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About a Decomposition of the Space of Symmetric Tensors of Compact Support on a Riemannian Manifold

O. Gil-Medrano and A. Montesinos Amilibia

Abstract. Let M be a noncompact manifold and let $\Gamma_c^\infty(S^2(M))$ (respectively $\Gamma_c^\infty(T^1(M))$) be the LF space of 2-covariant symmetric tensor fields (resp. 1-forms) on M, with compact support. Given any Riemannian metric g on M, the first-order differential operator $\delta^* : \Gamma_c^\infty(T^1(M)) \to \Gamma_c^\infty(S^2(M))$ can be defined by $\delta^*\omega = 2\,\mathrm{symm}\,\nabla\omega$, where ∇ denotes the Levi-Civita connection of g.

The aim of this paper is to prove that the subspace $\mathrm{Im}\,\delta^*$ is closed and to show several examples of Riemannian manifolds for which $\Gamma_c^\infty(S^2(M)) \neq \mathrm{Im}\,\delta^* \oplus (\mathrm{Im}\,\delta^*)^\perp$, where orthogonal is taken with respect to the usual inner product defined by the metric.

1. Introduction

Let M be a smooth manifold. If M is compact, the space $\Gamma^\infty(S^2(M))$ of 2-covariant symmetric tensor fields on M, endowed with the C^∞-topology, is a Fréchet space. In 1969 Berger and Ebin [Be-Eb] studied some decompositions of that space and in particular they showed that, for any fixed Riemannian metric g on M, the space $\Gamma^\infty(S^2(M))$ splits into two orthogonal, complementary, closed subspaces. One of them is the image of the first-order differential operator δ_g^* defined on the space $\Gamma^\infty(T^1(M))$ of 1-forms on M by $\delta_g^*\omega = 2\ \mathrm{symm}\,\nabla\omega$, where ∇ is the Levi-Civita connection of g. The other subspace is $\ker\delta_g$, where δ_g is the divergence operator induced by the metric; it is the adjoint of δ_g^* with respect to the usual inner product of tensor fields defined by g.

This splitting has been used in the study of Riemannian functionals (see [Bes, Ch. 4]) because it is the infinitesimal version of the slice theorem for the space $\mathcal{M}$ of Riemannian metrics on M [Ebi]. Ebin's result asserts that if M is a compact, orientable manifold without boundary then, at each metric, there is a slice for the usual action of the group $\mathcal{G}$ of diffeomorphisms of M on the space $\mathcal{M}$. Since $\mathcal{M}$ is an open convex cone in $\Gamma^\infty(S^2(M))$ it carries a natural structure of Fréchet manifold such that, for each $g \in \mathcal{M}$ the tangent space at g, $T_g\mathcal{M}$, is $\Gamma^\infty(S^2(M))$;

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the tangent space to the orbit $\mathcal{O}_g$ through g is $T_g\mathcal{O}_g = \operatorname{Im}\delta_g^*$ (see [Ebi, p.25]). So, the decomposition result can be read as $T_g\mathcal{M} = (T_g\mathcal{O}_g) \oplus (T_g\mathcal{O}_g)^\perp$, where orthogonal is taken with respect to the $\mathcal{G}$-invariant metric G given by

$$G_g(h,k) = \int_M \operatorname{Tr}(g^{-1}hg^{-1}k)\ dv_g.$$

Ebin's slices are of the form $\exp_g(U)$ where U is an open neighbourhood of zero in $(T_g\mathcal{O}_g)^\perp$ and $\exp_g$ is the exponential map, at g, of the metric G.

If the manifold M is not compact $\mathcal{M}$ can be endowed with a differentiable structure such that $T_g\mathcal{M}$ is the LF-space $\Gamma_c^\infty(S^2(M))$ of sections with compact support (see 2.5 for a description). The subspace $\delta_g^*(\Gamma_c^\infty(T^1(M)))$ is also the tangent space to the orbit $\mathcal{O}_g$ and the $\mathcal{G}$-invariant metric G on $\mathcal{M}$ can be defined as in the compact case; the geometry of $(\mathcal{M}, G)$, in particular its exponential map, is completely analogous (see [GM-Mi]). Nevertheless in this paper we show that concerning the splitting of $\Gamma_c^\infty(S^2(M))$ the behaviour in the noncompact case is rather different.

Let us recall that in [Be-Eb] the algebraic decomposition $\Gamma^\infty(S^2(M)) = \operatorname{Im}\delta_g^* \oplus \ker\delta_g$ is obtained by using the theory of elliptic differential operators on sections of vector bundles over a compact manifold. The closedness of $\operatorname{Im}\delta_g^*$ follows essentially as a consequence of the decomposition; so, that direct sum is also topological.

For a noncompact manifold that procedure is not available; in fact we give in §7 several examples of Riemannian manifolds for which the algebraic decomposition does not hold.

The first kind of examples are those manifolds admitting an infinitesimal affine transformation which is not a Killing vector field; $\mathbb{R}^n$ with the Euclidean metric among them. By means of the solutions of the corresponding elliptic boundary problem on a compact, connected, orientable manifold with boundary we find that for the interior of such a manifold the decomposition is never true.

With the same technique we also obtain (Corollary 7.20) a characterization of the decomposable elements of $\Gamma_c^\infty(S^2(M))$, valid for every noncompact M.

The greater part of the work is devoted to show that $\operatorname{Im}\delta^*$ is a closed subspace of $\Gamma_c^\infty(S^2(M))$ when M is a noncompact manifold. Our result is not a generalization of the corresponding one in [Be-Eb] because our method does not apply if M is compact: in some sense, it should be considered as being complementary.

To obtain that result we first need a description of the involved topologies in terms of a given Riemannian metric and its Levi-Civita connection; this is done in §2. In paragraphs 3 and 4 we prove several results concerning the operator δ^* and some related topics that are used in §5 to study the restriction of δ^* to $\Gamma_K^\infty(T^1(N))$ where N is a submanifold of M of the form $N = M \setminus G$, with G an open subset of M such that ∂G is compact and regular and N is connected, and where K is a compact subset of N, $K \neq N$. We obtain that, under these hypotheses, $\delta^*(\Gamma_K^\infty(T^1(N)))$ is closed in $\Gamma_K^\infty(S^2(N))$ and that δ^* is a homeomorphism onto its image (Corollary 5.5).

These results and several lemmas of diverse nature allow us to prove the closedness of $\operatorname{Im}\delta^*$ (Proposition 6.6).

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2. Definitions of Some Topologies in Spaces of Tensor Fields

Several topologies can be defined on sets of differentiable maps between manifolds by using the adequate jet space. In this paragraph we will consider the special case of sections of a tensor bundle over a manifold and we will describe some of these topologies in terms of a Riemannian metric and its Levi-Civita connection.

Let M be a differentiable manifold and let $T^s(M)$ denote the vector bundle of s-times covariant tensors on M. For l being a nonnegative integer (or $l = \infty$), $\Gamma^l(T^s(M))$ will represent the sections of class C^l, we will use $\Gamma^l_A(T^s(M))$ for those with support in $A \subset M$ and $\Gamma^l_c(T^s(M))$ for those with compact support.

