1)$ where r is a non-negative integer. Without loss of generality, let $W \cap V(A_{(i,m_i)} \setminus \{v_i\}) = \emptyset$.

Now, we define a vertex set $Y = \{y_{(i,j)} \in V(A_{(i,j)}) \mid y_{(i,j)}$ is adjacent to a vertex of degree 1 in $A_{(i,j)}, 1 \leq i \leq p, 1 \leq j \leq m_i\}$. Note that, in $G \rhd_o P_n$, the vertex $(y_{(i,j)}, p_1)$ is a stem. Now, we assume that for $1 \leq i \leq p$ and $1 \leq j \leq m_i$, $(y_{(i,j)}, p_1)$ satisfies Lemmas 4.2 and 4.3. Let $\mathcal{C} \subset V(G \rhd_o P_n)$ be the vertex set of all vertices in path branches of $(y_{(i,j)}, p_1)$ with $1 \leq i \leq p$ and $1 \leq j \leq m_i$. So, we obtain that $|B \cap \mathcal{C}| = \sum_{i=1}^{p} m_i$.

For $a \in V(G)$, let q_a be a projection of all vertices of $H(a)$. Let Q be a graph with $V(Q) = \{q_a \mid a \in V(G)\}$ and $q_a q_b \in E(Q)$ whenever $ab \in E(G)$. Let $B^* = \{q_v \in V(Q) \mid (v, w) \in B\}$ and $\mathcal{C}^* = \{q_v \in V(Q) \mid (v, w) \in \mathcal{C}\}$. So, $|B^* \cap \mathcal{C}^*| = \sum_{i=1}^{p} m_i$. Note that for every $i \in \{1, 2, \ldots, p\}$, $|\mathcal{A}(q_{v_i}) \cap B^*| = m_i$. Let B^{**} be the set of all vertices of B^* except for vertex in the m_i-th path branch of q_{v_i} with $1 \leq i \leq p$. Since $\beta(G \rhd_o P_n) \leq \beta(G) + p - 1 = r + p - 1 + \sum_{i=1}^{p}(m_i - 1) = r - 1 + \sum_{i=1}^{p} m_i$, we have that $|B^{**}| \leq r - 1 + \sum_{i=1}^{p}(m_i - 1) = \beta(G) - 1$. So, B^{**} is not a resolving set of Q

which implies that also B^* cannot resolve $V(Q)$. It follows that B is not a basis of $G \rhd_o P_n$, a contradiction. $\square$

5. Proof of Theorem 1.5

For $n \geq 3$, we define H_n as a graph with vertex set $V(H_n) = V_1 \cup V_2$ where $V_1 = \{u_i \mid 1 \leq i \leq n\}$, $V_2 = \{v_i \mid 1 \leq i \leq n\}$ and edge set $E(H_n) = \{u_i u_j \mid 1 \leq i < j \leq n\} \cup \{u_i v_i \mid 1 \leq i \leq n\}$. Note that, an induced subgraph of H_n by V_1 is isomorphic to a complete graph K_n. Also, H_n does not contain a stem. First, we will determine the metric dimension of H_n which can be seen in the next lemma.

Lemma 5.1. *For $n \geq 3$, the metric dimension of H_n is $n - 1$.*

Proof. For the upper bound, let us consider $W \subset V(H_n)$ where $W = \{v_i \mid 1 \leq i \leq n - 1\}$. Note that, $|W| = n - 1$. Now, we will show that W is a resolving set of H_n. Let x and y be two distinct vertices of H_n. We distinguish three cases.

1. $x, y \in V_1$
   Let $x = u_i$ and $y = u_j$ with $i \neq j$. So, we have $v_i \in W$ or $v_j \in W$. Now, we assume that $v_i \in W$. Since $d_{H_n}(y, v_i) = d_{H_n}(x, v_i) + 1$, we obtain $r(x \,|\, W) \neq r(y \,|\, W)$.
2. $x, y \in V_2$
   Then we obtain $x \in W$ or $y \in W$, which trivially implies $r(x \,|\, W) \neq r(y \,|\, W)$.
3. $x \in V_1$ and $y \in V_2$
   If $y \in W$, then we have $r(x \,|\, W) \neq r(y \,|\, W)$. Otherwise, we have two possibilities for x. If $x = u_i$ with $i \in \{1, 2, \ldots, n-1\}$, then $d_{H_n}(y, v_i) = d_{H_n}(x, v_i) + 2$. If $x = u_n$, then for every $w \in W$, we have $d_{H_n}(y, w) = d_{H_n}(x, w) + 1$. From both possibilities, we obtain $r(x \,|\, W) \neq r(y \,|\, W)$.

