

Convergence of the p -Series for Stationary Sequences

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ABSTRACT. Let (X_n) be a stationary sequence. We prove the following

(i) If the variables (X_n) are iid and $\mathbb{E}(|X_1|) < \infty$ then

$$\lim_{p \rightarrow 1^+} \left((p-1) \left(\sum_{n=1}^{\infty} \frac{|X_n(x)|^p}{n^p} \right) \right)^{1/p} = \mathbb{E}(|X_1|), \text{ a.e.}$$

(ii) If $X_n(x) = f(T^n x)$ where (X, \mathcal{F}, μ, T) is an ergodic dynamical system, then

$$\lim_{p \rightarrow 1^+} \left((p-1) \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right) \right)^{1/p} = \int f d\mu \quad \text{a.e. for } f \geq 0, f \in L \log L.$$

Furthermore the maximal function,

$$\sup_{1 < p < \infty} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p} \text{ is integrable for functions } f \geq 0, f \in L \log L.$$

These limits are linked to the maximal function $N^*(x) = \left\| \left(\frac{X_n(x)}{n} \right) \right\|_{1, \infty}$.

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1. Introduction

Let Z_n be a sequence of independent, identically distributed random variables and (a_n) a sequence of positive real numbers. The a.e. convergence of the weighted averages

$$(*) \quad \frac{\sum_{n=1}^N a_n Z_n}{A_n},$$

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where $A_n = \sum_{k=1}^n a_k$, has been characterized by B. Jamison, S. Orey and W. Pruitt ([JOP]). They proved that the condition

$$(0) \quad \sup_n \frac{\tilde{N}_n}{n} < \infty$$

where $\tilde{N}_n = \#\{k : \frac{a_k}{A_k} \geq \frac{1}{n}\}$ is necessary and sufficient for the a.e. convergence of the weighted averages (*) to $\mathbb{E}(Z_1)$. In [A1], interested by the a.e. convergence (y) of averages of the form

$$\frac{\sum_{n=1}^N X_n(x)g(S^n y)}{N},$$

we considered the maximal function $N^*(x) = \sup_n \frac{N_n(x)}{n}$ where $N_n(x) = \#\{k : \frac{X_k(x)}{k} \geq \frac{1}{n}\}$, ($X_k \geq 0$). We proved the following:

- (1) If X_n are iid random variables and $\mathbb{E}(|X_1|) < \infty$ then $N^*(x)$ is finite a.e.
- (2) If the X_n are given by an ergodic dynamical system (i.e., $X_n(x) = f(T^n x)$ where (X, \mathcal{F}, μ, T) is an ergodic dynamical system and f a measurable non-negative function) then for all p , $1 < p < \infty$ there exists a finite constant C_p such that

$$(**) \quad \mu\{x : N^*(x) > \lambda\} \leq \frac{C_p}{\lambda^p} \int |f|^p d\mu \quad \text{for all } \lambda > 0.$$

Furthermore for all p , $1 < p < \infty$, for all $f \in L_+^p$ we have $\lim_{n \rightarrow \infty} \frac{N_n(x)}{n} = \int f d\mu$ a.e.

(A closer inspection of the proof of (**)) shows that the constant C_p is of the form $\frac{C}{p-1}$ where C is an absolute constant independent of p .)

If $0 < p < \infty$, and $(x_i)_{i \geq 1}$ is a sequence of nonnegative real numbers, set

$$\|(x_i)\|_{p,\infty} = \left(\sup_{\lambda > 0} \lambda^p \#\{i \geq 1; |x_i| > \lambda\} \right)^{1/p}.$$

It is easily seen that for $r < p$

$$\|(x_i)\|_{p,\infty} \leq \left(\sum_i |x_i|^p \right)^{1/p} \leq \left(\frac{p}{p-r} \right)^{1/p} \|(x_i)\|_{r,\infty}$$

(cf. [SW]). In particular, for all p , $1 < p \leq 2$ we have

$$(3) \quad (p-1)^{1/p} \left(\sum_i |x_i|^p \right)^{1/p} \leq p^{1/p} \|(x_i)\|_{1,\infty}.$$

As $\|(x_i)\|_{1,\infty} \sim \sup_n \frac{\#\{k: x_k \geq 1/n\}}{n}$, for bounded sequences the previous inequality applied pointwise to a stationary sequence (X_n) of integrable functions gives us not only the existence of the p -series

$$(p-1)^{1/p} \left(\sum_i \left| \frac{X_i(x)}{i} \right|^p \right)^{1/p} \quad \text{for all } p, 1 < p \leq \infty,$$

but also the inequality

$$(4) \quad \sup_{1 < p < \infty} (p-1)^{1/p} \left(\sum_i \left| \frac{X_i(x)}{i} \right|^p \right)^{1/p} \leq 2 \left\| \left(\frac{X_i(x)}{i} \right) \right\|_{1, \infty}$$

if $\left\| \left(\frac{X_i(x)}{i} \right) \right\|_{1, \infty} < \infty$. The inequality (4) and some of our previous results suggest the study of the limit when p tends to 1^+ of the series

$$(p-1)^{1/p} \left(\sum_{i=1}^{\infty} \left| \frac{X_i(x)}{i} \right|^p \right)^{1/p}.$$

Definition. Let (X_n) be a stationary sequence of integrable functions. The p series associated to this sequence is the a.e. series (when it exists):

$$(p-1)^{1/p} \left(\sum_{i=1}^{\infty} \left| \frac{X_i(x)}{i} \right|^p \right)^{1/p}.$$

In this note, using an elementary lemma on sequence of real numbers, we will show that for (X_n) iid with $\mathbb{E}(|X_1|) < \infty$ the p -series

$$(p-1)^{1/p} \left(\sum_{i=1}^{\infty} \left| \frac{X_i(x)}{i} \right|^p \right)^{1/p} \quad \text{converges a.e. to } \mathbb{E}(|X_1|)$$

when p tends to 1^+ .