2.1. Let g be a given Riemannian metric on M, and ∇ its Levi-Civita connection. If $L \in \Gamma(T^r(M))$ and $x \in M$ we take $\|L(x)\|$ to be the usual norm in $\mathcal{L}^r(T_xM;\mathbb{R})$ when T_xM is considered with the norm induced by g_x. For each $0 < l < \infty$, given $T \in \Gamma^l(T^s(M))$, one has the covariant derivative of T, $\nabla T \in \Gamma^{l-1}(T^{s+1}(M))$ and in general if $0 < j \leq l$, $\nabla^j T \in \Gamma^{l-j}(T^{s+j}(M))$ is defined recurrently; then, for $x \in M$, $(T(x), \dots, \nabla^l T(x))$ is an element of $\mathcal{L}^s(T_xM;\mathbb{R}) \times \cdots \times \mathcal{L}^{s+l}(T_xM;\mathbb{R})$ and $\|(T(x), \dots, \nabla^l T(x))\|$ will be the norm in the product, given by the maximum of the norms in each space, and will be denoted by $|T|_{C^l,x}$.

Then, for each compact $K \subset M$ and for each $0 \leq l < \infty$, we can define in $\Gamma^l(T^s(M))$ the following semi-norm

$$|T|_{C^l,K} = \sup\{\ |T|_{C^l,x}\ ;\ x \in K\}.$$

When restricted to $\Gamma^l_K(T^s(M))$ it is a norm, the induced topology is the C^l topology and hence does not depend on the given metric g. Sequential convergence is the uniform convergence of the tensor field and its derivatives up to the order l. $\Gamma^l_K(T^s(M))$ is a Banach space.

2.2. For each l we have the inclusion $\Gamma^\infty_K(T^s(M)) \subset \Gamma^l_K(T^s(M))$, but the subspace is not closed and consequently it is not a Banach space. One can then define on it the quasi-norm:

$$\|T\|_K = \sum_{l=0}^{\infty} \frac{1}{2^l}\ \frac{|T|_{C^l,K}}{1 + |T|_{C^l,K}}.$$

The topology given by that quasi-norm is the C^∞ topology; it can also be described as the weak topology defined by the inclusions; sequential convergence is the uniform convergence of the tensor field and all of its derivatives. $\Gamma^\infty_K(T^s(M))$ is a Fréchet space.

The quasi-norm $\|\ \|_K$ is equivalent to the quasi-norm $|T|_K = \sup\{\ |T|_x\ ;\ x \in K\}$, where

$$|T|_x = \sum_{l=0}^{\infty} \frac{1}{2^l}\ \frac{|T|_{C^l,x}}{1 + |T|_{C^l,x}}.$$

2.3. $\Gamma^\infty(T^s(M))$ can also be endowed with a Fréchet space structure. When M is compact it is enough to take $K = M$; in the noncompact case, let $\{K_n\}_{n\in\mathbb{N}}$ be an increasing sequence of compact sets whose interiors cover M, and then take the topology defined by the quasi-norm:

$$\sum_{n=0}^{\infty} \frac{1}{2^n}\ \frac{\|T\|_{K_n}}{1 + \|T\|_{K_n}}.$$

It gives the C^∞-compact topology and sequential convergence is the uniform convergence of the tensor field and all of its derivatives on each compact.

2.4. For a noncompact manifold, the subspace $\Gamma_c^\infty(T^s(M))$ is not closed, and then it is not a Fréchet space. Nevertheless, for any two compact subsets of M such that $K \subset K'$ one has that $\Gamma_K^\infty(T^s(M))$ is included in $\Gamma_{K'}^\infty(T^s(M))$ as a subspace; thus one can consider $\Gamma_c^\infty(T^s(M))$ as the (strict) inductive limit of these Fréchet spaces; it is then a complete LF-space. In that case the topology is included in the coherent topology and it can be described as follows: a basis of neighbourhoods of 0 consists in those convex, balanced subsets such that the intersection with each $\Gamma_K^\infty(T^s(M))$ is open.

The same topology is obtained by using any family of compact subsets of M, which is cofinal for the direction given by the inclusion.

2.5. Finally, $\Gamma^\infty(T^s(M))$ is the disjoint union of subsets of the form $T+\Gamma_c^\infty(T^s(M))$ and one can then consider the disjoint union topology, which is finer than the C^∞-compact topology. It admits then a structure of locally affine manifold modelled on the convenient vector space $\Gamma_c^\infty(T^s(M))$ (see [Fr-Kr, pp. 71 and 132] for definitions).

2.6. Similar constructions can be done for any sub-bundle of $T^s(M)$ or in general for any vector bundle over M by using a fibre metric and a fibre connection instead of a Riemannian metric.

Remark. In the sequel we are going to use also sections of tensor bundles over submanifolds of the form $N = M \setminus G$, where G is an open set of M with regular boundary; we will understand by an element of $\Gamma^l(T^s(N))$ the restriction to N of a C^l section of $T^s(M)$ defined in an open set containing N. That definition of differentiability is, in that case, equivalent to the other possible usual definitions of differentiability for manifolds with boundary (cf. [Val, pp. 354 and 369]).

3. The Operator δ^*

The aim of this paragraph is to survey the definitions and several results concerning the operator δ^* and also to establish the notations that will be used in the sequel. Many of the results are well known and others are obtained by direct computation; therefore we will give them without proof.

3.1. Let M be a manifold and let us denote by $S^2(M)$ the bundle of 2-times covariant, symmetric tensors on M; the fibre of $T^2(M)$ at a point $x \in M$ will be considered, without changing the notation, either as $\mathcal{L}^2(T_xM;\mathbb{R})$ or as $\mathcal{L}(T_xM, T_x^*M)$, and analogously for the bundle of 2-times contravariant tensors on M.

For a given metric g, one can define the operator $\delta^* : \Gamma^\infty(T^1(M)) \to \Gamma^\infty(S^2(M))$ given by $\delta^*(\omega) = 2\,\mathrm{symm}\,\nabla\omega$. It is easy to see that $\delta^*\omega$ is equal to $\mathcal{L}_X g$, the Lie derivative of the metric tensor in the direction of the vector field associated, by the metric, with ω, that is, $X = g^{-1}\omega$. We have then the following

Definition. We will say that a 1-form ω is a Killing form if and only if $\delta^*\omega = 0$.

3.2. One can also define the divergence operator $\delta : \Gamma^\infty(S^2(M)) \to \Gamma^\infty(T^1(M))$ given by $\delta h = -2\,\mathrm{Tr}(\nabla h)$. Its expression in local coordinates is then $(\delta h)_k = -2g^{ij}(\nabla_j h)_{ik}$.

REMARK. The operators δ^*, δ can be defined analogously in general $\delta^* : \Gamma^\infty(S^k(M)) \to \Gamma^\infty(S^{k+1}(M))$ and $\delta : \Gamma^\infty(S^{k+1}(M)) \to \Gamma^\infty(S^k(M))$; δ is the formal adjoint of δ^* (see [Bes, p. 35]).