From the three previous cases, we can say that W is a resolving set of H_n.

For the lower bound, suppose that $\beta(H_n) \leq n - 2$. Let B be a basis of H_n. So, there exist i and j where $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j$ such that $u_i, v_i, u_j, v_j \notin B$. For $z \in V(H_n) \setminus \{u_i, v_i, u_j, v_j\}$, if $z \in V_1$, then $d_{H_n}(u_i, z) = 1 = d_{H_n}(u_j, z)$, otherwise $d_{H_n}(u_i, z) = 2 = d_{H_n}(u_j, z)$. It follows that $r(u_i \mid B) = r(u_j \mid B)$, a contradiction. $\square$

Now, we are ready to prove Theorem 1.5.

5.1 Proof of Theorem 1.5

For $m \geq 2$ and $n \geq 3$, let P_m be a path graph with $V(P_m) = \{p_i \mid 1 \leq i \leq m\}$ and $E(P_m) = \{p_i p_{i+1} \mid 1 \leq i \leq m - 1\}$, and $G \cong H_n$. We consider a comb product $G \rhd_o P_m$ where $o = p_1$. By Lemma 5.1, we have $\beta(G) = n - 1$. However, we will prove that $\beta(G \rhd_o P_m) = n$ which is greater than the lower bound in Theorem 1.3.

Since $G \rhd_o P_m$ contains n stems having 2 path branches, by Lemma 4.1, we obtain $\beta(G \rhd_o P_m) \geq n$. Now, we will show that $\beta(G \rhd_o P_m) \leq n$. We define $W = \{(v, p_m) \mid v \in V_2\}$. We will show that W is a resolving set of $G \rhd_o P_m$. Let x

and y be two distinct vertices of $G \rhd_o P_m$. For $z_1, z_2 \in V(G)$, we assume $x \in P_m(z_1)$ and $y \in P_m(z_2)$. We distinguish two cases.

1. $z_1 = z_2$
   We assume $x = (z_1, p_r)$ and $y = (z_2, p_t)$ where $1 \le r < t \le m$. If $z_1 = z_2 \in V_2$, then we obtain $d_{G \rhd_o P_m}(x, (z_1, p_m)) = d_{G \rhd_o P_m}(y, (z_1, p_m)) + t - r$. Otherwise, we have $d_{G \rhd_o P_m}(y, (z_1, p_m)) = d_{G \rhd_o P_m}(x, (z_1, p_m)) + t - r$.
2. $z_1 \neq z_2$
   We distinguish three subcases.
   (a) $z_1, z_2 \in V_2$
   Then we obtain $d_{G \rhd_o P_m}(y, (z_1, p_m)) = d_{G \rhd_o P_m}(x, (z_1, p_m)) + r + t + 3$.
   (b) $z_1, z_2 \in V_1$
   Let $z_1 = u_i$ and $z_2 = u_j$ where $i, j \in \{1, 2, \ldots, n\}$. If $d_{G \rhd_o P_m}(y, (v_j, p_m)) \neq d_{G \rhd_o P_m}(x, (v_j, p_m))$, then we have nothing to prove. Otherwise, we consider that $d_{G \rhd_o P_m}(y, (v_i, p_m)) = d_{G \rhd_o P_m}(x, (v_i, p_m)) + t - r + 1$.
   (c) $z_1 \in V_1$ and $z_2 \in V_2$
   Let $z_2 = v_j$ where $j \in \{1, 2, \ldots, n\}$. If $z_1 = u_j$, then we obtain
   $$d_{G \rhd_o P_m}(x, (z_2, p_m)) = d_{G \rhd_o P_m}(y, (z_2, p_m)) + r + t + 1.$$
   Otherwise, $d_{G \rhd_o P_m}(x, (z_2, p_m)) = d_{G \rhd_o P_m}(y, (z_2, p_m)) + r + t + 2$.

From the two previous cases, we can say that W with $|W| = n$ is a resolving set of $G \rhd_o P_m$. □

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