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On a Class of Toeplitz + Hankel Operators

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Abstract. In this paper we study operators of the form $M(\phi) = T(\phi) + H(\phi)$ where $T(\phi)$ and $H(\phi)$ are the Toeplitz and Hankel operators acting on ℓ_2. We investigate the connection between Fredholmness and invertibility of $M(\phi)$ for functions $\phi \in L^\infty(\mathbb{T})$. Using this relationship we establish necessary and sufficient conditions for the invertibility of $M(\phi)$ with piecewise continuous ϕ. Finally, we consider several stability problems related to $M(\phi)$, in particular the stability of the finite section method.

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1. Introduction

This paper is devoted to the study of operators of the form

$$M(\phi) \quad = \quad T(\phi) + H(\phi). \tag{1}$$

Here $\phi \in L^\infty(\mathbb{T})$ is a Lebesgue measurable and essentially bounded function on the unit circle $\mathbb{T}$ with Fourier coefficients ϕ_n. The Toeplitz and Hankel operators are defined as usual in terms of the infinite matrices

$$T(\phi) \ = \ \left(\phi_{j-k}\right)_{j,k=0}^{\infty}, \qquad H(\phi) \ = \ \left(\phi_{j+k+1}\right)_{j,k=0}^{\infty}. \tag{2}$$

The operators are considered as acting on the Hilbert space $\ell_2 = \ell_2(\mathbb{Z}_+)$ of one-sided square-summable sequences.

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The operators $M(\phi)$ represent a special case of Toeplitz + Hankel operators of the form $T(\phi)+H(\psi)$ with arbitrary functions ϕ and ψ in $L^\infty(\mathbb{T})$. Unfortunately, it seem hopeless to ask for invertibility in this general setting, even if ϕ and ψ are continuous functions. However, in this paper we will show that the operators $M(\phi)$ possess certain properties that make it possible to solve invertibility and related problems for piecewise continuous ϕ.

In the next section we establish basic properties of $M(\phi)$. Among other things, it turns out that in the case of even functions ϕ the invertibility of $M(\phi)$ is almost trivial. In the general case we obtain the remarkable result that $M(\phi)$ is invertible if and only if $M(\phi)$ is Fredholm and has index zero. As in the Toeplitz case, this is the key for studying invertibility.

In the third section we establish necessary and sufficient conditions for the invertibility of $M(\phi)$ in the case of piecewise continuous ϕ. These results are based on Fredholm criteria that can be obtained from the work of Power [6] or Böttcher and Silbermann [3]. Additional considerations are needed in order to determine the Fredholm index. We also give a geometric description of the spectrum of $M(\phi)$.

As an application of the invertibility results, we investigate in the fourth section the stability of the sequence of the finite sections

$$M_n(\phi) \quad = \quad T_n(\phi)+H_n(\phi). \tag{3}$$

Here the $n\times n$ Toeplitz and Hankel matrices are defined by

$$T_n(\phi) \;=\; \big(\phi_{j-k}\big)_{j,k=0}^{n-1}, \qquad H_n(\phi) \;=\; \big(\phi_{j+k+1}\big)_{j,k=0}^{n-1}. \tag{4}$$

The function ϕ is assumed to be piecewise continuous. The proof of the stability criterion relies on results obtained by Roch and Silbermann [7] and Spitkovsky and Tashbaev [8].

Another application is treated in the last section. There we examine the stability problem for sequences $\{M(\phi_\lambda)\}_\lambda$ as $\lambda\to\infty$. Here $\{\phi_\lambda\}_\lambda$ are certain sequences of (smooth) functions related to a given piecewise continuous function ϕ. For example, the function ϕ_λ may arise from any approximate identity of ϕ. The proofs are based on results of [5].

The investigations taken up in this paper (in particular the afore-mentioned applications) are motivated by a recent paper [2]. There the asymptotic behavior of the determinants of Toeplitz + Hankel matrices, $\det M_n(\phi)$, for certain piecewise continuous functions ϕ is studied. The interest in the asymptotic behavior of these determinants came from random matrix theory. For certain matrix ensembles one needs to find the asymptotic formula for the determinant of a sum of Wiener–Hopf and Hankel integral operators, which is the continuous analogue of the problem considered here. The desired asymptotic formula allows one to compute the distribution function for a discontinuous random variable. The discontinuous case is particularly important because it corresponds to counting the number of eigenvalues of a random matrix in an interval. For more details on this topic we refer the reader to the literature given in [2].

