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# A UNIFIED THEORY OF PERFECT AND RELATED FUNCTIONS

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**Abstract.** A unified theory has been developed on the basis of the similarity in properties of perfect and allied types of functions. The theory intromites as a starting point a certain subset of $\mathcal{P}(X)$, the power set of a nonvoid set $X$, and an operator $\alpha$ on $\mathcal{P}(X)$; a second operator $\beta$ is also brought into action. This theory of $\beta$-perfect functions includes the theories of perfect, $\theta$-perfect and $\delta$-perfect functions and is seen to generate many new types of functions when different pairs of operators take the roles of the pair $(\alpha, \beta)$.

## 1. Introduction

Perfect functions and certain allied types of functions reffered to in the literature have been studied thoroughly by several authors. These types of functions have similarity in many aspects. Certain characterizations are quite analogous. The analogous features existing in their definitions and the types of results obtained in the process of their study point to the necessity of promoting a unified theory. The purpose of this paper is thus to present a unified theory by agglomeration of the individual findings.

In the next section, we display the development of the unified theory in a general perspective, while in Section 3 we bring about certain concepts viz. those of $\beta$-sets, $\gamma$-sets and $\beta$-continuity to find other characterizations of $\beta$-perfect functions from an altogether different perspective. In the process, we show up the individual behaviours in particularized settings where the underlying spaces under consideration are topological spaces.

## 2. $\beta$-perfect functions

DEFINITION 2.1. Let $X$ be a nonempty set and let $\mathcal{B}$ be a family of subsets of $X$ such that each point of $X$ belongs to some member of $\mathcal{B}$, and that $\mathcal{B}$ is closed under finite intersection. The set of all members of $\mathcal{B}$, each containing a given point

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$x$ of $X$ is denoted by $\mathcal{B}_x$. In addition, let us consider an operator $\alpha$ on the power set $\mathcal{P}(X)$ of $X$ such that

$$A \subset B\,(\subset X) \implies \alpha(A) \subset \alpha(B). \qquad (*)$$

Henceforth by a space $X$ (or $Y$) or simply by $X$ (or $Y$) we mean a nonempty set $X$ (respectively $Y$) endowed with an operator $\alpha$ and associated with a collection $\mathcal{B}$ of subsets of $X$ (or $Y$), where $\alpha$ and $\mathcal{B}$ are defined as above.

Definition 2.2. A point $x$ of a space $X$ is called a $\beta$-adherent point of a set $A \subset X$ if for each $U \in \mathcal{B}_x$, $\alpha(U) \cap A \neq \emptyset$. The set of all $\beta$-adherent points of $A$ is denoted by $[A]_\beta$.

Remark 2.3. It is clear from the above definition that the operator $\beta$ is increasing, i.e. $[A]_\beta \subset [B]_\beta$ whenever $A \subset B$ $(\subset X)$.

When we consider a function $f$ from a space $X$ to a space $Y$, we use the same notations $\alpha$, $\beta$ etc. for the space $Y$ as well, with the hope that the context will leave no scope for confusion.

Definition 2.4. A point $x \in X$ is called a $\beta$-adherent point of a filterbase $\mathcal{F}$ on $X$, written as $x \in \beta\text{-ad}\,\mathcal{F}$, if $x \in \bigcap\{\,[F]_\beta\,:\,F \in \mathcal{F}\,\}$.

Definition 2.5. A filterbase $\mathcal{F}$ on $X$ is said to $\beta$-converge to a point $x$ of $X$, denoted by $\mathcal{F} \xrightarrow{\beta} x$, if for each $U \in \mathcal{B}_x$, there exists an $F \in \mathcal{F}$ such that $F \subset \alpha(U)$.

Definition 2.6. A filterbase $\mathcal{F}$ on $X$ is said to be $\beta$-directed to a subset $A$ of $X$, denoted by $\mathcal{F} \xrightarrow{\beta-\mathrm{d}} A$, if for each filterbase $\mathcal{G}$ finer than $\mathcal{F}$, $A \cap \beta\text{-ad}\,\mathcal{G} \neq \emptyset$.

Let us now illustrate the above unified definitions in ceratian particular and well known settings. Suppose $\mathcal{B}_x$ denotes the set of all open neighbourhoods of any point $x$ of a topological space $X$, and $\mathcal{B} = \bigcup_{x\in X}\mathcal{B}_x$. We shall restrict our discussion of particularization to only three cases as follows.