The same argument shows that the p series

$$(p-1)^{1/p} \left(\sum_{i=1}^{\infty} \left| \frac{\prod_{j=1}^H X_{j,i}(x_j)}{i} \right|^p \right)^{1/p} \quad \text{converges a.e. to } \prod_{j=1}^H \mathbb{E}(|X_{j,1}|)$$

where $(X_{j,n})_n$ are iid random variables satisfying the condition $\mathbb{E}(|X_{j,1}|) < \infty$, and the variables x_j are selected in a universal way specified in [A1].

We can remark that for each p the function $G_p(x) = (p-1)^{1/p} \left(\sum_{i=1}^{\infty} \left| \frac{X_i(x)}{i} \right|^p \right)^{1/p}$ is not *integrable*, as $G_p(x) \geq (p-1)^{1/p} \sup_i \left| \frac{X_i(x)}{i} \right|$, and for (X_i) iid with $\mathbb{E}(|X_1| \log |X_1|) = \infty$, the function $\sup_i \left| \frac{X_i(x)}{i} \right|$ is not integrable, as shown by D. Burkholder in [B]. So $F^*(x) = \sup_{1 < p < \infty} G_p(x)$ is a supremum of nonintegrable functions. This makes the handling of the function $F^*(x)$ somewhat delicate.

In the second part of this note we will focus on the ergodic stationary case. We will consider an ergodic dynamical system (X, \mathcal{F}, μ, T) and a nonnegative measurable function f . Using (2) we will show first that

$$\frac{N_n(f)(x)}{n} = \frac{\#\{k : \frac{f(T^k x)}{k} \geq 1/n\}}{n}$$

converges in L^1 norm to $\int f d\mu$. Then using extrapolation methods we will show that

$$(5) \quad \left\| \left\| \left(\frac{f(T^k x)}{k} \right) \right\|_{1,\infty} \right\|_1 < \infty \text{ for } f \in L(\text{Log } L).$$

One of our interests in (5) lies in the following observation: If we denote by $\frac{f(T^{n^*} x)}{n^*}$ a decreasing rearrangement of the sequence $\frac{f(T^n x)}{n}$, then we have

$$(6) \quad \left\| \left(\frac{f(T^k x)}{k} \right) \right\|_{1,\infty} = \sup_n n \frac{f(T^{n^*} x)}{n^*}.$$

Hence for $f \in L(\text{Log } L)$, (6) provides us with some information on the decreasing rate of the sequence $\frac{f(T^n x)}{n}$.

Using (5), we will prove that for $f \in L \log L$, $f \geq 0$,

$$(6') \quad M_1^*(x) = \sup_{1 < p < \infty} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p},$$

and

$$(7) \quad \lim_{p \rightarrow 1^+} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p} = \int f d\mu \text{ a.e., } (\mu).$$

The integrability of $M_1^*(x)$ for f in $L \log L$ extends the results on the integrability of the $\sup_n \frac{f(T^n x)}{n}$ in the ergodic case. We do not know at the present time if (7) holds for $f \in L^1$. Finally, in the third part of this paper we will study the connection between the maximal operators

$$M_1^*(f)(x) = \sup_{1 < p < \infty} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p}, \quad M_2^*(f)(x) = \sup_N \frac{1}{N} \sum_{n=1}^N f(T^n x)$$

and

$$\left\| \left(\frac{f(T^n x)}{n} \right) \right\|_{1,\infty} = N^*(f)(x).$$

If there is no ambiguity we will simply denote these maximal functions by $M_1^*(x)$, $M_2^*(x)$ and $N^*(x)$.

2. Convergence of the p -series for iid sequences

2.1. The one dimensional case. The next elementary lemma will be useful for the convergence we are looking for.

Lemma 1. Let $(x_n)_n$ be a sequence of nonnegative numbers such that $\frac{x_k}{k} \rightarrow 0$ and $\frac{\#\{k : \frac{x_k}{k} \geq 1/n\}}{n} \mapsto \bar{x}$, then

- (a) $\lim_{p \rightarrow 1^+} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{x_n}{n} \right)^p \right)^{1/p} = \bar{x}$.
 (b) If $\frac{x_{k^*}}{k^*}$ is a decreasing rearrangement of the sequence $\left(\frac{x_k}{k} \right)_k$ then $k \cdot \frac{x_{k^*}}{k^*}$ converges to \bar{x} .

