

## ON THE CONVERGENCE OF A THREE-LEVEL VECTOR SOR SCHEME

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**Abstract.** In this paper we consider a vector alternating directions difference scheme for solving multidimensional wave equation. The scheme reduces to a modified block successive overrelaxation (SOR) algorithm. The stability and the convergence of the scheme are investigated.

As a model problem let us consider the first initial-boundary value problem (IBVP) for the multidimensional wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \Delta u + f, & (x, t) \in Q = \Omega \times (0, T) = (0, 1)^n \times (0, T), \\ u(x, 0) &= u_0(x), & \frac{\partial u(x, 0)}{\partial t} = u_1(x), & x \in \Omega, \\ u(x, t) &= 0, & x \in \Gamma = \partial\Omega, & t \in (0, T). \end{aligned} \quad (1)$$

We assume that the generalized solution of IBVP (1) belongs to the Sobolev space  $W_2^s(Q)$ ,  $s \geq 2$  [7]. In this case there exists a trace  $u|_{t=t'} \in W_2^{s-1/2}(\Omega) \subset L_2(\Omega)$  for  $t' \in [0, T]$ . We also assume that the solution  $u$  can be oddly extended in space variables outside the domain  $\Omega$ , preserving the Sobolev class.

Let  $\bar{\omega}$  be uniform mesh in  $\bar{\Omega}$  with step size  $h$ . Let us denote  $\omega = \bar{\omega} \cap \Omega$ ,  $\gamma = \bar{\omega} \setminus \omega$  and  $\omega_i = \omega \cup \{x = (x_1, \dots, x_n) \in \gamma \mid x_i = 0\}$ . Let  $\bar{\theta}$  be uniform mesh on  $[-\tau/2, T]$  with step size  $\tau$ , and  $\theta = \bar{\theta} \cap (0, T)$ . Finally, let  $\bar{Q}_{h\tau} = \bar{\omega} \times \bar{\theta}$ . For a function  $v$  defined on the mesh  $\bar{Q}_{h\tau}$  we define the finite-difference operators  $v_{x_i}$ ,  $v_{\bar{x}_i}$ ,  $v_t$  and  $v_{\bar{t}}$  in the usual manner [8]. Let us denote  $v = v(x, t)$ ,  $\hat{v} = v(x, t + \tau)$  and  $\check{v} = v(x, t - \tau)$ .

Let  $H_h$  be the set of discrete functions defined on the mesh  $\bar{\omega}$ , which vanish on  $\gamma$ . Let us denote

$$A_i v = \begin{cases} -v_{x_i \bar{x}_i}, & x \in \omega \\ 0, & x \in \gamma \end{cases} \quad \text{and} \quad Av = \sum_{i=1}^n A_i v.$$

The identity operator on  $H_h$  will be denoted by  $I$ .

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We introduce the following discrete inner product  $(v, w)_\omega = h^n \sum_{x \in \omega} v(x) w(x)$  and the norms

$$\|v\|_\omega = (v, v)_\omega^{1/2} = \left( h^n \sum_{x \in \omega} v^2(x) \right)^{1/2} \quad \text{and} \quad \|v\|_{\omega_i} = \left( h^n \sum_{x \in \omega_i} v^2(x) \right)^{1/2}.$$

For a linear, selfadjoint and nonnegative operator  $A$  on  $H_h$  with  $\|v\|_A$  we denote so called "energy" seminorm  $\|v\|_A = (Av, v)_\omega^{1/2}$ . In particular

$$\|v\|_{A_i} = (A_i v, v)_\omega^{1/2} = \|v_{x_i}\|_{\omega_i}.$$

With  $T_i$  and  $T_t$  we denote the Steklov averaging operators in space variables  $x_i$  and time variable  $t$  (see [4])

$$T_i f(x, t) = \frac{1}{h} \int_{x_i-h/2}^{x_i+h/2} f(x_1, \dots, x'_i, \dots, x_n, t) dx'_i,$$

$$T_t f(x, t) = \frac{1}{\tau} \int_{t-\tau/2}^{t+\tau/2} f(x_1, \dots, x_n, t') dt'.$$

Finally,  $C$  will stand for a positive generic constant, independent of  $h$  and  $\tau$ .