3.3. For a 1-form ω the covariant derivative of $\delta^*\omega$ is given by

$$(\nabla\delta^*\omega)(Y,Z,W) = \nabla^2\omega(Y,Z,W) + \nabla^2\omega(Y,W,Z).$$

3.4. It will be useful to consider the operator $\mathcal{A} : \Gamma^\infty(T^1(M)) \to \Gamma^\infty(T^3(M))$ given by:

$$\mathcal{A}(\omega)(W,Y,Z) = \frac{1}{2}\{(\nabla\delta^*\omega)(W,Y,Z) + (\nabla\delta^*\omega)(Y,Z,W) - (\nabla\delta^*\omega)(Z,W,Y)\}.$$

Proposition 3.5. *The operator $\mathcal{A}$ satisfies the following equalities:*

a) $\mathcal{A}(\omega)(W,Y,Z) = \nabla^2\omega(W,Y,Z) + \omega(R(Y,Z,W))$.
b) $(\nabla\delta^*\omega)(Y,Z,W) = \mathcal{A}(\omega)(Y,Z,W) + \mathcal{A}(\omega)(Y,W,Z)$.
c) $\mathcal{A}(\omega)(W,Y,Z) = g((\mathcal{L}_X\nabla_W - \nabla_W\mathcal{L}_X - \nabla_{[X,W]})(Y),Z)$, *where* $X = g^{-1}\omega$.

Proof. Let $R(Y,Z,W) = -\nabla_Y\nabla_Z W + \nabla_Z\nabla_Y W + \nabla_{[Y,Z]}W$ be the curvature tensor; then

$$\nabla^2\omega(Y,Z,W) - \nabla^2\omega(Z,Y,W) = \omega(R(Y,Z,W)).$$

Now, a) follows from 3.3 and the properties of the curvature tensor and b) is obtained from part a) and 3.3. If $X = g^{-1}\omega$ a direct computation shows that

$$\nabla^2\omega(W,Y,Z) = g((\mathcal{L}_X\nabla_W - \nabla_W\mathcal{L}_X - \nabla_{[X,W]})(Y),Z) + g(R(X,W,Y),Z),$$

so part c) follows from a). □

Corollary 3.6. *For a* 1*-form ω the following statements are equivalent:*

a) $\mathcal{A}(\omega) = 0$.
b) $\delta^*\omega$ *is parallel.*
c) *The vector field $X = g^{-1}\omega$ is an infinitesimal affine transformation.*

Proof. The equivalence between a) and b) is obtained from part b) of Proposition 3.5 and the definition of $\mathcal{A}$. It is known (see [Ko-No, Vol. I, p. 231]) that the vanishing of the right-hand term in Proposition 3.5, c) is equivalent to X being an infinitesimal affine transformation and that shows the equivalence between a) and c). □

So, we give the following

Definition. We will say that a 1-form is an affine form if and only if $\mathcal{A}(\omega) = 0$.

3.7. We will denote by L the operator $L : \Gamma^\infty(T^1(M)) \to \Gamma^\infty(T^1(M))$ given by $L\omega = \delta\delta^*\omega$; it is elliptic and by definition of δ we have $L\omega = -2\,\mathrm{Tr}(\nabla\delta^*\omega)$, so we have the following

Corollary. *Every affine form is an element of* $\ker L$.

3.8. Let us denote by (,) the usual pointwise inner product induced by the metric, g, on tensor fields. In particular for $\alpha, \beta \in \Gamma^\infty(T^1(M))$, $(\alpha, \beta) = g^{-1}(\alpha, \beta)$ and for $h, k \in \Gamma^\infty(S^2(M))$, $(h, k) = \mathrm{Tr}(g^{-1}hg^{-1}k)$.

It is easy to see that $(\delta^*\omega, h) = (\omega, \delta h) + \mathrm{div}\ Y$, where Y is the vector field given by $Y = 2g^{-1}hg^{-1}\omega$.

3.9. When restricted to sections with compact support, the pointwise inner product (,) gives, by integration over the Riemannian manifold, an inner product that we are going to represent by $\langle\ ,\ \rangle$. If M is a compact manifold without boundary then, by 3.8 one can see that $\langle \delta^*\omega, h\rangle = \langle \omega, \delta h\rangle$; the operators δ^* and δ (or their restrictions to sections with compact support, in the noncompact case) are adjoint to each other and $(\delta^*(\Gamma_c^\infty(T^1(M))))^\perp = \ker\delta \bigcap \Gamma_c^\infty(S^2(M))$.

3.10. The elliptic operator L is also self-adjoint and using the theory of such operators in a compact manifold one has

Proposition ([Be-Eb]). *For a compact Riemannian manifold M, $\Gamma^\infty(S^2(M))$ can be decomposed as the orthogonal direct sum of the two closed subspaces* $\mathrm{Im}\,\delta^*$ *and* $\ker\delta$.

4. The Variation of $|\omega|_{C^1,x}$ along a Geodesic

Proposition 4.1. *Let (M, g) be a Riemannian manifold, ω a 1-form in M, $\gamma\big|_{[a,b]}$ a segment of a normalized geodesic, and $K = \gamma([a, b])$. Then for each $L \geq$* $\max\{1,\ |R|_{C^0,K}\}$ *the following inequality holds*

$$|\omega|_{C^1,\gamma(t)} \ \leq\ |\omega|_{C^1,\gamma(a)}\ e^{L(t-a)} + |\mathcal{A}(\omega)|_{C^0,K}\ \frac{e^{L(t-a)}-1}{L},$$

for all $t \in [a, b]$.

Proof. Let $p = \gamma(b)$. We define the map

$$h : [a, b] \to \mathcal{L}(T_pM, \mathbb{R}) \times \mathcal{L}^2(T_pM; \mathbb{R}),$$

with $h = (h^1, h^2)$ given by:

$$h^1(t)(v) = \omega(\gamma(t))(P_b^t(v)),$$
$$h^2(t)(v, w) = (\nabla\omega)(\gamma(t))(P_b^t(v), P_b^t(w)),$$

for $v, w \in T_pM$, where $P_{t_1}^{t_2}$ represents the parallel displacement, along γ, from t_1 to t_2.

Using that γ is a geodesic and that, for each $v \in T_pM$, the vector field along γ, $V(t) = P_b^t(v)$, is parallel we obtain that

$$(h^1)'(t)(v) = (\nabla\omega)(\gamma(t))(\gamma'(t), P_b^t(v)),$$

and

$$(h^2)'(t)(v, w) = (\nabla^2\omega)(\gamma(t))(\gamma'(t), P_b^t(v), P_b^t(w)).$$

Let us now define the operator

$$\mathcal{R} : [a,b] \times \mathcal{L}(T_pM,\mathbb{R}) \times \mathcal{L}^2(T_pM;\mathbb{R}) \to \mathcal{L}(T_pM,\mathbb{R}) \times \mathcal{L}^2(T_pM;\mathbb{R})$$

given by

$$\mathcal{R}^1(t,\alpha,\beta)(v) = \beta(\gamma'(b),v),$$
$$\mathcal{R}^2(t,\alpha,\beta)(v,w) = \alpha(\mathcal{R}_t(v,w)),$$

where

$$\mathcal{R}_t(v,w) = -P_t^b(R(P_b^t(v),P_b^t(w),\gamma'(t))).$$

Then, it is immediate that

$$(h^1)'(t) - \mathcal{R}^1(t,h^1(t),h^2(t)) = 0,$$

and using 3.5, a)

$$(h^2)'(t)(v,w) - \mathcal{R}^2(t,h^1(t),h^2(t))(v,w) = \mathcal{A}(\omega)(\gamma(t))(\gamma'(t),P_b^t(v),P_b^t(w)).$$