Proof. We denote by $R_n = \{k : \frac{x_k}{k} \geq 1/n\}$ and $N_n = \#\{k : \frac{x_k}{k} \geq 1/n\} = \#R_n$. To prove (a) it is enough to show that

$$\lim_{p \rightarrow 1^+} (p-1) \left(\sum_{n=1}^{\infty} \left(\frac{x_n}{n} \right)^p \right) = \bar{x}.$$

We can write the series $(p-1) \left(\sum_{n=1}^{\infty} \left(\frac{x_n}{n} \right)^p \right)$ in the following way;

$$\begin{aligned} (p-1) \left(\sum_{n=1}^{\infty} \left(\frac{x_n}{n} \right)^p \right) &= (p-1) \left[\sum_{n \in R_1} \left(\frac{x_n}{n} \right)^p + \sum_{n \in \mathbb{N}^* \setminus R_1} \left(\frac{x_n}{n} \right)^p \right] \\ &= A_p + B_p. \end{aligned}$$

As $\lim_{p \rightarrow 1} A_p = 0$ we just need to consider $B_p = (p-1) \sum_{n \in \mathbb{N}^* \setminus R_1} \left(\frac{x_n}{n} \right)^p$. But we have

$$(p-1) \sum_{n=1}^{\infty} \frac{N_{n+1} - N_n}{(n+1)^p} \leq B_p \leq (p-1) \sum_{n=1}^{\infty} \frac{N_{n+1} - N_n}{n^p}.$$

It is then enough to prove that B_p is squeezed into two terms tending to the same limit \bar{x} . We will only prove that the term $(p-1) \sum_{n=1}^{\infty} \frac{N_{n+1} - N_n}{n^p}$ converges to \bar{x} . The same argument shows the same conclusion for the second term $(p-1) \sum_{n=1}^{\infty} \frac{N_{n+1} - N_n}{(n+1)^p}$.

We have

$$\begin{aligned} (p-1) \sum_{n=1}^{\infty} \frac{N_{n+1} - N_n}{n^p} &= (p-1) \left(-\frac{N_1}{1^p} + \sum_{n=2}^{\infty} \frac{N_n(n^p - (n-1)^p)}{n^p(n-1)^p} \right) \\ &= (p-1) \left[-\frac{N_1}{1^p} + \sum_{n=2}^{\infty} \frac{N_n(1 - (\frac{n-1}{n})^p)}{(n-1)^p} \right] \\ &\sim (p-1) \left[-\frac{N_1}{1^p} + p \sum_{n=2}^{\infty} \frac{N_n}{n} \cdot \frac{1}{(n-1)^p} \right]. \end{aligned}$$

As $\frac{N_n}{n}$ converges to \bar{x} and $\sum_{n=2}^{\infty} \frac{1}{(n-1)^p} \sim \frac{1}{p-1}$ we conclude that

$$\lim_{p \rightarrow 1^+} (p-1) \sum_{n=1}^{\infty} \frac{N_{n+1} - N_n}{n^p} = \bar{x}.$$

(b) To obtain the convergence of the sequence $k \frac{x_k^*}{k^*}$ to \bar{x} we can observe that

$$\lim_{t \rightarrow \infty} \frac{\#\{\ell : \frac{x_\ell}{\ell} \geq \frac{1}{t}\}}{t} = \bar{x}.$$

If we take the increasing sequence $t_k = \frac{k^*}{x_k^*}$ where $\frac{x_k^*}{k^*}$ is the k^{th} term of the decreasing rearrangement of the sequence $\frac{x_k^*}{k^*}$ we can see that

$$\frac{x_{k^*}}{k^*} \cdot \#\{\ell : \frac{x_\ell}{\ell} \geq \frac{x_{k^*}}{k^*}\} = k \cdot \frac{x_k^*}{k^*} \quad \text{converges to } \bar{x}.$$

This ends the proof of this lemma. \square

In this part we only consider sequences X_n of iid nonnegative random variables such that $\mathbb{E}(X_1) < \infty$. This assumption can be made in view of the nature of our p series.

Theorem 2. *Let (X_n) be a sequence of iid nonnegative random variables such that $\mathbb{E}(X_1) < \infty$. Then we have*

$$(a) \quad \lim_{n \rightarrow \infty} \frac{N_n(x)}{n} = \mathbb{E}(X_1) \quad a.e. \quad \left(\text{with } N_n(x) = \#\left\{k : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\} \right),$$

$$(b) \quad \lim_{p \rightarrow 1^+} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{X_n(x)}{n} \right)^p \right)^{1/p} = \mathbb{E}(X_1), \quad a.e.$$

Proof. By the previous lemma, (b) is an immediate consequence of (a), so we are left with proving (a).

In our proof of Lemma 1 in [A1], we showed that we have

$$\left\| \frac{X_n(x)}{n} \right\|_{1,\infty} < \infty \quad a.e., \quad \text{because} \quad \overline{\lim}_{n \rightarrow \infty} \frac{\#\{k : \frac{X_k(x)}{k} \geq \frac{1}{n}\}}{n} = \mathbb{E}(X_1).$$

We proved this by noting that

$$N_n(x) = \#\left\{k : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\} = \sum_{n=1}^{\infty} \mathbf{1}\left\{x : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\}.$$

Then we considered

$$V_n(x) = \sum_{n=1}^{\infty} \mathbf{1}\left\{x : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\} - \mu\left\{x : \frac{X_k(x)}{k} \geq \frac{1}{n}\right\}.$$

Kolmogorov's inequality for sums of independent random variables leads to the following inequality for each $\epsilon > 0$.

$$\sum_{n=1}^{\infty} \mu\left\{ \left| \frac{N_{n^2}(x) - \mathbb{E}(N_{n^2})}{n^2} \right| \geq \epsilon \right\} < \infty.$$

An application of the Borel-Cantelli lemma gave us

$$\overline{\lim} \frac{N_{n^2}(x)}{n^2} = \lim_n \frac{\mathbb{E}(N_{n^2})}{n^2} = \mathbb{E}(X_1).$$

Then a simple interpolation allowed us to claim that

$$(8) \quad \overline{\lim}_n \frac{N_n(x)}{n} = \mathbb{E}(X_1).$$

But also in [A1], Theorem 3 shows that for each p , $1 < p \leq \infty$ we have

$$(9) \quad \lim_{n \rightarrow \infty} \frac{\#\{k : \frac{Y_k(x)}{k} \geq 1/n\}}{n} = \mathbb{E}(Y_1)$$

for (Y_n) sequence of iid random variables where $\mathbb{E}(|Y_1|^p) < \infty$ for some $1 < p \leq \infty$.

