

TWO ESTIMATES FOR INTEGRAL MEANS OF ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this note two estimates for integral means of analytic functions in the unit disc are obtained. It is also shown that both estimates are sharp.

If f is an analytic function in the unit disc $D = \{z : |z| < 1\}$, as usual, integral mean of order p , $0 < p < \infty$, of function f , is defined by

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \quad 0 < r < 1.$$

To prove our main result we need two Littlewood-Paley's theorems.

THEOREM 1. [2, p. 315] *Let $0 < p \leq 2$. Then there exists a constant $C > 0$ such that*

$$M_p^p(r, f) \leq C \int_0^1 (1 - \rho)^{p-1} M_p^p(r\rho, f') d\rho, \quad (1)$$

for every f analytic in D .

THEOREM 2. [2, p. 316] *Let $2 \leq p < \infty$. There exists a constant $C > 0$ such that*

$$M_p^2(r, f) \leq C \int_0^1 (1 - \rho) M_p^2(r\rho, f') d\rho, \quad (2)$$

for every f analytic in D .

The following Lemma will also be needed.

LEMMA 1. [2] *If $0 < p < \infty$, then there exists a constant $C > 0$ such that*

$$\int_0^1 (1 - \rho)^{p-1} (1 - r\rho)^{-p} d\rho \leq C \log \frac{1}{1-r}, \quad 0 < r < 1.$$

THEOREM. Let $M_p(r, f') = O((1-r)^{-1})$. Then

a) if $0 < p \leq 2$, then $M_p^p(r, f) = O(\log(1-r)^{-1})$;

b) if $2 \leq p < \infty$, then $M_p^2(r, f) = O(\log(1-r)^{-1})$.

Both estimates are sharp.

Note that from the same assumption $M_p(r, f') = O((1-r)^{-1})$ two different conclusions follows:

$$M_p(r, f) = O\left(\log^{1/p} \frac{1}{1-r}\right), \quad \text{if } 0 < p \leq 2,$$

$$M_p(r, f) = O\left(\log^{1/2} \frac{1}{1-r}\right), \quad \text{if } 2 \leq p < \infty.$$

Proof. a) Using (1) we find that

$$M_p^p(r, f) \leq C \int_0^1 (1-\rho)^{p-1} (1-r\rho)^{-p} d\rho.$$

Now a) follows by Lemma 1.

We use C to denote a positive constant not necessarily the same on any two occurrences.

b) By assumption,

$$M_p(r\rho, f') \leq \frac{C}{1-r\rho}. \quad (3)$$

From (2) and (3) we conclude that

$$M_p^2(r, f) \leq C \int_0^1 (1-\rho)(1-r\rho)^{-2} d\rho.$$

Using Lemma 1 ($p = 2$) and (3) we find that $M_p^2(r, f) \leq C \log((1-r)^{-1})$.

We show that both estimates are sharp in the sense that there exists f , analytic in D , such that $M_p(r, f') \leq C(1-r)^{-1}$, $0 < p \leq 2$, and

$$C^{-1} \log \frac{1}{1-r} \leq M_p^p(r, f) \leq C \log \frac{1}{1-r}, \quad (4)$$

and that there exists g , analytic in D , such that $M_p(r, g') \leq C(1-r)^{-1}$, $2 \leq p < \infty$, and

$$C^{-1} \log \frac{1}{1-r} \leq M_p^2(r, f) \leq C \log \frac{1}{1-r}. \quad (5)$$

We show that the functions f and g defined by $f(z) = (1-z)^{-1/p}$ and $g(z) = \sum_{n=0}^{\infty} z^{2^n}$ satisfy conditions mentioned above. Using Lemma [1, p. 65] we find that

$$M_p(r, f') \leq C \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dt}{|1-re^{it}|^{p+1}} \right)^{1/p} \leq \frac{C}{1-r}.$$

On the other hand, $M_p^p(r, f) = (2\pi)^{-1} \int_0^{2\pi} dt/|1 - re^{it}|$. It is easily seen that there exists a constant $C > 0$ so that

$$C^{-1} \log \frac{1}{1-r} \leq \int_0^{2\pi} \frac{dt}{|1 - re^{it}|} \leq C \log \frac{1}{1-r}.$$

Hence, we obtain (4).

Using inequality $M_p(r, g') \leq M_\infty(r, g')$, where $M_\infty(r, g') = \sup_{0 \leq t \leq 2\pi} |g'(re^{it})|$, we find that

$$M_p(r, g') \leq \sum_{n=0}^{\infty} 2^n r^{2^n - 1} \leq C \frac{1}{1-r}$$

(for the last inequality see [1, p. 66]). In [2, p. 316] it is proved that (5) is valid.

This completes the proof of the Theorem. ■

REFERENCES

- [1] P. Duran, *Theory of H^p spaces*, Academic Press, New York 1970.
- [2] A. Zygmund, *Trigonometric Series*, Volume III, University Press, Cambridge 1959.

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