Consequently,

$$\|h'(t) - \mathcal{R}(t,h(t))\| \leq |\mathcal{A}(\omega)|_{C^0,K}.$$

On the other hand, for each $t \in [a,b]$ the map $\mathcal{R}(t,\cdot)$ is linear, and its norm satisfies

$$\|\mathcal{R}(t,\cdot)\| \leq \max\{1,\ |R|_{C^0,K}\},$$

as can be easily verified. So, any L as in the statement is a Lipschitz constant for all the maps $\mathcal{R}(t,\cdot)$. Moreover, the map $g(t) \equiv 0$ is a solution of the equation $g'(t) = \mathcal{R}(t,g(t))$; then, it is well known (see for instance, [Lan, p. 68]) that

$$\|h(t)\| \leq \|h(a)\| + L\int_a^t (\|h(s)\| + \frac{|\mathcal{A}(\omega)|_{C^0,K}}{L})ds.$$

The result is obtained now by using Gronwall's Lemma and that, for all t,

$$|\omega|_{C^1,\gamma(t)} = \|h(t)\|.$$

□

4.2. As an easy consequence, we obtain the covariant version of a well known result about infinitesimal affine transformations (see [Ko-No, Vol. I, p. 232]).

Corollary. *Let ω be an affine 1-form in a connected manifold M. If there is some $q \in M$ such that $(\omega(q),\nabla\omega(q)) = 0$ then ω must vanish everywhere.*

Proof. Under the assumptions, the closed subset $\{p \in M\ ;\ (\omega(p),\nabla\omega(p)) = 0\}$ is nonempty. From the above Proposition it is also open, because it contains every normal neighbourhood of each of its points. □

5. The Closedness of δ^* Restricted to Sections with Support in a Fixed Compact

Along this paragraph M will be a Riemannian manifold and N will be a submanifold with boundary of the form $N = M \setminus G$, where G is an open subset of M such that ∂G is compact and regular and N is connected.

Lemma 5.1. *Given any compact $K \subset N$ there is a subset S open in N, of compact closure, with $K \subset S$, and there exists $d \in \mathbb{R}$ such that for all $p, q \in K$ there is a piecewise geodesic from p to q, contained in S and of length less than d.*

Proof. Let us assume first that $G = \emptyset$. For each $x \in K$ choose $\epsilon_x > 0$, such that the geodesic ball centered at x and of radius ϵ_x, $B_{\epsilon_x}(x)$, has compact closure and is a normal neighbourhood of x, i. e. every point in $B_{\epsilon_x}(x)$ can be joined to x by a geodesic, contained in $B_{\epsilon_x}(x)$ and of length less than ϵ_x. By the compactness of K, there is a finite subset $\{x_i\}_{i=1}^k \subset K$ such that

$$K \subset \bigcup_{i=1}^{k} B_{\epsilon_i}(x_i) = S.$$

Let us denote $B_i = B_{\epsilon_i}(x_i)$, $d = 2(\epsilon_1 + \cdots + \epsilon_k)$ and let us assume that K is connected. In that case, given $p, q \in K$ there is a simple chain (that we are going to denote $\{B_i\}_{i=1}^j$ for simplicity) from p to q, that is $p \in B_1$, $q \in B_j$, and $B_i \cap B_{i+1} \neq \emptyset$. If we take

$$p_0 = p, \quad p_1 \in B_1 \cap B_2, \dots, \; p_i \in B_i \cap B_{i+1}, \dots, \; p_j = q,$$

then by construction, both p_i and p_{i+1} can be joined to x_{i+1} by a geodesic segment in B_{i+1} of length less than ϵ_{i+1}. So, p can be joined to q by a piecewise geodesic in S of length less than $2(\epsilon_1 + \cdots + \epsilon_j) \leq d$.

If K is not connected one can consider a compact, connected set K', such that $K \subset K'$ and then apply the argument above to K'. This compact, connected set can be obtained, for instance, as the union of $\overline{S}$, which has at most k connected components, and the image of curves connecting these components.

For $G \neq \emptyset$ and $K \subset M \setminus \overline{G}$ the same proof can be used by taking ϵ_x small enough to have $B_{\epsilon_x}(x) \subset M \setminus \overline{G}$.

Finally, if $K \bigcap \partial G \neq \emptyset$, there is a positive real number λ such that the closed outer tube around ∂G of radius λ, T_λ, has the property that any point of T_λ can be joined to ∂G by a geodesic of length smaller than or equal to λ, and that $\partial T_\lambda \bigcap K \neq \emptyset$. Let then $\tilde{K}$ be the compact subset $(K \setminus (T_\lambda)^\circ) \bigcup \partial T_\lambda$. $\tilde{K}$ is disjoint from ∂G, and there are $\tilde{S}$, $\tilde{d}$ obtained as above. Now, take $S = \tilde{S} \bigcup ((T_\lambda)^\circ \bigcap N)$ and $d = \tilde{d} + 2\lambda$. □

Remark. The number of segments of the piecewise geodesic from p to q is also bounded, by an integer independent of the points.

Proposition 5.2. *Let K be a compact subset of N, $K \neq N$. Then, there is $a_1 \in \mathbb{R}$ such that for all $\omega \in \Gamma_K^\infty(T^1(N))$ the following inequality holds*

$$|\omega|_{C^1,K} \leq \; a_1 \; |\delta^*\omega|_{C^1,K}.$$

Proof. Let p be an element of $N \setminus K$, and let us take $\tilde{K} = K \bigcup \{p\}$. Let S and d be obtained by applying the Lemma to $\tilde{K}$, and let $L = \max\{1,\ |R|_{C^0,\overline{S}}\}$. For a given $q \in K$ let $\gamma = \{\gamma_i\}_{i=1}^r$ be the piecewise geodesic, that we can take normalized, that exists by the Lemma. For $i = 1, \dots, r-1$ let us denote by p_i the endpoint of γ_i and the beginning of γ_{i+1}, and by d_i the length of γ_i.

Applying now Proposition 4.1 to γ_1 and having in mind that $|\omega|_{C^1,p} = 0$ we have

$$|\omega|_{C^1,p_1} \ \leq \ |\mathcal{A}(\omega)|_{C^0,K} \ \frac{\mathrm{e}^{Ld_1} - 1}{L},$$

and after the r steps needed to reach q one obtains that

$$|\omega|_{C^1,q} \ \leq \ |\mathcal{A}(\omega)|_{C^0,K} \ \frac{\mathrm{e}^{L(d_1+\cdots+d_r)} - 1}{L}.$$

Since by definition of $\mathcal{A}$ (3.4)

$$|\mathcal{A}(\omega)|_{C^0,K} \leq \frac{3}{2} \ |\delta^*\omega|_{C^1,K},$$

we can take $a_1 = \frac{3}{2} \ \frac{\mathrm{e}^{Ld}-1}{L}$. □

Proposition 5.3. *Let K be a compact subset of N, $K \neq N$. Then for each $l \geq 2$ there is $a_l \in \mathbb{R}$ such that for all $\omega \in \Gamma^\infty_K(T^1(N))$ the following holds*

$$|\omega|_{C^l,K} \leq \ a_l \ |\delta^*\omega|_{C^{l-1},K}.$$

Proof. Let $q \in K$; we have

$$|\omega|_{C^2,q} = \max\{|\omega|_{C^1,q}, \|\nabla^2\omega(q))\|\}.$$

Now, from the above Proposition

$$|\omega|_{C^1,q} \leq \ a_1 \ |\delta^*\omega|_{C^1,K},$$

and by 3.5, a)

$$\begin{aligned}\|\nabla^2\omega(q))\| &\leq \|\mathcal{A}(\omega)(q)\| + \|\omega(q)\| \ \|R(q)\| \\ &\leq |\mathcal{A}(\omega)|_{C^0,K} + a_1 \ |\delta^*\omega|_{C^1,K} \ |R|_{C^0,K} \\ &\leq (\frac{3}{2} + a_1|R|_{C^0,K}) \ |\delta^*\omega|_{C^1,K}.\end{aligned}$$