2. Basic properties of $M(\phi)$

In this section we present basic properties of $M(\phi)$ with symbols $\phi\in L^\infty(\mathbb{T})$. We first introduce some notation. Let $\ell_2(\mathbb{Z})$ be the Hilbert space of two-sided square-summable sequences. The Laurent operator acting on $\ell_2(\mathbb{Z})$ with generating function $\phi \in L^\infty(\mathbb{T})$ is defined by the infinite matrix

$$L(\phi) \quad = \quad \left(\phi_{j-k}\right)_{j,k=-\infty}^{\infty}. \tag{5}$$

It is well known that $\|L(\phi)\| = \|\phi\|_\infty$. We also need the projection

$$P : (x_k)_{k\in\mathbb{Z}} \mapsto (y_k)_{k\in\mathbb{Z}}, \qquad y_k \;=\; \begin{cases} x_k & \text{if } k \geq 0 \\ 0 & \text{if } k < 0, \end{cases} \tag{6}$$

the associated projection $Q = I - P$, and the flip operator

$$J : (x_k)_{k\in\mathbb{Z}} \mapsto (x_{-1-k})_{k\in\mathbb{Z}}. \tag{7}$$

Note that $JP = QJ$, $J^2 = I$, and that P, Q and J are selfadjoint. Moreover, we have $JL(\phi)J = L(\widetilde{\phi})$ where $\widetilde{\phi}$ is the function defined by $\widetilde{\phi}(t) = \phi(1/t)$, $t \in \mathbb{T}$. It is easy to observe that

$$T(\phi) \;=\; PL(\phi)P, \qquad H(\phi) \;=\; PL(\phi)JP, \tag{8}$$

when identifying the image of P with the space $\ell_2 = \ell_2(\mathbb{Z}_+)$. Hence

$$M(\phi) \quad = \quad PL(\phi)(I + J)P. \tag{9}$$

In the following, it will occasionally be convenient to identify ℓ_2 with the Hardy space $H^2(\mathbb{T})$ and $\ell_2(\mathbb{Z})$ with the Lebesgue space $L^2(\mathbb{T})$ and to consider the operators as acting on these spaces. In this setting, P is the Riesz projection, $L(\phi)$ represents the multiplication operator, and J is the operator that maps a function $f(t)$ into the function $1/t \cdot f(1/t)$.

As a first result, we establish an estimate of the norm of $M(\phi)$.

Proposition 2.1. *Let $\phi \in L^\infty(\mathbb{T})$. Then* $\|\phi\|_\infty \leq \|M(\phi)\| \leq \sqrt{2}\,\|\phi\|_\infty$.

Proof. Notice that

$$\|(I+J)P\|^2 = \|P^*(I+J)^*(I+J)P\| = 2\,\|P(I+J)P\| = 2\,\|P\| = 2.$$

Hence $\|(I+J)P\| = \sqrt{2}$. Taking (9) into account, the upper estimate follows.

Now let U_n denote the multiplication operator $f(t) \mapsto t^n f(t)$ on $L^2(\mathbb{T})$. Since $U_nU_{-n} = I$ and $JU_n = U_{-n}J$, it is easy to see that

$$U_{-n}M(\phi)U_n \;=\; (U_{-n}PU_n)L(\phi)(U_{-n}PU_n) + (U_{-n}PU_{-n})L(\phi)J(U_{-n}PU_n).$$

Because $U_{-n}PU_n \to I$ and $U_{-n}PU_{-n} \to 0$ strongly on $L^2(\mathbb{T})$ as $n \to \infty$, we have

$$U_{-n}M(\phi)U_n \quad \to \quad L(\phi) \tag{10}$$

strongly on $L^2(\mathbb{T})$ as $n \to \infty$. Note that $U_{\pm n}$ are isometries on $L^2(\mathbb{T})$, and therefore (10) implies the lower estimate. $\square$

The examples of functions $\phi(t) = 1$ and $\phi(t) = t$ show that the lower and upper estimates can in general not be improved.