We approximate IBVP (1) by the following alternating direction finite-difference scheme (FDS) (see [1], [9])

$$v_{i\bar{t}}^i + \frac{1}{2} \sum_{j=1}^{i-1} (A_j \hat{v}^j + A_j \check{v}^j) + \frac{1}{2\sigma} A_i (\hat{v}^i + \check{v}^i)$$

$$+ \left(1 - \frac{1}{\sigma}\right) A_i v^i + \sum_{j=i+1}^n A_j v^j = T_1 \cdots T_n T_t f, \quad t \in \theta, \quad (2)$$

$$v^i|_{t=\mp\tau/2} = T_1 \cdots T_n (u_0 \mp 0.5 \tau u_1), \quad i = 1, 2, \dots, n.$$

Here  $\sigma$  is a positive free parameter. Equation (2) can be rewritten in the form

$$\left(I + \frac{\tau^2}{2\sigma} A_i\right) v_{i\bar{t}}^i + \frac{\tau^2}{2} \sum_{j=1}^{i-1} A_j v_{i\bar{t}}^j + \sum_{j=1}^n A_j v^j = T_1 \cdots T_n T_t f, \quad t \in \theta.$$

FDS (2) is economic since the evaluation of  $v^i$  on the next time level reduces to the inversion of the operator  $I + \frac{\tau^2}{2\sigma} A_i$ , represented by threediagonal matrix. It can be treated as a operator (vector) variant of the successive overrelaxation method [3]. A similar FDS, close to Jacobi overrelaxation method (JOR) was considered in [2] and [5].

The errors defined as  $z^i = T_1 \cdots T_n u - v^i$  satisfy FDS

$$\left(I + \frac{\tau^2}{2\sigma} A_i\right) z_{i\bar{t}}^i + \frac{\tau^2}{2} \sum_{j=1}^{i-1} A_j z_{i\bar{t}}^j + \sum_{j=1}^n A_j z^j = \varphi^i, \quad t \in \theta, \quad (3)$$

$$z_t^i|_{t=-\tau/2} = \beta, \quad 0.5(z^i + \hat{z}^i)|_{t=-\tau/2} = \delta, \quad i = 1, 2, \dots, n$$

where

$$\begin{aligned}\varphi^i &= \xi + \sum_{j=1}^n \eta^j - \frac{\tau^2}{2} \sum_{j=1}^{i-1} \zeta^j - \frac{\tau^2}{2\sigma} \zeta^i, \\ \xi &= T_1 \cdots T_n \left( u_{t\bar{t}} - T_t \frac{\partial^2 u}{\partial t^2} \right), \\ \eta^j &= T_1 \cdots T_n \left( T_t \frac{\partial^2 u}{\partial x_j^2} - u_{x_j \bar{x}_j} \right), \\ \zeta^j &= \frac{\tau^2}{2} T_1 \cdots T_n u_{x_j \bar{x}_j t \bar{t}}, \\ \beta &= T_1 \cdots T_n \left( T_t \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} \right) \Big|_{t=0}, \\ \delta &= 0.5 T_1 \cdots T_n \left( u|_{t=-\tau/2} - 2 u|_{t=0} + u|_{t=\tau/2} \right).\end{aligned}$$

To prove the stability and the convergence of FDS (2) we represent equation (3) in the matrix form

$$\begin{aligned}\left[ \left( \mathbf{I} + \frac{\tau^2}{2} \left( \mathbf{L} + \frac{1}{\sigma} \mathbf{I} \right) \Lambda \right) \mathbf{z}_{t\bar{t}} + \mathbf{E} \Lambda \mathbf{z} \right] &= \Phi, \quad t \in \theta, \\ \mathbf{z}_t|_{t=-\tau/2} &= \mathbf{b}, \quad 0.5(\mathbf{z} + \hat{\mathbf{z}})|_{t=-\tau/2} = \mathbf{d},\end{aligned}\tag{4}$$

where  $\mathbf{z} = (z^1, \dots, z^n)^T$ ,  $\Phi = (\varphi^1, \dots, \varphi^n)^T$ ,  $\mathbf{I} = \text{diag}(I, \dots, I)$ ,  $\Lambda = \text{diag}(\Lambda_1, \dots, \Lambda_n)$ ,  $\mathbf{b} = (\beta, \dots, \beta)^T$ ,  $\mathbf{d} = (\delta, \dots, \delta)^T$ ,

$$\mathbf{L} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ I & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I & I & \dots & I & 0 \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} 0 & I & \dots & I & I \\ 0 & 0 & \dots & I & I \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{E} = \mathbf{L} + \mathbf{I} + \mathbf{U}.$$