We can then take

$$a_2 = \max\{a_1, \frac{3}{2} + a_1|R|_{C^0,K}\}.$$

By covariant derivation of 3.5, a), a_l can be obtained in a similar way for each l, as a function of $\{a_1, \dots, a_{l-1}, |R|_{C^{l-2},K}\}$. □

Corollary 5.4. *Let K be a compact subset of N, $K \neq N$. Then every sequence $\{\omega_i\}$ in $\Gamma_K^\infty(T^1(N))$ such that $\{\delta^*\omega_i\}$ converges is also convergent.*

Proof. It is obtained from the definitions of the topology (2.2), the fact that the spaces are complete and using Propositions 5.2 and 5.3. $\square$

Corollary 5.5. *Let K be a compact subset of N, $K \neq N$. Then, $\delta^*(\Gamma_K^\infty(T^1(N)))$ is closed in $\Gamma_K^\infty(S^2(N))$ and $\delta^* : \Gamma_K^\infty(T^1(N)) \to \delta^*(\Gamma_K^\infty(T^1(N)))$ is a homeomorphism.*

Proof. The first assertion and the closedness of δ^* are obtained from Corollary 5.4. The map is continuous and onto and it is also injective by Corollary 4.2. $\square$

6. The Closedness of the Image of δ^*

In order to prove that $\delta^*(\Gamma_c^\infty(T^1(M)))$ is closed we need some lemmas; 6.1, 6.2 and their Corollary 6.3 are of topological nature, Lemma 6.4 is a property of the operator δ^* and finally Lemma 6.5 gives a sufficient condition for a subspace to be closed in the strict inductive limit of a countable family of Fréchet spaces.

Lemma 6.1. *Let M be a connected manifold, and let $K \subset M$ be compact. Then, there exists a compact K' with $K \subset K'$ and such that none of the connected components of $M \setminus K'$ is of compact closure.*

Proof. We assume that $M \setminus K \neq \emptyset$ because otherwise there is nothing to prove. Let us denote $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) the family of connected components of $M \setminus K$ of compact closure (resp. of noncompact closure), and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

Each $C \in \mathcal{C}$ is open in M and closed in $M \setminus K$ and the union of K with the union of any subfamily of $\mathcal{C}$ is closed.

We only need to show that

$$K' = K \cup \big(\cup_{C \in \mathcal{C}_1} C \big)$$

is compact, the other conditions being trivially satisfied.

Let G be an open set of compact closure, such that $K \subset G$; then $\mathcal{C}_1' = \{C \in \mathcal{C}_1 \ ; \ C \cap (M \setminus G) \neq \emptyset\}$ is finite because any $C \in \mathcal{C}_1'$ should cut ∂G which is a compact subset of the locally connected space $M \setminus K$. Now K' can be written as

$$K' = K \cup \big(\cup_{C \in \mathcal{C}_1 \setminus \mathcal{C}_1'} C \big) \cup \big(\cup_{C \in \mathcal{C}_1'} C \big).$$

Then, since

$$K \cup \big(\cup_{C \in \mathcal{C}_1 \setminus \mathcal{C}_1'} C \big)$$

is closed and it is included in $\overline{G}$, the compactness of K' follows from the fact that $\overline{C}$ is compact, for all $C \in \mathcal{C}_1'$ and that $\overline{C} \cup K = C \cup K$. $\square$

Lemma 6.2. *Let K be a compact proper subset of M. Then there is an open G with $K \subset G$ and such that $\overline{G} = G \cup \partial G$ is compact, ∂G is a regular submanifold of M and none of the connected components of $M \setminus G$ is of compact closure.*

Proof. Let U be an open subset of compact closure with $K \subset U$. There is $f \in C^\infty(M)$ such that $f\big|_K \equiv 1$ and $\operatorname{supp} f \subset U$. From Sard Theorem, there is a regular value of f, $a \in (0,1)$ and then, by the proof of Lemma 6.1, the open set G obtained by the union of $G_1 = f^{-1}((a,\infty))$ and the connected components of compact closure of $M \setminus \overline{G}_1$ has the required properties. □

Corollary 6.3. *On every connected manifold M there is an open covering $\{G_n\}_{n\in\mathbb{N}}$ such that if $K_n = \overline{G}_n$ then for all $n \in \mathbb{N}$:*

a) *K_n is compact and ∂K_n is a regular submanifold of M.*
b) *None of the connected components of $M \setminus K_n$ is of compact closure.*
c) *$K_n \subset G_{n+1}$.*

Lemma 6.4. *Let M be a Riemannian manifold and let $K \subset M$ be a compact subset. If a 1-form ω has $\operatorname{supp} \delta^*(\omega) \subset K$ then, either $\operatorname{supp} \omega \subset K$, or there exists a connected component C of $M \setminus K$ such that $C \subset \operatorname{supp} \omega$.*

Proof. Let us denote $V_\omega = \{x \in M \,;\, \omega_x \neq 0\}$; then if $\operatorname{supp}\omega$ is not included in K, V_ω should intersect $M \setminus K$ and so there is a connected component C of $M \setminus K$ such that $C \cap V_\omega \neq \emptyset$. Then, ω restricted to C is a non identically zero Killing form of the connected manifold C and from 4.2 one should have $(M \setminus \operatorname{supp}\omega) \cap C = \emptyset$. □

Lemma 6.5. *Let E be the strict inductive limit of a countable family $\{E_n\}_{n\in\mathbb{N}}$ of Fréchet spaces. Assume that A is a subspace of E such that:*

a) *$A_n = A \cap E_n$ is closed in E_n for all $n \in \mathbb{N}$ and*
b) *$E_{n-1} + A_n$ is closed in E_n for all $n \in \mathbb{N}$, $n > 1$.*

Then, A is closed in E.

Proof. Let $p \in E \setminus A$. We can assume without loss of generality that $p \in E_1$. Then as A_1 is closed in E_1 there is λ_1 in the topological dual E_1^*, such that $\lambda_1(p) = 1$ and $A_1 \subset \ker \lambda_1$.

We can construct $\alpha_2 : E_1 + A_2 \to \mathbb{R}$ given by $\alpha_2(e+a) = \lambda_1(e)$; it is well defined and linear and we are going to see that $\ker \alpha_2 = \ker \lambda_1 + A_2$ is closed and then $\alpha_2 \in (E_1+A_2)^*$. In fact, let $l : E_1 \times A_2 \to E_1 + A_2$ be given by $l(e,a) = e + a$; $E_1 \times A_2$ and $E_1 + A_2$ are Fréchet spaces, l is linear, continuous and onto, whence by the open map theorem, l is open and in particular $l((E_1 \setminus \ker\lambda_1) \times A_2)$ is open in $E_1 + A_2$. It is not difficult to see that $l((E_1 \setminus \ker\lambda_1) \times A_2) = (E_1 + A_2) \setminus l(\ker\lambda_1 \times A_2) = (E_1 + A_2) \setminus \ker\alpha_2$ given that l is onto and that by construction $A_2 \cap E_1 = A_1 \subset \ker\lambda_1$.