Recall that for Toeplitz and Hankel operators the following relations hold:

$$T(\phi\psi) \;=\; T(\phi)T(\psi) + H(\phi)H(\widetilde{\psi}), \tag{11}$$

$$H(\phi\psi) \;=\; T(\phi)H(\psi) + H(\phi)T(\widetilde{\psi}). \tag{12}$$

Adding both equations, it follows that

$$M(\phi\psi) \;=\; T(\phi)M(\psi) + H(\phi)M(\widetilde{\psi}),$$

and hence

$$M(\phi\psi) = M(\phi)M(\psi) + H(\phi)M(\widetilde{\psi} - \psi). \tag{13}$$

This formula is in some sense the analogue of (11).

Let $\overline{H^\infty(\mathbb{T})}$ be the Hardy space of all functions $\phi \in L^\infty(\mathbb{T})$ for which $\phi_n = 0$ for each $n > 0$. Moreover, denote by $L^\infty_{\text{even}}(\mathbb{T})$ the C*-algebra of all functions $\phi \in L^\infty(\mathbb{T})$ for which $\phi = \widetilde{\phi}$. If $\phi \in \overline{H^\infty(\mathbb{T})}$ or $\psi \in L^\infty_{\text{even}}(\mathbb{T})$, then (13) simplifies to

$$M(\phi\psi) = M(\phi)M(\psi). \tag{14}$$

This means that there exist two classes of functions for which $M(\phi)$ is multiplicative. If $\phi \in \overline{H^\infty(\mathbb{T})}$, then $M(\phi) = T(\phi)$. The properties of such Toeplitz operators are well known. The second class, which consists of even functions, is considered next.

Theorem 2.2. *The mapping $\Lambda : \phi \mapsto M(\phi)$ is a $*$-isomorphism from $L^\infty_{\text{even}}(\mathbb{T})$ onto a $*$-subalgebra of $\mathcal{L}(\ell_2)$.*

Proof. Using the fact that $JL(\phi) = L(\widetilde{\phi})J$ and $J^2 = I$, and equation (9), we can write

$$M(\phi) = 1/2\, P(I+J)L(\phi)(I+J)P.$$

Now it is easy to see that $\Lambda(\phi)^* = \Lambda(\overline{\phi})$, where $\overline{\phi}$ denotes the complex conjugate of ϕ. The fact that Λ is multiplicative follows from (14). Because of Proposition 2.1, the mapping Λ has a trivial kernel. Hence Λ is a $*$-isomorphism. □

We remark that this result implies in particular that $\|M(\phi)\| = \|\phi\|_\infty$ for even ϕ. This improvement of Proposition 2.1 can be proved directly by invoking the above identity along with $\|(I+J)P\| = \|P(I+J)\| = \sqrt{2}$.

We see from Theorem 2.2 that the case of operators $M(\phi)$ with even symbols is especially simple. In particular, one can solve the invertibility problem completely.

In what follows, for a Banach algebra B, the notation GB stands for the group of all invertible elements in this Banach algebra.

Corollary 2.3. *Let $\phi \in L^\infty_{\text{even}}(\mathbb{T})$. Then $M(\phi)$ is invertible if and only if $\phi \in GL^\infty_{\text{even}}(\mathbb{T})$. Moreover, if this is true, then the inverse is given by $M(\phi^{-1})$.*

In general, the invertibility problem is much more delicate. We will see below how invertibility can be related to Fredholmness. A necessary Fredholm condition is given next.