Let us also define the inner product and the associated norm of vector-functions

$$(\mathbf{z}, \mathbf{w}) = \sum_{i=1}^n (z^i, w^i)_\omega, \quad \|\mathbf{z}\| = (\mathbf{z}, \mathbf{z})^{1/2}.$$

For a three-level FDS in the form

$$\mathbf{C} \mathbf{z}_{t\bar{t}} + \mathbf{A} \mathbf{z} = \Psi,\tag{5}$$

using the energy method [8] one easily prove the following proposition.

**LEMMA 1.** *If  $\mathbf{A} = \mathbf{A}^* \geq \mathbf{0}$ , and  $\mathbf{C} - 0.25 \tau^2 \mathbf{A} \geq \mathbf{D} = \mathbf{D}^* > \mathbf{0}$  then FDS (5) is stable and the a priori estimate*

$$\max_{t \in \theta} N(\mathbf{z}) \leq N(\mathbf{z})|_{t=-\tau/2} + \tau \sum_{t \in \theta} \|\Psi\|_{\mathbf{D}^{-1}},$$

where

$$N^2(\mathbf{z}) = \|\mathbf{z}_t\|_{\mathbf{C}-0.25\tau^2\mathbf{A}}^2 + \left\| \frac{\mathbf{z} + \hat{\mathbf{z}}}{2} \right\|_{\mathbf{A}}^2$$

holds.

Applying  $\Lambda$  to (4) we obtain a FDS in the canonical form (5), where  $\mathbf{A} = \Lambda \mathbf{E} \Lambda = \mathbf{A}^* \geq \mathbf{0}$ ,  $\mathbf{C} = \Lambda + \frac{\tau^2}{2} \Lambda \mathbf{L} \Lambda + \frac{\tau^2}{2\sigma} \Lambda^2$  and  $\Psi = \Lambda \Phi$ . One can easily acknowledge that

$$((\mathbf{C} - 0.25\tau^2\mathbf{A})\mathbf{z}, \mathbf{z}) = (\Lambda\mathbf{z}, \mathbf{z}) + \tau^2 \frac{2-\sigma}{4\sigma} (\Lambda\mathbf{z}, \Lambda\mathbf{z}),$$

which means that for  $0 < \sigma < 2$ ,  $\mathbf{C} - 0.25\tau^2\mathbf{A} \geq \Lambda > \mathbf{0}$ , and, consequently, FDS (5) is stable.

From lemma 1 we obtain the a priori estimate

$$\max_{t \in \theta} N(\mathbf{z}) \leq N(\mathbf{z})|_{t=-\tau/2} + \tau \sum_{t \in \theta} \|\Psi\|_{\Lambda^{-1}}. \quad (6)$$