We take M a positive constant; using (8) and (9) we get

$$\begin{aligned} \mathbb{E}(X_1 \wedge M) &= \lim_n \frac{\#\{k : \frac{X_k(x) \wedge M}{k} \geq 1/n\}}{n} \\ &\leq \underline{\lim} \frac{\#\{k : \frac{X_k(x)}{k} \geq 1/n\}}{n} \\ &= \overline{\lim} \frac{\#\{k : \frac{X_k(x)}{k} \geq 1/n\}}{n} \\ &= \mathbb{E}(X_1). \end{aligned}$$

As $\lim_M \mathbb{E}(X_1 \wedge M) = \mathbb{E}(X_1)$ we have obtained a proof of (a) from which (b) now follows easily. \square

2.2. The multidimensional case. The previous situation can be extended to a more general situation. In [A1] we proved the following:

Given H a positive integer and a nonnegative iid sequence $(X_{1,n})_n$ on the probability measure space $(\Omega_1, \mathcal{F}_1, \mu_1)$ satisfying the condition $\mathbb{E}(X_{1,1}) < \infty$, it is possible to find a set of full measure $\tilde{\Omega}_1$ such that if $x_1 \in \tilde{\Omega}_1$ the following holds:

For all probability measure spaces $(\Omega_2, \mathcal{F}_2, \mu_2)$ and all nonnegative iid sequences $(X_{2n})_n$ such that $\mathbb{E}(X_{2,1}) < \infty$ it is possible to find a set of full measure $\tilde{\Omega}_2$ such that if $x_2 \in \tilde{\Omega}_2$ the following holds:

For all probability measure spaces $(\Omega_H, \mathcal{F}_H, \mu_H)$ and all iid sequences $(X_{H,n})_n$ of nonnegative random variables satisfying $\mathbb{E}(X_{H,1}) < \infty$ we can find a set of full measure $\tilde{\Omega}_H$ for which if $x_H \in \tilde{\Omega}_H$ we have

$$(10) \quad \overline{\lim}_n \frac{\#\{k : \frac{\prod_{i=1}^H X_{i,k}(x_i)}{k} \geq \frac{1}{n}\}}{n} = \prod_{i=1}^H \mathbb{E}(X_{i,1}).$$

The difficulty resides in the way those sets of full measure $\tilde{\Omega}_i$ are obtained; they are independent of the incoming variables $(X_{j,n})$ for $j > i$.

We want to prove that in (10) we actually have convergence to $\prod_{i=1}^H \mathbb{E}(X_{i,1})$. More precisely we have:

Theorem 3. Given H a positive integer and a nonnegative sequence of iid variables $(X_{1n})_n$ on the probability measure space $(\Omega_1, \mathcal{F}_1, \mu_1)$ satisfying the condition $\mathbb{E}(X_{1,1}) < \infty$, it is possible to find a set of full measure $\tilde{\Omega}_1$ such that if $x_1 \in \tilde{\Omega}_1$ the following holds:

For all probability measure spaces $(\Omega_2, \mathcal{F}_2, \mu_2)$ and all nonnegative iid sequences $(X_{2,n})_n$ such that $\mathbb{E}(X_{2,1}) < \infty$, it is possible to find a set of full measure $\tilde{\Omega}_2$ such that if $x_2 \in \tilde{\Omega}_2$ the following holds:

For all probability measure spaces $(\Omega_H, \mathcal{F}_H, \mu, H)$ and all iid sequences $(X_{H,n})_n$ of nonnegative random variables satisfying $\mathbb{E}(X_{H,1}) < \infty$ we can find a set of full measure $\tilde{\Omega}_H$ for which if $x_H \in \tilde{\Omega}_H$ we have

$$(11) \quad \lim_n \frac{\#\{k : \frac{\prod_{i=1}^H X_{i,k}(x_i)}{k} \geq \frac{1}{n}\}}{n} = \prod_{i=1}^H \mathbb{E}(X_{i,1})$$

and

$$(12) \quad \lim_{p \rightarrow 1^+} \left((p-1) \left(\sum_{n=1}^{\infty} \left(\frac{\prod_{i=1}^H X_{i,n}(x_i)}{n} \right)^p \right)^{1/p} = \prod_{i=1}^H \mathbb{E}(X_{i,1}).$$

Proof. As previously we just need to prove (11) to get (12). We use induction to prove (11). The result is true for $H = 1$, as shown in the previous theorem.