Proposition 2.4. *Let $\phi \in L^\infty(\mathbb{T})$, and suppose that $M(\phi)$ is Fredholm. Then $\phi \in GL^\infty(\mathbb{T})$.*

Proof. The proof is based on standard arguments. If $M(\phi)$ is a Fredholm operator, then there exist a $\delta > 0$ and a (finite rank) projection K on the kernel of $M(\phi)$ such that

$$\|M(\phi)f\|_{H^2(\mathbb{T})} + \|Kf\|_{H^2(\mathbb{T})} \geq \delta\|f\|_{H^2(\mathbb{T})}$$

for all $f \in H^2(\mathbb{T})$. Putting Pf instead of f, this implies that

$$\|M(\phi)f\|_{L^2(\mathbb{T})} + \|PKPf\|_{L^2(\mathbb{T})} + \delta\|(I-P)f\|_{L^2(\mathbb{T})} \geq \delta\|f\|_{L^2(\mathbb{T})}$$

for all $f \in L^2(\mathbb{T})$. Replacing f by $U_n f$ and observing again that $U_{\pm n}$ are isometries on $L^2(\mathbb{T})$, it follows that

$$\|U_{-n}M(\phi)U_n f\|_{L^2(\mathbb{T})} + \|PKPU_n f\|_{L^2(\mathbb{T})} + \delta\|U_{-n}(I-P)U_n f\|_{L^2(\mathbb{T})} \geq \delta\|f\|_{L^2(\mathbb{T})}.$$

Now we take the limit $n \to \infty$. Because $U_n \to 0$ weakly, we have $PKPU_n \to 0$ strongly. It remains to apply (10) and again the fact that $U_{-n}PU_n \to I$ strongly. We obtain

$$\|L(\phi)f\|_{L^2(\mathbb{T})} \quad \geq \quad \delta\|f\|_{L^2(\mathbb{T})}.$$

This implies the desired assertion. □

The next result is concerned with an assertion about the kernel and cokernel of $M(\phi)$. It is the analogue of Coburn's result for Toeplitz operators. We begin with the following observation. Given $\phi \in L^\infty(\mathbb{T})$, define

$$K \quad = \quad \left\{\, t \in \mathbb{T} \;\middle|\; \phi(t) = \phi(t^{-1}) = 0 \,\right\}. \tag{15}$$

This definition of course depends on the choice of representatives for ϕ, however the following remarks are independent of that choice. Note that the characteristic function χ_K is real and even. Thus $M(\chi_K)$ is a selfadjoint projection (Theorem 2.2). Since $M(\phi)M(\chi_K) = M(\phi\chi_K) = 0$, we obtain that

$$\operatorname{im} M(\chi_K) \quad \subseteq \quad \ker M(\phi). \tag{16}$$

If the set K has Lebesgue measure zero, then obviously $M(\chi_K) = 0$, whereas if K has a positive Lebesgue measure, then the image of $M(\chi_K)$ is infinite dimensional. The latter fact can be seen by decomposing K into pairwise disjoint and even sets $K_1, \ldots, K_n$ with positive Lebesgue measure. Then $M(\chi_K) = M(\chi_{K_1}) + \cdots + M(\chi_{K_n})$, where $M(\chi_{K_1}), \ldots, M(\chi_{K_n})$ are mutually orthogonal projections which are all nonzero (by Proposition 2.1). This shows that $\dim \operatorname{im} M(\chi_K) \geq n$ for all n.

Theorem 2.5. *Let $\phi \in L^\infty(\mathbb{T})$ and let K be as above. Then $\ker M(\phi) = \operatorname{im} M(\chi_K)$ or $\ker M^*(\phi) = \{0\}$.*

Proof. As before we consider the operator $M(\phi)$ as being defined on $H^2(\mathbb{T})$. Then $M(\phi)$ and its adjoint can be written in the form:

$$M(\phi) \;=\; PL(\phi)(I+J)P, \qquad\qquad M^*(\phi) \;=\; P(I+J)L(\overline{\phi})P.$$

Suppose that we have functions $f_+, g_+ \in H^2(\mathbb{T})$ such that $M(\phi)f_+ = 0$ and $M^*(\phi)g_+ = 0$ with $g_+ \neq 0$. We have to show that $f_+ \in \operatorname{im} M(\chi_K)$. Introducing the functions

$$\begin{aligned} f(t) &= f_+(t) + t^{-1}f_+(t^{-1}), & f_-(t) &= \phi(t)f(t), \\ g(t) &= \overline{\phi}(t)g_+(t), & g_-(t) &= g(t) + t^{-1}g(t^{-1}), \end{aligned}$$