Since $E_1 + A_2$ is closed in E_2, the form $\alpha_2 \in (E_1 + A_2)^*$ can be extended to $\lambda_2 \in E_2^*$, such that $\lambda_2(p) = 1$, $A_2 \subset \ker\lambda_2$ and $\lambda_2\big|_{E_1} = \lambda_1$. In this way we can construct, by recurrence, a sequence $\{\lambda_n\}$, $\lambda_n \in E_n^*$, with $\lambda_n(p) = 1$, $A_n \subset \ker\lambda_n$ and such that if $n_1 \leq n_2$ then $\lambda_{n_2}\big|_{E_{n_1}} = \lambda_{n_1}$.

That gives a well defined $\lambda \in E^*$, such that $\lambda(p) = 1$ and $A \subset \ker\lambda$; so, we can conclude that A is closed. □

Proposition 6.6. *Let M be a noncompact manifold. Then $\delta^*(\Gamma_c^\infty(T^1(M)))$ is a closed subspace of $\Gamma_c^\infty(S^2(M))$.*

Proof. Let $\{G_n\}_{n\in\mathbb{N}}$ be an open covering as in Corollary 6.3. Then, $\Gamma_c^\infty(S^2(M))$ is the strict inductive limit of the Fréchet spaces $E_n = \Gamma_{K_n}^\infty(S^2(M))$. If we take $A = \delta^*(\Gamma_c^\infty(T^1(M)))$ we only need to show that A satisfies the hypotheses of Lemma 6.5.

To show a) it is enough to see that $A_n = \delta^*(\Gamma_{K_n}^\infty(T^1(M)))$, the latter being closed by Corollary 5.5 because K_n is a proper subset of M. In fact, if $\delta^*\omega \in \Gamma_{K_n}^\infty(S^2(M))$ for $\omega \in \Gamma_c^\infty(T^1(M))$ then, from Lemma 6.4, either $\omega \in \Gamma_{K_n}^\infty(T^1(M))$, or there exists a connected component C of $M \setminus K_n$ such that $C \subset \operatorname{supp}\omega$, which is impossible, by property b) of G_n; then $A_n \subset \delta^*(\Gamma_{K_n}^\infty(T^1(M)))$. The other inclusion is obvious.

To show b) let us first characterize the elements of $E_{n-1} + A_n$. An element $h \in E_n$ is in $E_{n-1} + A_n$ if and only if there exists $\alpha \in \Gamma_{K_n}^\infty(T^1(M))$ such that $h\big|_N = \delta^*\alpha\big|_N$, where $N = M \setminus G_{n-1}$. This is clear because if $h \in E_{n-1} + A_n$ then there is $h_1 \in E_{n-1}$ and $\alpha \in \Gamma_{K_n}^\infty(T^1(M))$ such that $h = h_1 + \delta^*\alpha$; but $h_1\big|_N = 0$ and then h and $\delta^*\alpha$ should coincide on N. Conversely, if $h \in E_n$ and if there is $\alpha \in \Gamma_{K_n}^\infty(T^1(M))$ such that $h\big|_N = \delta^*\alpha\big|_N$, then $h_1 = h - \delta^*\alpha \in E_{n-1}$.

Let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence in $E_{n-1} + A_n$ that converges to $h \in E_n$. Then, there is a sequence $\{\alpha_i\}_{i\in\mathbb{N}}$ in $\Gamma_{K_n}^\infty(T^1(M))$ such that $h_i\big|_N = \delta^*\alpha_i\big|_N$; let us denote $\omega_i = \alpha_i\big|_N \in \Gamma_K^\infty(T^1(N))$, with $K = K_n \setminus G_{n-1}$. (We can assume that N is connected, otherwise we can apply the following argument to each connected component). The sequence $\{\delta^*\omega_i\}_{i\in\mathbb{N}}$ converges to $h\big|_N$ and $K \neq N$. Then, from Corollary 5.4 there is $\omega \in \Gamma_K^\infty(T^1(N))$ such that $\{\omega_i\}_{i\in\mathbb{N}}$ converges to ω and, by continuity of δ^*, $h\big|_N = \delta^*\omega$. Let $\tilde\omega \in \Gamma_{K_n}^\infty(T^1(M))$ be any differentiable extension of ω; then, $h\big|_N = \delta^*\tilde\omega\big|_N$, and $h \in E_{n-1} + A_n$. □

7. About the Decomposition of the Space $\Gamma_c^\infty(S^2(M))$

Let us represent by $\mathcal{T}(g)$ the closed subspace $\delta^*(\Gamma_c^\infty(T^1(M)))$ and by $\mathcal{N}(g)$ the closed subspace $\ker\delta \bigcap \Gamma_c^\infty(S^2(M))$. We know (3.9) that $\mathcal{N}(g) = \mathcal{T}(g)^\perp$.

In this paragraph we are going to show several examples of noncompact Riemannian manifolds for which $\Gamma_c^\infty(S^2(M)) \neq \mathcal{N}(g) \oplus \mathcal{T}(g)$.

Remarks. The inner product $\langle\ ,\ \rangle$, defined on $\Gamma_c^\infty(T^1(M))$ (see 3.9), extends to a bilinear map from $\Gamma_c^\infty(T^1(M)) \times \Gamma^\infty(T^1(M))$ to $\mathbb{R}$ (or from $\Gamma^\infty(T^1(M)) \times \Gamma_c^\infty(T^1(M))$ to $\mathbb{R}$) that we are going to denote with the same symbol; analogously for the inner product defined on $\Gamma_c^\infty(S^2(M))$.

If E is a subspace of $\Gamma^\infty(S^2(M))$, by abuse of notation we will write $E^\perp$ to mean $\{h \in \Gamma_c^\infty(S^2(M))\ ;\ \langle h, k\rangle = 0,\ \forall k \in E\}$.

For $\langle\delta^*\omega, h\rangle = \langle\omega, \delta h\rangle$ to be true it is not necessary that both sections involved have compact support; it is sufficient that either $\omega \in \Gamma_c^\infty(T^1(M))$ or $h \in \Gamma_c^\infty(S^2(M))$.