Further

$$\begin{aligned} N^2(\mathbf{z}) &= \|\mathbf{z}_t\|_{\mathbf{C}-0.25\tau^2\mathbf{A}}^2 + \left\| \frac{\mathbf{z} + \hat{\mathbf{z}}}{2} \right\|_{\mathbf{A}}^2 \geq \|\mathbf{z}_t\|_{\Lambda}^2 + \left\| \Lambda \frac{\mathbf{z} + \hat{\mathbf{z}}}{2} \right\|_{\mathbf{E}}^2 \\ &= \sum_{i=1}^n \|z_t^i\|_{\Lambda_i}^2 + \left\| \sum_{i=1}^n \Lambda_i \frac{z^i + \hat{z}^i}{2} \right\|_{\omega}^2 \equiv \|\mathbf{z}\|_2^2, \\ N^2(\mathbf{z})|_{t=-\tau/2} &= \|\mathbf{b}\|_{\mathbf{C}-0.25\tau^2\mathbf{A}}^2 + \|\mathbf{d}\|_{\mathbf{A}}^2 \\ &= \sum_{i=1}^n \left( \|\beta\|_{\Lambda_i}^2 + \tau^2 \frac{2-\sigma}{4\sigma} \|\Lambda_i \beta\|_{\omega}^2 \right) + \left\| \sum_{i=1}^n \Lambda_i \delta \right\|_{\omega}^2, \\ \|\Psi\|_{\Lambda^{-1}} &= \|\Phi\|_{\Lambda} = \left( \sum_{i=1}^n \|\varphi^i\|_{\Lambda_i}^2 \right)^{1/2}. \end{aligned}$$

Substituting in (6), when  $\tau \asymp h$  (i.e.  $C_1 h \leq \tau \leq C_2 h$ ), we obtain

$$\max_{t \in \theta} \|\mathbf{z}\|_2 \leq C \sum_{i=1}^n \left( \|\beta_{x_i}\|_{\omega_i} + \|\delta_{x_i \bar{x}_i}\|_{\omega} + \tau \sum_{t \in \theta} \|\varphi_{x_i}^i\|_{\omega_i} \right). \quad (7)$$

To prove the convergence of FDS (2) we must estimate the terms  $\varphi_{x_i}^i = \xi_{x_i} + \sum_{j=1}^n \eta_{x_i}^j - \frac{\tau^2}{2} \sum_{j=1}^{i-1} \zeta_{x_i}^j - \frac{\tau^2}{2\sigma} \zeta_{x_i}^i$ ,  $\beta_{x_i}$  and  $\delta_{x_i \bar{x}_i}$ . The value of  $\xi_{x_i}$  in the node  $(x, t) \in \omega_i \times \theta$  is a bounded linear functional of  $u \in W_2^s(e)$ , where  $e = \prod_{l=1}^n (x_l - 2h, x_l + 2h) \times (t - \tau, t + \tau)$  and  $s \geq 2$ . Moreover,  $\xi_{x_i}^j$  vanishes on the polynomials of the fourth degree. Using the Bramble-Hilbert lemma and the methodology proposed in [6] and developed in [4], for  $\tau \asymp h$ , we obtain

$$|\xi_{x_i}| \leq C h^{s-4-n/2} |u|_{W_2^s(e)}, \quad 3 \leq s \leq 5.$$

From here, by summation over the meshes  $\omega_i$  and  $\theta$ , it follows

$$\tau \sum_{t \in \theta} \|\xi_{x_i}\|_{\omega_i} \leq C h^{s-3} \|u\|_{W_2^s(Q)}, \quad 3 \leq s \leq 5.$$

In the same manner we can estimate  $\eta_{x_i}^j$ ,  $\zeta_{x_i}^j$ ,  $\beta_{x_i}$  and  $\delta_{x_i \bar{x}_i}$ . From these estimates and the inequality (7) we obtain the following convergence rate estimate for FDS (2):

$$\max_{t \in \theta} \|\mathbf{z}\|_2 \leq C h^{s-3} \|u\|_{W_2^s(Q)}, \quad 3 \leq s \leq 5. \quad (8)$$

Another group of convergence rate estimates can be obtained in the following way. Denoting

$$z = \Lambda^{-1} \sum_{i=1}^n A_i z^i,$$

from (3) we obtain

$$\left(I + \frac{\tau^2}{2\sigma} A_i\right) z_{t\bar{t}}^i + \frac{\tau^2}{2} \sum_{j=1}^{i-1} A_j z_{t\bar{t}}^j + \Lambda z = \varphi^i.$$