it follows at once that $f_-, g_- \in H^2_-(\mathbb{T})$, where $H^2_-(\mathbb{T})$ denotes the kernel of P. We obtain from the definition of g_- that $t^{-1}g_-(t^{-1}) = g_-(t)$. Hence $g_- = 0$ by checking the Fourier coefficients. It follows that $t^{-1}g(t^{-1}) = -g(t)$. On the other hand, the definition of f says that $t^{-1}f(t^{-1}) = f(t)$. This implies that $(f\overline{g})(t^{-1}) = -(f\overline{g})(t)$. Also from the above relations we conclude that $f\overline{g} = f\phi\overline{g}_+ = f_-\overline{g}_+$. Because $f_- \in H^2_-(\mathbb{T})$ and $\overline{g}_+ \in \overline{H^2(\mathbb{T})}$, we have $f_-\overline{g}_+ \in H^1_-(\mathbb{T})$. Considering again the Fourier coefficients, this shows $f\overline{g} = f_-\overline{g}_+ = 0$. Since $g_+ \in H^2(\mathbb{T})$ and $g_+ \neq 0$ the F. and M. Riesz Theorem says that $g_+(t) \neq 0$ almost everywhere on $\mathbb{T}$, and thus we obtain $f_- = 0$. Hence $\phi f = 0$. Because $t^{-1}f(t^{-1}) = f(t)$, we have also $\widetilde{\phi}f = 0$. The definition of the set K now implies that $(1-\chi_K)f = 0$. Noting that $M(1-\chi_K)f_+ = 0$, we finally arrive at $f_+ \in \operatorname{im} M(\chi_K)$. □

Combining Theorem 2.5 with Proposition 2.4 we obtain the following result.

Corollary 2.6. *Let $\phi \in L^\infty(\mathbb{T})$. If $M(\phi)$ is Fredholm, then $M(\phi)$ has a trivial kernel or a trivial cokernel.*

We remark that the previous assertion need not be true for Toeplitz + Hankel operators in general. A counterexample is given by $T(\phi)+H(\psi)$ with $\phi(t)=1$ and $\psi(t)=-t$.

Another conclusion is the next statement, which reduces the invertibility problem for $M(\phi)$ to the Fredholm problem and the index computation.

Corollary 2.7. *Let $\phi \in L^\infty(\mathbb{T})$. Then $M(\phi)$ is invertible if and only if $M(\phi)$ is Fredholm and has index zero.*

Note that in the case of a continuous function ϕ, the operator $M(\phi)$ is just $T(\phi)$ plus a compact operator. This in conjunction with the well known invertibility result for Toeplitz operators puts us in position to obtain the following simple result. If ϕ is continuous, then $M(\phi)$ is invertible if and only if ϕ does not vanish on $\mathbb{T}$ and has winding number zero.

Finally, we give a sufficient invertibility condition that relies on factorization. This condition follows directly from equation (14). Suppose that $\phi \in L^\infty(\mathbb{T})$ can be factored in the form $\phi = \phi_-\phi_0$ with $\phi_- \in G\overline{H^\infty(\mathbb{T})}$ and $\phi_0 \in GL^\infty_{\mathrm{even}}(\mathbb{T})$. Then $M(\phi)$ is invertible and the inverse equals

$$M^{-1}(\phi) \quad = \quad M(\phi_0^{-1})T(\phi_-^{-1}). \tag{17}$$

It seems very likely that one can formulate a necessary and sufficient invertibility (or Fredholm) criterion for $M(\phi)$ in terms of a factorization of the symbol with precise conditions on the factors as is well known in the Toeplitz case. However, we must admit that we have not been able to accomplish such a result.

3. Fredholmness and invertibility for *PC* symbols

Let PC stand for the C*-algebra of all piecewise continuous functions on the unit circle, i.e., functions ϕ for which the one-sided limits $\phi(\tau \pm 0) := \lim_{\theta \to +0} \phi(\tau e^{\pm i\theta})$ exist for each $\tau \in \mathbb{T}$. In this section we want to establish Fredholmness and invertibility criteria for $M(\phi)$ with ϕ piecewise continuous. With regard to Corollary 2.7, it is necessary in this connection to determine the Fredholm index of $M(\phi)$.