Let us assume that the result is true for $H - 1$. Hence if $c_k = \prod_{i=1}^{H-1} X_{ik}(x_i)$ where $x_i \in \tilde{\Omega}_i$ we have

$$(13) \quad \lim_{n \rightarrow \infty} \frac{\#\{k : \frac{c_k}{k} \geq \frac{1}{n}\}}{n} = \prod_{i=1}^{H-1} \mathbb{E}(X_{i,1}).$$

The idea of the proof is the same as in Lemma 1 in [A1]. We have for $x_i \in \tilde{\Omega}_i$, $1 \leq i \leq H - 1$, $(X_{H,n})$ a sequence of nonnegative iid random variables and for all $\epsilon > 0$

$$(14) \quad \sum_{n=1}^{\infty} \mu \left\{ x_H : \left| \frac{N_{n^2}(x_H) - \mathbb{E}(N_{n^2})}{n^2} \right| \geq \epsilon \right\} < \infty$$

where

$$N_{n^2}(x_H) = \frac{\#\{k : \frac{c_k X_{H,k}(x_H)}{k} \geq 1/n^2\}}{n^2}.$$

The inequality (14) is obtained by applying Kolmogorov's inequality to the series of independent random variables

$$\sum_{k=1}^{\infty} \mathbf{1} \left\{ x_H : \frac{c_k X_{H,k}(x_H)}{k} \geq 1/n \right\} - \mu \left\{ x_H : \frac{c_k X_{H,k}(x_H)}{k} \geq 1/n \right\}.$$

The Borel-Cantelli lemma applied to (14) gives us

$$\lim_{n \rightarrow \infty} \frac{N_{n^2}(x_H) - \mathbb{E}(N_{n^2})}{n^2} = 0 \quad \text{a.e. } (x_H).$$

As $\lim_{n \rightarrow \infty} \frac{\mathbb{E}(N_{n^2})}{n^2} = \prod_{i=1}^H \mathbb{E}(X_{i,1})$ we have

$$\lim_{n \rightarrow \infty} \frac{N_{n^2}(x_H)}{n^2} = \prod_{i=1}^H \mathbb{E}(X_{i,1}) \quad \text{a.e. } (x_H).$$

The monotonicity of N_n gives us for $p_n^2 \leq n \leq (p_{n+1})^2$

$$\frac{N_{p_n^2}(x_H)}{p_n^2} \leq \frac{N_n(x_H)}{p_n^2} \leq \frac{N_{(p_{n+1})^2}(x_H)}{p_n^2} = \frac{N_{(p_{n+1})^2}(x_H)}{(p_{n+1})^2} \cdot \frac{(p_{n+1})^2}{(p_n)^2}.$$

This last chain of inequalities implies that

$$\lim_{n \rightarrow \infty} \frac{N_n(x_H)}{n} = \lim_{n \rightarrow \infty} \frac{N_{p_n^2}(x_H)}{p_n^2} = \prod_{i=1}^H \mathbb{E}(X_{i,1}) \quad \text{as } \frac{p_n^2}{n} \rightarrow 1.$$

□

3. Convergence of the p -series for ergodic stationary sequences

In this part the sequence X_n will be given by an ergodic dynamical system (X, \mathcal{F}, μ, T) on a probability measure space (X, \mathcal{F}, μ) . The sequence is defined by the relation $X_n(x) = f(T^n x)$ where f is a nonnegative integrable function.

Proposition 4. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system and f a nonnegative integrable function. We have*

$$\lim_{n \rightarrow \infty} \left\| \frac{N_n(f)}{n} - \int f d\mu \right\|_1 = 0, \quad \text{where}$$

$$\frac{N_n(f)(x)}{n} = \frac{\#\{k : \frac{f(T^k x)}{k} \geq 1/n\}}{n}$$

Proof. We know that $\lim_{n \rightarrow \infty} \frac{N_n(f)}{n} = \int f d\mu$ a.e. for $f \in L_+^p$ for some p , $1 < p \leq \infty$ (see Theorem 3 in [A1]). The difficulty at this level comes from the nature of the function of f , $N_n(f)$; the map N_n is not linear nor positively homogeneous. But we have the following properties:

- (A) $\left\| \frac{N_n(f)}{n} \right\|_\infty \leq \|f\|_\infty$,
- (B) If f, g are nonnegative functions with disjoint support then we have $\frac{N_n(f+g)}{n} = \frac{N_n(f)}{n} + \frac{N_n(g)}{n}$ for all $n \geq 1$.
- (C) For all $f \geq 0$ integrable functions we have $\left\| \frac{N_n(f)}{n} \right\|_1 \leq \|f\|_1$.

(A) and (B) are easy to check.

To establish (C) we take $f \in L^1$ for which we can find for each ϵ nonnegative numbers $(\alpha_i)_i$ and sets $(A_i)_i$ such that $f \leq \sum \alpha_i \mathbf{1}_{A_i}$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\int \sum \alpha_i \mathbf{1}_{A_i} d\mu \leq (1 + \epsilon) \int f d\mu$. We have

$$\frac{N_n(f)}{n} \leq \frac{N_n(\sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{A_i})}{n} \quad \text{by monotonicity.}$$

Thus

$$\begin{aligned} \left\| \frac{N_n(f)}{n} \right\|_1 &\leq \left\| \frac{N_n(\sum_{i=1}^{\infty} \alpha_i \mathbf{1}_{A_i})}{n} \right\|_1 \\ &= \left\| \sum_{i=1}^{\infty} \frac{N_n(\alpha_i \mathbf{1}_{A_i})}{n} \right\|_1 \quad \text{by (B)} \\ &= \sum_{i=1}^{\infty} \left\| \frac{N_n(\alpha_i \mathbf{1}_{A_i})}{n} \right\|_1. \end{aligned}$$