When $\alpha$ stands for the identity (closure, interior-closure) operator, the operator $\beta$ becomes the closure (resp. $\theta$-closure [11], $\delta$-closure [11]) operator. Then the notion of $\beta$-adherent point of a filterbase reduces to its adherent point (resp. $\theta$-adherent point [3,2], $\delta$-adherent point [11,7]). A filterbase is $\beta$-directed towards a set $A$ or a point $x$ becomes equivalent to it being directed [12] (resp. $\theta$-directed [2], $\delta$-directed [7]) towards $A$ or $x$. The $\beta$-closure of a set takes the meaning of its closure (resp. $\theta$-closure [11] or almost closure [3], $\delta$-closue [11]) and $\beta$-convergence of a filterbase becomes its ordinary convergence (resp. $\theta$-convergence [11] or almost convergence [3], $\delta$-convergence [11]).

Stated below is a list of results elaborating the unified concepts defined so far. We shall omit the proofs as they are quite analogous to those concerning ordinary convergence and adherence of filterbases in topological spaces, and can be constructed without much difficulty.

Proposition 2.7. *For a filterbase $\mathcal{F}$ on $X$, if $\mathcal{F}$ $\beta$-converges to $x \in X$, then $x \in \beta\text{-ad}\,\mathcal{F}$.*

PROPOSITION 2.8. *If $x \in [A]_\beta$, where $x \in X$ and $A \subset X$, then there exists a filterbase $\mathcal{F}$ on $A$ such that $\mathcal{F} \xrightarrow{\beta} X$.*

PROPOSITION 2.9. *If $x \in \beta\text{-}ad\,\mathcal{F}$, for some filterbase $\mathcal{F}$ on $X$ (where $x \in X$), then there exists a filterbase $\mathcal{G}$ on $X$ finer than $\mathcal{F}$ such that $\mathcal{G} \xrightarrow{\beta} x$.*

PROPOSITION 2.10. *For a filterbase $\mathcal{F}$ on $X$ and $x \in X$, $\mathcal{F} \xrightarrow{\beta} x$ iff $\mathcal{F} \xrightarrow{\beta-\mathrm{d}} x$.*

REMARK 2.11. We see that the above propositions generalize the corresponding results in the particular cases under consideration. For example, Proposition 2.10 encompasses Result 2(d) of [12] and Proposition 2.9 of [3] (with $A$ a singleton).

DEFINITION 2.12. Let $X$ and $Y$ be two spaces. A function $f\colon X \to Y$ is called $\beta$-perfect iff for each filterbase $\mathcal{F}$ on $f(X)$, $\mathcal{F} \xrightarrow{\beta-\mathrm{d}} B\,(\subset f(X))$ implies $f^{-1}(\mathcal{F}) \xrightarrow{\beta-\mathrm{d}} f^{-1}(B)$.

We note, at this stage, that the three widely studied types of functions viz. perfect [12,10], $\theta$-perfect [2,3] and $\delta$-perfect [7] functions are achieved as special cases of $\beta$-perfect functions in the three respective particular cases under consideration.

THEOREM 2.13. *For a function $f\colon X \to Y$, the following are equivalent:*

*(a) $f$ is $\beta$-perfect.*

*(b) For each filterbase $\mathcal{F}$ on $f(X)$ such that $\mathcal{F} \xrightarrow{\beta} y\,(\in f(X))$, $f^{-1}(\mathcal{F}) \xrightarrow{\beta-\mathrm{d}} f^{-1}(y)$.*

*(c) For any filterbase $\mathcal{F}$ on $X$, $\beta\text{-}ad\, f(\mathcal{F}) \subset f(\beta\text{-}ad\,\mathcal{F})$.*

*Proof.* '(a) $\implies$ (b)' follows from Definition 2.12 and Proposition 2.10. The proof of '(b) $\implies$ (c)' is quite similar to Theorem 3.1 of [7] by using Proposition 2.9 and $(*)$, while that of '(c) $\implies$ (a)' goes parallel to the proof of Theorem 3.1 ($(b) \implies (c)$) of [3].

REMARK 2.14. The above theorem along with Proposition 2.10 gives a generalized version of each of Result 2(e) [12], Theorem 3.1 [3] and Theorem 3.1 [7].

DEFINITION 2.15. A function $f\colon X \to Y$ is called $\beta$-closed iff $[f(A)]_\beta \subset f([A]_\beta)$, for each $A \subset X$.