The $*$-subalgebra of $\mathcal{L}(\ell_2)$ generated by Toeplitz and Hankel operators with piecewise continuous symbols has been examined by Power [6]. The description given there allows us to derive necessary and sufficient conditions for Fredholmness of operators of the form $T(\phi)+H(\psi)$ with $\phi, \psi \in PC$. A similar examination was made also by Böttcher and Silbermann [3, Sect. 4.95–4.102], using a different method.

We recall these results for the special case $\phi = \psi$. For this purpose we introduce the set $\mathbb{T}_+ = \{\, t \in \mathbb{T} \mid \operatorname{Im} t > 0 \,\}$.

Proposition 3.1. *Let $\phi \in PC$. Then $M(\phi)$ is Fredholm if and only if*

$$b_{\tau,\mu}(\phi) := \phi(\tau+0)\phi(\bar\tau+0)\mu + \phi(\tau-0)\phi(\bar\tau-0)(1-\mu)$$

is nonzero for all $\mu \in [0,1]$ and all $\tau \in \mathbb{T}_+$ and

$$b_{\tau,\mu}(\phi) := \phi(\tau+0)\mu + \phi(\tau-0)(1-\mu) - i\tau\Big(\phi(\tau+0)-\phi(\tau-0)\Big)\sqrt{\mu(1-\mu)}$$

is nonzero for all $\mu \in [0,1]$ *and all* $\tau \in \{-1,1\}$.

Proof. In the above references, the condition for $\tau \in \{-1,1\}$ is stated explicitly. In addition, for $\tau \in \mathbb{T}_+$ one has the requirement that the matrix $B_{\tau,\mu}(\phi)$ given by

$$\begin{pmatrix} \phi(\tau+0)\mu + \phi(\tau-0)(1-\mu) & \Big(\phi(\tau+0)-\phi(\tau-0)\Big)\sqrt{\mu(1-\mu)} \\ \Big(\phi(\bar\tau-0)-\phi(\bar\tau+0)\Big)\sqrt{\mu(1-\mu)} & \phi(\bar\tau+0)\mu + \phi(\bar\tau-0)(1-\mu) \end{pmatrix}$$

be invertible for each $\mu \in [0,1]$. But it is easy to see that $\det B_{\tau,\mu}(\phi) = b_{\tau,\mu}(\phi)$. □

Corollary 3.2. *Let* $\phi \in PC$. *Then* $M(\phi)$ *is Fredholm if and only if* $\phi(\tau \pm 0) \neq 0$ *for each* $\tau \in \mathbb{T}$ *and the following is satisfied:*

$$\frac{1}{2\pi}\arg\left(\frac{\phi(\tau-0)\phi(\bar\tau-0)}{\phi(\tau+0)\phi(\bar\tau+0)}\right) \notin \mathbb{Z}+\frac{1}{2} \quad \textit{for each } \tau \in \mathbb{T}_+, \tag{18}$$

$$\frac{1}{2\pi}\arg\left(\frac{\phi(\tau-0)}{\phi(\tau+0)}\right) \notin \mathbb{Z}+\frac{\tau}{4} \quad \textit{for each } \tau \in \{-1,1\}. \tag{19}$$

Proof. If $\tau \in \mathbb{T}_+$ is fixed and μ runs from 0 to 1, then the value of $b_{\tau,\mu}(\phi)$ runs along the line segment from $\phi(\tau-0)\phi(\bar\tau-0)$ to $\phi(\tau+0)\phi(\bar\tau+0)$. Similarly, if $\tau \in \{-1,1\}$ is fixed and μ runs from 0 to 1, then the value of $b_{\tau,\mu}(\phi)$ runs along the half circle with endpoints $\phi(\tau-0)$ and $\phi(\tau+0)$ in positive ($\tau=1$) or negative ($\tau=-1$) direction, respectively. □

The preceding corollary settles completely the problem of Fredholmness for $M(\phi)$ with piecewise continuous ϕ. However, we have to carry the considerations a little bit further in order to obtain information about the index of $M(\phi)$.