As

$$\begin{aligned} \frac{N_n(\alpha_i \mathbf{1}_{A_i})}{n} &= \frac{\#\{k : \frac{\mathbf{1}_{A_i}(T^k x)}{k} \geq \frac{1}{n\alpha_i}\}}{n} \\ &= \frac{\sum_{k=1}^{[n\alpha_i]} \mathbf{1}_{A_i}(T^k x)}{n} \quad \text{we have} \\ \left\| \frac{N_n(\alpha_i \mathbf{1}_{A_i})}{n} \right\|_1 &= \sum_{k=1}^{[n\alpha_i]} \frac{\mu(A_i)}{n} \leq \frac{(n\alpha_i)\mu(A_i)}{n} = \alpha_i \mu(A_i). \end{aligned}$$

So

$$\left\| \frac{N_n(f)}{n} \right\|_1 \leq \sum_{i=1}^{\infty} \alpha_i \mu(A_i) \leq (1 + \epsilon) \int f d\mu.$$

As ϵ is arbitrary we have reached a proof of (C).

We are now in a position to prove Proposition 4.

For each positive real number M we can write $f = f \wedge M + g_M$ with $f \wedge M$ and g_M nonnegative functions with disjoint support.

We have

$$\frac{N_n(f)}{n} - \int f d\mu = \frac{N_n(f \wedge M)}{n} - \int f \wedge M d\mu + \frac{N_n(g_M)}{n} - \int g_M d\mu.$$

Hence

$$\begin{aligned} \overline{\lim}_n \left\| \frac{N_n(f)}{n} - \int f d\mu \right\|_1 &\leq \overline{\lim}_n \left\| \frac{N_n(f \wedge M)}{n} - \int (f \wedge M) d\mu \right\|_1 \\ &\quad + \overline{\lim}_n \left\| \frac{N_n(g_M)}{n} \right\|_1 + \int g_M d\mu. \end{aligned}$$

By the theorem mentioned at the beginning of this proof, associating the a.e. convergence of $\frac{N_n(f)(x)}{n}$ to $\int f d\mu$ for functions in L^p for some p , we conclude that

$$\overline{\lim}_n \left\| \frac{N_n(f \wedge M)}{n} - \int f \wedge M d\mu \right\|_1 = 0.$$

Hence

$$\overline{\lim}_n \left\| \frac{N_n(f)}{n} - \int f d\mu \right\|_1 \leq 2 \int g_M d\mu, \quad \text{by (C).}$$

As $\int g_M d\mu \xrightarrow{M} 0$, the proof of this proposition is complete. \square

Theorem 5. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system and $f \in L \log L$, $f \geq 0$. Then we have*

$$(a) \quad \left\| \left\| \left(\frac{f(T^k x)}{k} \right) \right\|_{1, \infty} \right\|_1 = \left\| \sup_n n \cdot \frac{f(T^{n^*} x)}{n^*} \right\|_1 < \infty$$

where $\frac{f(T^{n^*} x)}{n^*}$ is for μ a.e. x a decreasing rearrangement of the sequence $\frac{f(T^n x)}{n}$.

$$(b) \quad \lim_{n \rightarrow \infty} \frac{N_n(f)(x)}{n} = \int f d\mu, \quad \mu \text{ a.e.}$$

$$(c) \quad \lim_{p \rightarrow 1^+} \left((p-1) \sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p} = \int f d\mu, \quad \mu \text{ a.e.}$$

Proof. First we can make the following observations:

For all measurable sets A we have

$$(15) \quad \begin{aligned} \left\| \left(\frac{\mathbf{1}_A(T^k x)}{k} \right) \right\|_{1, \infty} &= \sup_{t > 0} \frac{\#\{k : \frac{\mathbf{1}_A(T^k x)}{k} \geq 1/t\}}{t} = \sup_n \frac{\#\{k : \frac{\mathbf{1}_A(T^k x)}{k} \geq 1/n\}}{n} \\ &= \sup_n \frac{N_n(\mathbf{1}_A)(x)}{n} \\ &= N^*(\mathbf{1}_A)(x). \end{aligned}$$

Because of the maximal inequality for the ergodic averages we have

$$(16) \quad \mu\{x : N^*(\mathbf{1}_A)(x) > \lambda\} \leq \frac{1}{\lambda} \cdot \mu(A) \quad \text{for all } \lambda > 0.$$

(Note that $N^*(\mathbf{1}_A)(x) \leq 1$, hence for all $p \geq 1$ we also have

$$(17) \quad \mu\{x : N^*(\mathbf{1}_A)(x) > \lambda\} \leq \frac{1}{\lambda^p} \cdot \mu(A).$$

For all positive real numbers y we have:

$$(18) \quad y^{\frac{i}{i+1}} = y^{\frac{i}{i+1}} \cdot \frac{(i+1)^{1/i+1}}{(i+1)^{1/i+1}} \leq \frac{y(i+1)^{1/i}}{(i+1)} + \frac{1}{(i+1)^2}$$

(apply the inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, for $a = y^{i/i+1} \cdot (i+1)^{1/i+1}$, $b = \frac{1}{(i+1)^{1/i+1}}$, $p = \frac{i+1}{i}$ and $q = \frac{p}{p-1} = i+1$).

We proceed now with the proof of Theorem 5 (a).