Proposition 7.2. *For every metric g on M the space $\mathcal{N}(g) \oplus \mathcal{T}(g)$ is included in $(\delta^*(\ker L))^\perp$.*

Proof. If $h \in \mathcal{N}(g) \oplus \mathcal{T}(g)$ then $h = h_0 + \delta^*\omega$ with $h_0 \in \mathcal{N}(g)$, $\omega \in \Gamma_c^\infty(T^1(M))$. Let now be $\alpha \in \ker L$; then we have

$$\langle h, \delta^*\alpha\rangle = \langle h_0, \delta^*\alpha\rangle + \langle\delta^*\omega, \delta^*\alpha\rangle = \langle\delta h_0, \alpha\rangle + \langle\omega, L\alpha\rangle = 0.$$

□

Corollary 7.3. *If for a metric g in M, $\delta^*(\ker L) \neq \{0\}$, then $\Gamma^\infty_c(S^2(M)) \neq \mathcal{N}(g) \oplus \mathcal{T}(g)$.*

Proof. Let α be an element of $\Gamma^\infty(T^1(M))$ such that $L\alpha = 0$ and $\delta^*\alpha \neq 0$, that exists by hypothesis. Let U be an open set on which $\delta^*\alpha$ is everywhere different from zero and let φ be a nonnegative element of $C^\infty(M)$, taking the value 1 in a nonempty open subset of U and with compact support in U. Then, $h = \varphi\delta^*\alpha \in \Gamma^\infty_c(S^2(M))$ and $\langle h, \delta^*\alpha\rangle \neq 0$. So, $h \notin (\delta^*(\ker L))^\perp$. □

Remark 7.4. $\delta^*(\ker L) \bigcap \Gamma^\infty_c(S^2(M)) = \{0\}$ because if $\delta^*\alpha$ has compact support then $\langle \delta^*\alpha, \delta^*\alpha\rangle = \langle \alpha, \delta\delta^*\alpha\rangle$ and if moreover $\alpha \in \ker L$, then $\langle \delta^*\alpha, \delta^*\alpha\rangle = 0$. In particular for a compact manifold $\delta^*(\ker L) = \{0\}$; this can also be concluded from the fact that for a compact manifold the decomposition holds.

For a noncompact manifold the space of sections of noncompact support $\delta^*(\ker L)$ can be seen as an obstruction to the decomposability of $\Gamma^\infty_c(S^2(M))$.

Proposition 7.5. *Let g be a metric on M admitting an affine form which is not a Killing form. Then, $\Gamma^\infty_c(S^2(M)) \neq \mathcal{N}(g) \oplus \mathcal{T}(g)$.*

Proof. It is a consequence of the previous result and of Corollary 3.7. □

In particular, for $\mathbb{R}^n$ with the Euclidean metric the decomposition of $\Gamma^\infty_c(S^2(M))$ does not hold.

Remark 7.6. It is known [Ko-No, Vol.I, p. 242] that on a complete irreducible manifold, different from $\mathbb{R}$, every infinitesimal affine transformation is a Killing vector field. As a consequence, 7.5 does not provide us with examples within this kind of manifolds.

In what follows we are going to show that if M is the interior of a compact, connected Riemannian manifold with boundary then $\delta^*(\ker L) \neq \{0\}$.

7.7. Let N be a compact Riemannian manifold with regular boundary $\partial N \neq \emptyset$ that, for simplicity, we assume to be orientable; let ν, $d\overline{v}$ be respectively, the unit normal to the boundary and the oriented Riemannian volume element of the boundary. Then, for $\omega \in \Gamma^\infty(T^1(N))$ and $h \in \Gamma^\infty(S^2(N))$ we have

$$\langle \delta^*\omega, h\rangle = \langle \omega, \delta h\rangle + \int_{\partial N} g(Y, \nu)\, d\overline{v}$$

where $Y = 2g^{-1}hg^{-1}\omega$ (see 3.8). It is easy to see that on ∂N, $g(Y, \nu) = 2(\omega, h(\nu))$ and then, the above equality can be written as

$$\langle \delta^*\omega, h\rangle = \langle \omega, \delta h\rangle + 2\langle \omega, h(\nu)\rangle_{\partial N}.$$

As a consequence, the operator L satisfies

$$\langle L\omega, \gamma\rangle = \langle \omega, L\gamma\rangle + 2\langle \omega, \delta^*\gamma(\nu)\rangle_{\partial N} - 2\langle \gamma, \delta^*\omega(\nu)\rangle_{\partial N},$$

for all $\omega, \gamma \in \Gamma^\infty(T^1(N))$.

7.8. Boundary problem. Let us denote by $\overline{\Gamma}^{\infty}(T^1(N))$ the space of C^∞ maps $\beta : \partial N \to T^1(N)$ such that $\pi \circ \beta = \mathrm{Id}$; for a given $\beta \in \overline{\Gamma}^{\infty}(T^1(N))$ we will consider $\Gamma^{\infty}_{\beta}(T^1(N)) = \{\omega \in \Gamma^\infty(T^1(N))\ ;\ \omega\big|_{\partial N} = \beta\}$. For each $(\alpha, \beta) \in \Gamma^\infty(T^1(N)) \times \overline{\Gamma}^{\infty}(T^1(N))$ we want to know if $\alpha \in L(\Gamma^\infty_\beta(T^1(N)))$; so we are concerned with the existence of $\omega \in \Gamma^\infty(T^1(N))$ such that:

$$(BP)\ \begin{cases} L\omega = \alpha & \text{on } N, \\ \omega = \beta & \text{on } \partial N. \end{cases}$$

Proposition 7.9. *The Boundary Problem* (BP) *is elliptic.*

Proof. That L is an elliptic operator is well known (see [Be-Eb]). That the boundary conditions are elliptic with respect to L (see [Hör, 10.6.2] for the definition) is a long and technical but straightforward computation. □

Applying then [Hör, p. 273] we have the following

Proposition 7.10. *The space* $\mathcal{N} = \ker L \cap \Gamma^\infty_0(T^1(N))$ *is finite dimensional. The space*

$$\mathcal{R} = \{(\alpha, \beta) \in \Gamma^\infty(T^1(N)) \times \overline{\Gamma}^{\infty}(T^1(N)) \ \ \textit{such that there is a solution of (BP)}\}$$

has finite codimension in $\Gamma^\infty(T^1(N)) \times \overline{\Gamma}^{\infty}(T^1(N))$ *and it is defined by a finite number of elements* $(\gamma_j, \eta_j) \in \Gamma^\infty(T^1(N)) \times \overline{\Gamma}^{\infty}(T^1(N))$ *through the conditions*

$$0 = \langle \alpha, \gamma_j \rangle + \langle \beta, \eta_j \rangle_{\partial N}.$$

Now we obtain a more useful characterization of the elements in $\mathcal{N}$ and $\mathcal{R}$.

7.11. Let $\mathcal{K}$ be the space $\mathcal{K} = \ker \delta^* \cap \Gamma^\infty_0(T^1(N))$. By a similar argument to that used in 7.4 one can conclude that $\mathcal{K} = \mathcal{N}$.

Lemma 7.12. *Let* $(\gamma, \eta) \in \Gamma^\infty(T^1(N)) \times \overline{\Gamma}^{\infty}(T^1(N))$. *If for every* $(\alpha, \beta) \in \mathcal{R}$ *the equality* $0 = \langle \alpha, \gamma \rangle + \langle \beta, \eta \rangle_{\partial N}$ *holds, then* $\gamma \in \mathcal{K}$ *and* $\eta = 0$.

Proof. The hypothesis on (γ, η) can be written with the help of 7.7 as follows:

$$\langle \omega, L\gamma \rangle + \langle \omega, \eta \rangle_{\partial N} + 2\langle \omega, \delta^*\gamma(\nu) \rangle_{\partial N} - 2\langle \gamma, \delta^*\omega(\nu) \rangle_{\partial N} = 0,$$

for all $\omega \in \Gamma^\infty(T^1(N))$.