The following result unifies Theorem 2.2 of [7] and Theorem 3.1.1((a) and (b)) of [3]. Its proof can be done in the same way as in Theorem 2.2 of [7] using Proposition 2.8, Theorem 2.13 and the fact that $\beta$ is increasing.

PROPOSITION 2.16. *A $\beta$-perfect function $f\colon X \to Y$ is $\beta$-closed.*

## 3. $\beta$- and $\gamma$-sets, $\beta$-continuity and $\beta$-perfect functions

We have just seen in Proposition 2.16 that a $\beta$-perfect function is $\beta$-closed. This motivates us to find conditions which along with the $\beta$-closedness of functions yield $\beta$-perfect functions. For this purpose we introduce $\beta$-sets as follows as a unified concept corresponding to compact sets, $\theta$-rigid sets [3] and $N$-sets [1] in the three particular situations under consideration, and ultimately characterize $\beta$-perfect functions in terms of this type of sets.

DEFINITION 3.1. A subset of a space $X$ is called a $\beta$-set iff for every filterbase $\mathcal{F}$ on $X$, $A \cap \beta\text{-ad}\,\mathcal{F} = \emptyset$ implies the existence of an $F \in \mathcal{F}$ such that $A \cap [F]_\beta = \emptyset$.

THEOREM 3.2. *If $f\colon X \to Y$ is $\beta$-closed such that $f^{-1}(y)$ is a $\beta$-set for each $y \in Y$, then $f$ is $\beta$-perfect.*

*Proof.* The proof is similar to that of Theorem 3.4 of [3], where we have to use Theorem 2.13 and Proposition 2.10.

The concept of $\theta$-continuity [4] ($\delta$-continuity [8]) is achieved by replacing the roles of identity and closure operators in the definition of continuity by respectively closure (interior-closure) and $\theta$-closure ($\delta$-closure) operators. The following definition unifies these concepts and the next result gives a unified form of Corollary 2.10.1 of [3], Th 2.2 ((1) $\implies$ (3)) of [8] and a well known result on continuity.

DEFINITION 3.3. A function $f\colon X \to Y$ is called $\beta$-continuous iff for each $x \in X$ and each $B \in \mathcal{B}_{f(X)}$ in $Y$, there exists $A \in \mathcal{B}_x$ in $X$ such that $f(\alpha(A)) \subset \alpha(B)$.

PROPOSITION 3.4. *If $f\colon X \to Y$ is $\beta$-continuous, then $f([A]_\beta) \subset [f(A)]_\beta$, for $A \subset X$.*

THEOREM 3.5. *If a $\beta$-continuous function $f\colon X \to Y$ is $\beta$-perfect, then $f$ is $\beta$-closed and $f^{-1}(y)$ is a $\beta$-set, for each $y \in Y$.*

*Proof.* A similar demonstration as in Theorem 3.4 of [3] using Theorem 2.12 and Proposition 2.15 furnishes the proof.

Combining Theorem 3.2 and Theorem 3.5 we obtain:

THEOREM 3.6. *A $\beta$-continuous function $f\colon X \to Y$ is $\beta$-perfect iff $f$ is $\beta$-closed and $f^{-1}(y)$ is a $\beta$-set, for each $y \in Y$.*

The following theorem which incidentally generalizes Theorem 3.4 of [3], follows at once from the above theorem and Proposition 3.4.

THEOREM 3.7 *A $\beta$-continuous function $f\colon X \to Y$ is $\beta$-perfect iff $f([A]_\beta) = [f(A)]_\beta$, for each $A \subset X$ and $f^{-1}(y)$ is a $\beta$-set, for each $y \in Y$.*

$QHC$ sets [9] and $N$-sets [1] are well known weaker forms of compact sets. The following definition of $\gamma$-sets takes all these concepts into account and gives a generalized version for the definitions of all such sets.

DEFINITION 3.8. A subset $A$ of $X$ is said to be a $\gamma$-set iff for every cover $\mathcal{U}$ of $A$ by members of $\mathcal{B}$, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subset \bigcup\{\, \alpha(B) : B \in \mathcal{U}_0 \,\}$.