We take $f \in L \log L$ and denote by A_i the set

$$A_i = \{2^i \leq f < 2^{i+1}\}.$$

We have

$$\begin{aligned} N^*(f) &\leq N^*\left(\sum_{i=1}^{\infty} 2^{i+1} \mathbf{1}_{A_i}\right) \leq \sum_{i=1}^{\infty} N^*(2^{i+1} \mathbf{1}_{A_i}) \\ &= 2 \sum_{i=1}^{\infty} 2^i \cdot N^*(\mathbf{1}_{A_i}). \end{aligned}$$

By taking the integral with respect to the measure μ we get

$$\|N^*(f)\|_1 \leq 2 \sum_{i=1}^{\infty} 2^i \|N^*(\mathbf{1}_{A_i})\|_1$$

Using (17) we get

$$\begin{aligned} \|N^*(\mathbf{1}_{A_i})\|_1 &\leq \frac{p}{(p-1)} \sup_{t>0} [t \cdot \mu\{x : n^*(\mathbf{1}_{A_i})(x) > t\}] \\ &\leq \frac{p}{(p-1)} \cdot (\mu(A_i))^{1/p} \quad \text{for all } p, 1 \leq p < \infty. \end{aligned}$$

((17) is combined with the inequality $\|g\|_{L^1} \leq \frac{p}{(p-1)} \sup_{t>0} [t \mu\{x : |g(x)| > t\}^{1/p}]$.)
Going back to the evaluation of $\|N^*(f)\|_1$ we get

$$\begin{aligned} \|N^*(f)\|_1 &\leq 2 \sum_{i=1}^{\infty} 2^i \frac{(i+1/i)}{1/i} (\mu(A_i))^{1/i+1} \\ &= 2 \sum_{i=1}^{\infty} 2^i (i+1) (\mu(A_i))^{i/i+1} \\ &= 2 \sum_{i=1}^{\infty} ((2^i(i+1))^{i+1/i} \mu(A_i))^{i/i+1}. \end{aligned}$$

Applying (18) to each term $((2^i(i+1))^{i+1/i} \mu(A_i))^{i/i+1}$ we get

$$\begin{aligned} \|N^*(f)\|_1 &\leq 2 \sum_{i=1}^{\infty} [(2^i(i+1))^{i+1/i} \cdot (\mu(A_i)) \frac{(i+1)^{1/i}}{i+1} i + \frac{1}{(i+1)^2}] \\ &\leq 4 \sum_{i=1}^{\infty} [2^i i \mu(A_i) \cdot (1+i)^{2/i} + \frac{1}{(i+1)^2}] \\ &\leq 12 \cdot \sum_{i=1}^{\infty} [2^i i \mu(A_i) + \frac{1}{(i+1)^2}] \\ &\leq \frac{12}{\ln 2} \left[\int f \log f d\mu + 1 \right]. \end{aligned}$$

Thus we have proved the following inequality

$$(19) \quad \|N^*(f)\|_1 \leq \frac{12}{\ln 2} \left[\int f \log f d\mu + 1 \right] \quad \text{for all } f \geq 0, f \in L \log L.$$

This clearly ends the proof of Theorem 5 (a).

It remains to show (b). Our goal is to prove that for $f \geq 0, f \in L \log L$

$$(20) \quad \lim_n N^*(f - f \wedge n) = 0 \text{ a.e.}$$

Using (19) we have for all $t > 0$,

$$\|N^*(t(f - f \wedge n))\|_1 \leq \frac{12}{\ln 2} \left[\int (t(f - f \wedge n)) \log(t(f - f \wedge n)) d\mu + 1 \right] \\ \text{for all } f \geq 0, f \in L \log L.$$

This last inequality gives us

$$\|N^*(f - f \wedge n)\|_1 \leq \frac{12}{\ln 2} \left[\int (f - f \wedge n) \log(t(f - f \wedge n)) d\mu + \frac{1}{t} \right].$$

At the expense of taking a subsequence, we derive from it

$$\lim_k \|N^*(f - f \wedge n_k)\|_1 \leq \frac{12}{\ln 2} \cdot \frac{1}{t}.$$

Then we easily get $\lim_n N^*(f - f \wedge n) = 0$ a.e. This proves (20).

As $\lim_n \frac{N_k(f \wedge n)}{k} = \int f \wedge n d\mu$, because $f \wedge n$ is clearly bounded we have

$$\frac{N_k(f \wedge n)}{k} \leq \frac{N_k(f)}{k} = \frac{N_k(f \wedge n)}{k} + \frac{N_k(f - f \wedge n)}{k}$$

and after taking the limits we obtain

$$\int f \wedge n d\mu \leq \frac{\lim N_k(f)}{k} \leq \overline{\lim} \frac{N_k(f)}{k} \leq \overline{\lim} \frac{N_k(f \wedge n)}{k} + N^*[f - f \wedge n] \\ = \int f \wedge n d\mu + N^*[f - f \wedge n].$$

Finally, by taking the $\lim \inf$ with respect to n we can conclude that

$$\lim_k \frac{N_k(f)}{k} = \int f d\mu \text{ a.e.}$$

This proves Theorem 5 (b). Theorem 5 (c) now follows easily from Lemma 1. This ends the proof of Theorem 5. □

Corollary 6. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system and $f \in L \log L$, $f \geq 0$. Then there exists an absolute constant C such that*

$$\left\| \sup_{1 < p < \infty} (p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p} \right\|_1 \leq C \left[\int f \log f d\mu + 1 \right]$$

Proof. By Theorem 5 we know that

$$\left\| \sup_n n \cdot \frac{f(T^{n^*} x)}{n^*} \right\|_1 < \infty.$$

As for μ a.e. x , for each p , we have

$$(p-1)^{1/p} \left(\sum_{n=1}^{\infty} \left(\frac{f(T^n x)}{n} \right)^p \right)^{1/p} \leq \left(\sup_n n \cdot \frac{f(T^{n^*} x)}{n^*} \right) \cdot \left((p-1) \sum_{n=1}^{\infty} 1/n^p \right)^{1/p},$$

the corollary follows easily. \square

REMARK.