After solving in  $z_{t\bar{t}}^i$ , applying the operator  $A_i$  and summing over  $i$  we obtain

$$\begin{aligned} \mathbf{z}_{t\bar{t}} + \mathbf{A} \mathbf{z} &= \Psi, & t \in \theta, \\ \mathbf{z}_t|_{t=-\tau/2} &= \beta, & 0.5(\mathbf{z} + \hat{\mathbf{z}})|_{t=-\tau/2} = \delta, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \mathbf{A} &= \sum_{i=1}^n A_i, & \Psi &= \Lambda^{-1} \sum_{i=1}^n \hat{A}_i \varphi^i. \\ A_i &= A_i \left(I + \frac{\tau^2}{2\sigma} A_i\right)^{-1} \prod_{j=1}^{i-1} \left(I + \frac{\tau^2}{2\sigma} A_j\right)^{-1} \left[I + \frac{\tau^2}{2} \left(\frac{1}{\sigma} - 1\right) A_j\right], \\ \hat{A}_i &= A_i \left(I + \frac{\tau^2}{2\sigma} A_i\right)^{-1} \prod_{j=i+1}^n \left(I + \frac{\tau^2}{2\sigma} A_j\right)^{-1} \left[I + \frac{\tau^2}{2} \left(\frac{1}{\sigma} - 1\right) A_j\right]. \end{aligned}$$

For  $0 < \sigma < 2$  we have

$$\begin{aligned} -\Lambda_i \leq A_i = A_i^* \leq \Lambda_i, & \quad -\Lambda_i \leq \hat{A}_i = \hat{A}_i^* \leq \Lambda_i, & \quad 0 < \mathbf{A} = \mathbf{A}^* \leq \Lambda \quad \text{and} \\ I - \frac{\tau^2}{4} \mathbf{A} &= \frac{1}{2} \prod_{j=1}^n \left(I + \frac{\tau^2}{2\sigma} A_j\right)^{-1} \left\{ \prod_{j=1}^n \left(I + \frac{\tau^2}{2\sigma} A_j\right) + \prod_{j=1}^n \left[I + \frac{\tau^2}{2} \left(\frac{1}{\sigma} - 1\right) A_j\right] \right\} \\ &\geq \prod_{j=1}^n \left(I + \frac{\tau^2}{2\sigma} A_j\right)^{-1} > 0. \end{aligned}$$

In such a way, accordingly to lemma 1, FDS (9) is unconditionally stable. In the case when  $\tau \asymp h$  we also have

$$I - 0.25 \tau^2 \mathbf{A} \geq cI, \quad \mathbf{A} \geq c\Lambda,$$

where  $c$  is a positive constant.

Using these relations and lemma 1 we obtain the a priori estimate

$$\begin{aligned} \max_{t \in \theta} \|z\|_1 &\equiv \max_{t \in \theta} \left( \|z_t\|_\omega^2 + \left\| \frac{z + \hat{z}}{2} \right\|_{\Lambda}^2 \right)^{1/2} \\ &\leq C \left( \|\beta\|_\omega + \sum_{i=1}^n \|\delta_{x_i}\|_{\omega_i} + \tau \sum_{t \in \theta} \sum_{i=1}^n \|\varphi^i\|_\omega \right). \end{aligned} \quad (10)$$

Similarly, applying operator  $\mathbf{A}^{k-1}$  ( $k = 2, 3, \dots$ ) to (9) and repeating the same procedure, we obtain

$$\begin{aligned} \max_{t \in \theta} \|z\|_k &\equiv \max_{t \in \theta} \left( \|z_t\|_{\Lambda^{k-1}}^2 + \left\| \frac{z + \hat{z}}{2} \right\|_{\Lambda^k}^2 \right)^{1/2} \\ &\leq C \left( \|\beta\|_{\Lambda^{k-1}} + \|\delta\|_{\Lambda^k} + \tau \sum_{t \in \theta} \sum_{i=1}^n \|\varphi^i\|_{\Lambda^{k-1}} \right). \end{aligned} \quad (11)$$

In such a way, the problem of deriving the convergence rate estimate for FDS (9), or (2), is now reduced to estimation of the right hand side terms in (10) and (11). Using the Bramble-Hilbert lemma, in the same manner as in the previous case, from (10–11) we obtain

$$\max_{t \in \theta} \|z\|_k \leq C h^{s-k-1} \|u\|_{W_2^s(Q)}, \quad k+1 \leq s \leq k+3; \quad k = 1, 2, \dots \quad (12)$$

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