Theorem 3.3. *Let* $\phi \in PC$. *Then* $M(\phi)$ *is Fredholm if and only if* ϕ *can be written in the form* $\phi(t) = t^{\varkappa}\exp(\psi(t))$ *with* $\varkappa \in \mathbb{Z}$ *and* $\psi \in PC$ *such that the following is satisfied*:

(i) $|\mathrm{Im}\,\Delta_\tau(\psi) + \mathrm{Im}\,\Delta_{\bar\tau}(\psi)| < 1/2$ *for each* $\tau \in \mathbb{T}_+$;
(ii) $-3/4 < \mathrm{Im}\,\Delta_1(\psi) < 1/4$ *and* $-1/4 < \mathrm{Im}\,\Delta_{-1}(\psi) < 3/4$.

Here $\Delta_\tau(\psi) := (2\pi)^{-1}\big(\psi(\tau-0)-\psi(\tau+0)\big)$. *Moreover, in this case we have* $\mathrm{ind}\, M(\phi) = -\varkappa$, *and* $M(\phi)$ *is invertible if* $\varkappa = 0$.

Proof. The "if" part is trivial. Now suppose that $M(\phi)$ is Fredholm and hence that the conditions of Corollary 3.2 are satisfied. For each point $\tau \in \mathbb{T}_+ \cup \{-1,1\}$ one can choose a piecewise continuous logarithm of ϕ which is defined and satisfies the conditions (i) and (ii) on an open neighborhood of $\{\tau,\bar\tau\}$. Because of compactness it suffices to consider only a finite covering of these neighborhoods. By gluing together the different pieces of logarithms of ϕ in a suitable way (by altering them locally by constants of $2\pi i\mathbb{Z}$ if necessary), it is possible to construct a function $\psi \in PC$ with $\phi = \exp\psi$ that satisfies the above conditions with the possible exception of one point, say $\tau = -1$. At this point, the size of the jump can be transformed into the proper value by replacing $\psi(t)$ by $\psi(t) - \varkappa\log t$ with suitable $\varkappa \in \mathbb{Z}$. But this yields the desired representation.

In order to prove the index formula, we rely on the representation just considered and introduce for each $\lambda \in [0,1]$ the functions $\phi_\lambda(t) = t^{\varkappa}\exp\big(\lambda\psi(t)\big)$. It easy to see that also the operators $M(\phi_\lambda)$ are Fredholm. Because the mapping $\lambda \in [0,1] \mapsto M(\phi_\lambda) \in \mathcal{L}(\ell_2)$ is continuous, the index of $M(\phi_\lambda)$ remains constant for $\lambda \in [0,1]$.

Note that $\phi_1 = \phi$ and $\phi_0 = t^{\varkappa}$. So we have $\operatorname{ind} M(\phi) = \operatorname{ind} M(t^{\varkappa}) = \operatorname{ind} T(t^{\varkappa}) = -\varkappa$. □

In applications, it is often customary to represent a piecewise continuous function as a certain product. Let $t_\beta \in PC$ with $\beta \in \mathbb{C}$ be the function

$$t_\beta(e^{i\theta}) \quad = \quad \exp(i\beta(\theta-\pi)), \qquad 0<\theta<2\pi. \tag{20}$$

This function is continuous and nonvanishing on $\mathbb{T}\setminus\{1\}$ and may have a jump at 1 whose size is characterized by $t_\beta(1-0)/t_\beta(1+0) = \exp(2\pi i\beta)$. Each invertible function $\phi \in PC$ with finitely many jump discontinuities can be written in the form

$$\phi(e^{i\theta}) \quad = \quad b(e^{i\theta}) \prod_{r=1}^{R} t_{\beta_r}(e^{i(\theta-\theta_r)}) \tag{21}$$

where b is a continuous nonvanishing function and $\theta_1, \dots, \theta_R \in (-\pi, \pi]$.