- 1) One can see that the limit when p tends to ∞ of the p -Series is equals to $\sup_n \frac{f(T^n x)}{n}$. This is the reason why we only focus on the existence of the limit when p tends to $1+$.
- 2) We proved in [A2] that if $N^*(f)(x)$ is a.e finite for all functions $f \in L^1_+$ then $M^*_2(f)(x)$ is also a.e finite for all functions $f \in L^1$.
- 3) The results obtained in this note can be extended to increasing sequences of integers $(p_n)_n$. The corresponding maximal function to consider is simply

$$\left\| \left(\frac{f(T^{p_n} x)}{n} \right) \right\|_{1, \infty}.$$

To illustrate this we have the following Proposition.

Proposition 7. *Let (X, \mathcal{F}, μ, T) be an ergodic dynamical system, p a fixed positive real number $1 < p < \infty$ and $(p_n)_n$ an increasing sequence of positive integers. Consider the following statements*

$$(a) \quad \left\| \left(\frac{f(T^{p_n} x)}{n} \right) \right\|_{1, \infty} < \infty \text{ a.e. for all } f \in L^p_+.$$

$$(b) \quad \sup_k k^{p-1} \sum_{n=k}^{\infty} \left(\frac{f(T^{p_n} x)}{n} \right)^p < \infty \text{ a.e. for all } f \in L^p_+.$$

$$(c) \quad \sup_N \frac{1}{N} \sum_{n=1}^N f(T^{p_n} x) < \infty \text{ a.e. for all } f \in L^p_+.$$

$$(d) \quad \sup_N \frac{1}{N} \sum_{n=1}^N f(T^{p_n} x) < \infty \text{ a.e. for all } f \in L(p, 1).$$

Then we have the following implications: (a) implies (d), (b) is equivalent to (c), (b) implies (a) and (c) implies (d).

Proof. The implications (c) implies (b) and (b) implies (a) can be proved the same way we did in [A1] for the usual Cesaro averages. (See the proof of Theorem 3 part b) in [A1]). The implication (c) implies (d) is a direct consequence of the structure of $L(p, q)$ spaces as shown in [SW].

It remains to prove the implications (a) implies (d) and (b) implies (c). For (a) implies (d), we can notice that (a) implies the existence of a finite constant C_p such that for all $f \in L^p_+$

$$(21) \quad \mu\{x : \left\| \left(\frac{f(T^{p^n} x)}{n} \right) \right\|_{1, \infty} > \lambda\} \leq \frac{C_p}{\lambda^p} \int |f|^p d\mu \quad \text{for all } \lambda > 0.$$

In the particular case of $f = \mathbf{1}_A$, (21) will give us the following

$$(22) \quad \mu\{x : \sup_N \frac{1}{N} \sum_{n=1}^N \mathbf{1}_A(T^{p^n} x) > \lambda\} \leq \frac{C_p}{\lambda^p} \mu(A) \quad \text{for all } \lambda > 0$$

because

$$\left\| \left(\frac{\mathbf{1}_A(T^{p^n} x)}{n} \right) \right\|_{1, \infty} = \sup_N \frac{1}{N} \sum_{n=1}^N \mathbf{1}_A(T^{p^n} x).$$

As this inequality is valid for all measurable sets A, we can conclude that the maximal operator

$$M^*(f)(x) = \sup_N \frac{1}{N} \sum_{n=1}^N f(T^{p^n} x)$$

is of restricted weak type (p,p) (see [SW]). In other words, the maximal operator M^* maps the characteristic function of any measurable set A from $L(p, 1)$ into $L(p, \infty)$. It is shown in [SW] that the nature of the maximal operator M^* and the existence of an equivalent norm on $L(p, \infty)$, making it a Banach space, M^* maps continuously all functions $f \in L(p, 1)$ into $L(p, \infty)$. This means the existence of a finite constant C_p such that for all $f \in L(p, 1)$,

$$(23) \quad \mu\{x : \sup_N \frac{1}{N} \sum_{n=1}^N f(T^{p^n} x) > \lambda\} \leq \frac{C_p}{\lambda^p} \|f\|_{p, 1}^p \quad \text{for all } \lambda > 0.$$

From this we can clearly derive (d).

The implication (b) implies (c) can be obtained by summation. For $f \in L^p_+$ let us denote by C_x the finite constant which dominates the sup on k. Then for each k, we have

$$k^{p-1} \sum_{n=k}^{2k} \left(\frac{f(T^{p^n} x)}{2k} \right)^p < C_x.$$

This implies the inequality

$$\sup_k \frac{1}{k} \sum_{n=k}^{2k} f(T^{p^n} x)^p < 2^p C_x.$$

From this we can derive by convexity the uniform boundedness of the averages $\frac{1}{k} \sum_{n=k}^{2k} f(T^{p^n} x)$. This property allows us to obtain (c) without difficulty. \square

REMARK. It would be interesting to know if (a) implies (c) for any increasing sequence p_n and any p , $1 \leq p < \infty$. For $p = 1$, $p_n = n$ we already mentioned that (a) implies (c) (see [A2]).

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