THEOREM 3.9. *For a subset $A$ of a space $X$, the following are equivalent:*

*(a) $A$ is a $\gamma$-set.*

*(b) Every maximal filterbase on $X$ which meets $A$, $\beta$-converges to some point of $A$.*

*(c) Every filterbase on $X$ which meets $A$, has a $\beta$-adherent point in $A$.*

*Proof.* $(a) \implies (b)$: Let $A$ be a $\gamma$-set and suppose that $\mathcal{U}$ is a maximal filterbase on $X$ which meets $A$, but does not $\beta$-converge to any point of $A$. Then for each $x \in A$, there exists a $B_x \in \mathcal{B}_x$ such that $U \cap (X \setminus \alpha(B_x)) \neq \emptyset$, for every $U \in \mathcal{U}$. The maximality of $\mathcal{U}$ implies that $(X \setminus \alpha(B_x)) \in \mathcal{U}$. Then $U_x \cap \alpha(B_x) = \emptyset$, for some $U_x \in \mathcal{U}$. Since $\{ B_x : x \in A\}$ is a cover of $A$ by members of $\mathcal{B}$ and $A$ is a $\gamma$-set, we have $A \subset \bigcup_{i=1}^{n} \alpha(B_{x_i})$, where $\{x_1, \dots, x_n\}$ is a finite subset of $A$. We can find $U \in \mathcal{U}$ such that $U \cap A \subset (\bigcap_{i=1}^{n} U_{x_i}) \cap (\bigcup_{i=1}^{n} \alpha(B_{x_i})) = \emptyset$, and this is a contradiction, since $\mathcal{U}$ meets $A$.

$(b) \implies (c)$: Let $\mathcal{F}$ be a filterbase on $X$ which meets $A$. Then $\mathcal{F}$ is contained in a maximal filterbase $\mathcal{F}_0$ which meets $A$. Since $\mathcal{F}_0$ $\beta$-converges to some $x \in A$ (by (b)), for every $V \in \mathcal{B}_x$ there exists an $F_0 \in \mathcal{F}_0$ such that $F_0 \subset \alpha(V)$. Since $F \cap F_0 \neq \emptyset$ for each $F \in \mathcal{F}$, we have $\alpha(V) \cap F \neq \emptyset$, for all $F \in \mathcal{F}$ and all $V \in \mathcal{B}_x$. Hence $x$ $(\in A)$ is a $\beta$-adherent point of $\mathcal{F}$.

$(c) \implies (a)$: If $A$ is not a $\gamma$-set, there is a cover $\mathcal{U}$ of $A$ by members of $\mathcal{B}$ such that for every finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$, $A \setminus \bigcup_{U \in \mathcal{U}_0} \alpha(U) \neq \emptyset$. Then

$$\mathcal{F} = \{X \setminus \bigcup_{U \in \mathcal{U}_0} \alpha(U) \ : \ \mathcal{U}_0 \text{ is a finite subfamily of } \mathcal{U}\,\}$$

is a filterbase on $X$ which meets $A$. By (c), there exists $x \in A$ such that $x \in \beta\text{-ad}\,\mathcal{F}$. Since $\mathcal{U}$ is a cover of $A$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. Then $U_x \in \mathcal{B}_x$ and $X \setminus \alpha(U_x) \in \mathcal{F}$, so that $x \notin [X \setminus \alpha(U_x)]_\beta$ which is a contradiction.

REMARK 3.10. From the above theorem we get Theorem 2.6 [3], Theorem 2 [5], Lemma 3.3 [7] and some well known characterizations of compact sets as immediate consequences.

THEOREM 3.11 *If $f\colon X \to Y$ is $\beta$-perfect, then inverse images of $\gamma$-sets are $\gamma$-sets.*

*Proof.* Proceeding similarly as in Corollary 3.1.1 (c) of [3] and using Theorems 3.9 and 2.13, and the fact that $\beta$ is increasing the proof can be constructed.

REMARK 3.12. In the three particular situations, i.e. when $\alpha$ is taken to mean the identity (resp. closure, interior-closure) operator, the above theorem yields Corollary 2 of [12], Theorem 3.1.1 (c) of [3] and Theorem 3.4 of [7] respectively.

In certain usual situations it is found that there remains no difference between the notions of $\beta$-sets and $\gamma$-sets. For example, as we have already observed, such a collapse takes place when the operator $\alpha$ is taken to stand for the identity operator and $\mathcal{B}$ for the collection of all open sets in a topological space $X$. Under such a condition which we shall denote by 'Condition C' (i.e., when the concepts of $\beta$-sets and $\gamma$-sets coincide) and the assumption that singletons are $\gamma$-sets (which is trivially true in all the three particular situations under consideration and in many other cases), the requirement of $\beta$-continuity in Theorem 3.6 can be dispensed with, i.e., we have:

THEOREM 3.13. *Under 'Condition C' on $X$, a function $f\colon X \to Y$ is $\beta$-perfect iff $f$ is $\beta$-closed and $f^{-1}(y)$ is a $\gamma$-set for each $y \in Y$ (assuming the singletons of $Y$ to be $\gamma$-sets).*

*Proof.* It follows from Proposition 2.16, Theorem 3.2 and Theorem 3.11.