Using this formula for conveniently chosen ω and by similar arguments to those in [Hör, p.264] one can show first that $L\gamma = 0$ and then that γ vanishes when restricted to the boundary; the conclusion is then obtained from 7.11. □

Proposition 7.13. *The space* $\mathcal{K}$ *is finite dimensional. Let* $(\alpha, \beta) \in \Gamma^\infty(T^1(N)) \times \overline{\Gamma}^{\infty}(T^1(N))$*; then,* $(\alpha, \beta) \in \mathcal{R}$ *if and only if* $\alpha \in \mathcal{K}^\perp$. *For a given* (α, β) *any two solutions of* (BP) *differ in an element of* $\mathcal{K}$.

Proof. The first and the last assertions are a consequence of 7.10 and 7.11. By 7.7 if $(\alpha, \beta) \in \mathcal{R}$ then $\alpha \in \mathcal{K}^{\perp}$. Now, if $\alpha \in \mathcal{K}^{\perp}$ using 7.12 and 7.10 we conclude that $(\alpha, \beta) \in \mathcal{R}$ for any $\beta \in \overline{\Gamma}^{\infty}(T^1(N))$. □

We have the following decomposition result for manifolds with boundary.

Corollary 7.14. *For each $\beta \in \overline{\Gamma}^{\infty}(T^1(N))$, every element of $\Gamma^{\infty}(S^2(N))$ can be written in a unique way as the sum of an element of $\delta^*(\Gamma^{\infty}_{\beta}(T^1(N)))$ and an element of $\ker \delta$. In particular, for $\beta = 0$ the two subspaces are orthogonal to each other and $\Gamma^{\infty}(S^2(N)) = \delta^*(\Gamma^{\infty}_0(T^1(N))) \oplus \ker \delta$.*

Proof. An element $h \in \Gamma^{\infty}(S^2(N))$ can be decomposed in that manner if and only if $(\delta h, \beta) \in \mathcal{R}$ or equivalently, if and only if $\delta h \in L(\Gamma^{\infty}_{\beta}(T^1(N)))$. Using 7.7 we have that $\delta h \in \mathcal{K}^{\perp}$ and then the result is an immediate consequence of 7.13. In the particular case of $\beta = 0$, the orthogonality is obtained from 7.7. □

7.15. We recall that, in a connected manifold, the value of a Killing vector field is completely determined by its value and that of its first jet at a single point [Ko-No, p. 232] and as a consequence the set of Killing 1-forms in a connected manifold is a finite dimensional vector space.

Proposition 7.16. *Let N be a compact, connected manifold with boundary. There is $\beta \in \overline{\Gamma}^{\infty}(T^1(N))$ such that there are no Killing forms in $\Gamma^{\infty}_{\beta}(T^1(N))$.*

Proof. Let $\{\omega_1, \dots, \omega_k\}$ be a basis of the real vector space of Killing 1-forms on M where M is the interior of N. By reordering, if necessary, there is a maximal integer l, $0 \leq l \leq k$, such that every ω_i, $1 \leq i \leq l$, can be extended to N; let then β_i, $1 \leq i \leq l$, be their restrictions to ∂N. Now, for any $\beta \in \overline{\Gamma}^{\infty}(T^1(N))$ which is not in the real vector space generated by $\{\beta_1, \dots, \beta_l\}$, there is no Killing 1-form taking the value β on the boundary. □

Proposition 7.17. *Let M be a noncompact manifold such that it is the interior of a compact, connected, orientable manifold with boundary. Then, $\delta^*(\ker L) \neq \{0\}$; consequently $\Gamma^{\infty}_c(S^2(M)) \neq \mathcal{N}(g) \oplus \mathcal{T}(g)$.*

Proof. Assume that M is the interior of N and let $\omega \in \ker L \cap \Gamma^{\infty}_{\beta}(T^1(N))$ with β as in 7.16. If $\tilde{\omega} = \omega\big|_M$ then $\tilde{\omega} \in \ker L$ and $\delta^* \tilde{\omega} \not\equiv 0$ because otherwise $\delta^* \omega \equiv 0$ which is impossible by the choice of β. □

7.18. Now we are going to use 7.13 to obtain a characterization of $\mathcal{N}(g) \oplus \mathcal{T}(g)$. It is immediate that $h \in \mathcal{N}(g) \oplus \mathcal{T}(g)$ if and only if the equation $L\omega = \delta h$ has a solution with compact support or, equivalently, if and only if $\delta h \in L(\Gamma^{\infty}_c(T^1(M)))$.

Proposition 7.19. *Let $\alpha \in \Gamma^{\infty}_c(T^1(M))$. Then, $\alpha \in L(\Gamma^{\infty}_c(T^1(M)))$ if and only if there is a compact K with $\operatorname{supp} \alpha \subset K$ such that α is orthogonal to every $\gamma \in \Gamma^{\infty}(T^1(M))$ with the property $L\gamma\big|_K = 0$.*

Proof. If $\alpha = L\omega$ with $\omega \in \Gamma_c^\infty(T^1(M))$, from 7.7, every compact K with regular boundary such that $\operatorname{supp}\omega \subset K$ satisfies the required conditions.

Conversely, let α be as in the statement and let K be the compact satisfying the hypothesis. Let us take a compact, connected submanifold N of M, with regular boundary and $K \subset N^\circ$, and let $\omega \in \Gamma_0^\infty(T^1(N))$ be such that $L\omega = \alpha$. The existence of such an ω is a consequence of 7.13 and the hypothesis on α. In what follows we are going to show that $\nabla^k\omega\big|_{\partial N} = 0$, for each k and so, ω can be extended to a smooth form in M vanishing outside N.

Let $\gamma \in \Gamma^\infty(T^1(N))$ be a solution of the boundary problem:

$$\begin{cases} L\gamma = 0 & \text{on } N, \\ \gamma = \delta^*\omega(\nu) & \text{on } \partial N. \end{cases}$$

The orthogonality property of α along with 7.7 gives that $\delta^*\omega(\nu) = 0$ on ∂N; also, $\omega\big|_{\partial N} = 0$ implies that $(\nabla_X\omega)\big|_{\partial N} = 0$ if X is tangent to ∂N. Both facts lead to $\nabla\omega\big|_{\partial N} = 0$. Using this and the relations with the curvature we have that $(\nabla^2\omega)(X, Y, .)\big|_{\partial N} = 0$ if either X or Y is tangent to ∂N. Since $L\omega$ vanishes, at least, in the open neighbourhood $N \setminus K$ of ∂N we have, after computation, $(\nabla^2\omega)(\nu, \nu, .)\big|_{\partial N} = 0$, that is $\nabla^2\omega\big|_{\partial N} = 0$. By using similar arguments it is easy to show by recurrence that $\nabla^k\omega\big|_{\partial N} = 0$, $k \geq 0$. Thus, ω extends to a smooth form in M with support in N. □

Corollary 7.20. *An element $h \in \Gamma_c^\infty(S^2(M))$ is in $\mathcal{N}(g) \oplus \mathcal{T}(g)$ if and only if there is a compact K with $\operatorname{supp}\delta h \subset K$ such that δh is orthogonal to every $\gamma \in \Gamma^\infty(T^1(M))$ with the property $L\gamma\big|_K = 0$.*

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Departamento de Geometría y Topología. Facultad de Matemáticas. Universidad de Valencia. 46100 Burjasot, Valencia. SPAIN.
gilo@vm.ci.uv.es

Departamento de Geometría y Topología. Facultad de Matemáticas. Universidad de Valencia. 46100 Burjasot, Valencia. SPAIN.
montesinos@mac.uv.es

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