REMARK 3.14. It is easy to see that the above theorem together with Theorem 2.13 ($(a) \iff (b)$) is a unified form of Corollary 1 and Theorem 3 of [12], and Theorem 3.5 of [7].

DEFINITION 3.15. A space $X$ is said to be $\alpha$-separated if for each pair of distinct points $x$, $y$ of $X$, there exists $U_x \in \mathcal{B}_x$ and $U_y \in \mathcal{B}_y$ such that $\alpha(U_x) \cap \alpha(U_y) = \emptyset$.

It is clear that the above definition is an offshoot of the consideration of the analogy in the definitions of Hausdorff and Urysohn separation axioms. The next theorem and the corollary thereafter crop up as generalized formulations of Theorems 3.7 and 3.7.1 of [3] in our unified setting. As the proofs of these results here follow from the corresponding proofs of the above mentioned results of [3] with obvious modifications concerning the operators $\alpha$ and $\beta$, we omit the proofs.

THEOREM 3.16. *Let $f\colon X \to Y$ be $\beta$-continuous and $Y$ be an $\alpha$-separated space. Then $f$ is $\beta$-perfect iff for every filterbase $\mathcal{F}$ on $X$ if $f(\mathcal{F}) \xrightarrow{\beta} y \in f(X)$ then $\beta$-ad$\mathcal{F} \neq \emptyset$.*

COROLLARY 3.17. *If $f\colon X \to Y$ is $\beta$-continuous, $X$ is a $\gamma$-set and $Y$ is $\alpha$-separated, then $f$ is $\beta$-perfect.*

THEOREM 3.18. *The composition of $\beta$-perfect functions $f\colon X \to Y$ and $g\colon Y \to Z$ is $\beta$-perfect if 'Condition C' is satisfied on $X$, and singletons of $Z$ are $\gamma$-sets.*

*Proof.* In view of Proposition 2.16, $g \circ f$ is clearly a $\beta$-closed function. If $z \in Z$, then $g^{-1}(z)$ is a $\gamma$-set. As $f$ is $\beta$-perfect, by Theorem 3.11 it follows that $f^{-1}(g^{-1}(z))$ is a $\gamma$-set in $X$. Hence $g \circ f$ is $\beta$-perfect, by Theorem 3.13.

REMARK 3.19. The above theorem unifies Theorem e (1.8) of [10] and Theorem 3.1 of [2].

## 4. Conclusions

In the last two sections we have shown as to how the three widely studied types of functions viz. perfect, $\theta$-perfect and $\delta$-perfect functions are achieved as special cases of $\beta$-perfect functions when the operator $\alpha$ stands respectively for the identity, closure and interior-closure operator. Also, the existing analogous results concerning these types of functions have been unified from an algebraic standpoint minus the aid to topological structure. Apart from the results encompassed so far, we now derive certain new results on $\delta$-perfect functions from Theorem 3.16 and Corollary 3.17, when $\alpha$ becomes the interior-closure operator. These are formulated as follows.

THEOREM 4.1. *Let $X$, $Y$ be topological spaces and $f\colon X \to Y$ be $\delta$-continuous with $Y$ a Hausdorff space. Then $f$ is $\delta$-perfect iff for any filterbase $\mathcal{F}$ on $X$, $f(\mathcal{F})$ $\delta$-converges to $y$ $(\in Y)$ implies* $\delta$-ad $\mathcal{F} \neq \emptyset$.

THEOREM 4.2. *Let $X$ be a nearly compact space and $Y$ a $T_2$ topological space. Then a $\delta$-continuous function $f\colon X \to Y$ is $\delta$-perfect.*

Moreover, Theorem 3.2 shows that in the converse of Theorem 3.4 of Dickman and Porter [3], the condition of $\theta$-continuity is superfluous; also the condition of almost closedness on $f$ can be weakened by the condition $[f(A)]_\theta \subset f([A]_\theta)$.

We conclude with the remark that many other perfect-like functions than those considered so far, could be accomplished by assigning different interpretations to the operators $\alpha$ and $\beta$ and/or by replacing the collection $\mathcal{B}$ of open sets by some other collection like $\alpha$-open sets [6] or regulary open sets etc.

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