Theorem 3.4. *Suppose that $\phi \in PC$ has finitely many jumps. Then $M(\phi)$ is invertible if and only if ϕ can be written in the form*

$$\phi(e^{i\theta}) \quad = \quad b(e^{i\theta}) t_{\beta^+}(e^{i\theta}) t_{\beta^-}(e^{i(\theta-\pi)}) \prod_{r=1}^{R} t_{\beta_r^+}(e^{i(\theta-\theta_r)}) t_{\beta_r^-}(e^{i(\theta+\theta_r)}) \tag{22}$$

where $\theta_1, \dots, \theta_R \in (0,\pi)$ are pairwise distinct points, b is a continuous nonvanishing function with winding number zero, and

(i) $|\operatorname{Re}(\beta_r^+ + \beta_r^-)| < 1/2$ *for each* $1 \le r \le R$;
(ii) $-3/4 < \operatorname{Re}\beta^+ < 1/4$ *and* $-1/4 < \operatorname{Re}\beta^- < 3/4$.

Proof. As stated in Theorem 3.3, $M(\phi)$ is invertible if and only if $\phi = \exp\psi$ with the above conditions on ψ. These conditions imply that also ψ has finitely many jumps, and hence one can always decompose $\psi = \psi_1 + \psi_2$ where $\psi_1(e^{i\theta})$ is continuous and $\psi_2(e^{i\theta})$ is piecewise linear with respect to θ. The factor $\exp\psi_1$ represents the function b, and the factor $\exp\psi_2$ can be identified with the product of the functions t_β. Observe that $t_\beta(e^{i\theta})$ is the exponential of a piecewise linear function. The conditions on the β's are just a restatement of the conditions on the Δ_τ's. □

Now we proceed with describing the essential spectrum of $M(\phi)$, i.e., the set of all $z \in \mathbb{C}$ for which $M(\phi) - zI$ is not Fredholm. We introduce the following sets. Let

$$\mathcal{C}_{\pm 1}(a;b) = \left\{ \, z \in \mathbb{C} \;\middle|\; z = a + (b-a)\Big(\mu \mp i\sqrt{\mu(1-\mu)}\Big), \;\; \mu \in [0,1] \, \right\}$$

$$\mathcal{L}(a;b) = \left\{ \, z \in \mathbb{C} \;\middle|\; z = a + (b-a)\mu, \;\; \mu \in [0,1] \, \right\}$$

be the half circles and the line segment, respectively, with endpoints a and b. Obviously, $\mathcal{C}_{\pm 1}(a;a) = \mathcal{L}(a;a) = \{a\}$ as a special case. We also define

$$\mathcal{H}(a_1,a_2;b_1,b_2) = \left\{ \, z \in \mathbb{C} \;\middle|\; (a_1-z)(a_2-z)(1-\mu) + (b_1-z)(b_2-z)\mu = 0, \;\; \mu \in [0,1] \, \right\}.$$

Note that $\mathcal{H}(a_1,a_2;b_1,b_2)$ is symmetric in a_1 and a_2 as well as in b_1 and b_2. In case of coinciding parameters we simply have $\mathcal{H}(a_1,c;b_1,c) = \mathcal{L}(a_1,b_1) \cup \{c\}$. In general $\mathcal{H}$ is more complicated. If one parameterizes $z = x + iy$ $(x, y \in \mathbb{R})$, then $\mathcal{H}(a_1, a_2; b_1, b_2)$ is a certain subset of the following cubic curve in x and y:

$$\operatorname{Im}\Big(A(x+iy)\overline{B(x+iy)}\Big) \;=\; 0. \tag{23}$$

Here $A(z) = (a_1 - z)(a_2 - z)$ and $B(z) = (b_1 - z)(b_2 - z)$. The entire cubic curve is obtained by taking μ in $\mathbb{R} \cup \{\infty\}$ instead of $[0,1]$. The reader can easily verify that the "geometric form" of $\mathcal{H}$ may be quite different.

Corollary 3.5. *Let $\phi \in PC$. Then the essential spectrum of $M(\phi)$ is equal to*

$$\bigcup_{\tau\in\mathbb{T}_+} \mathcal{H}\Big(\phi(\tau-0), \phi(\bar\tau-0); \phi(\tau+0), \phi(\bar\tau+0)\Big) \;\cup \bigcup_{\tau\in\{-1,1\}} \mathcal{C}_\tau\Big(\phi(\tau-0); \phi(\tau+0)\Big).$$

Proof. This follows from Proposition 3.1. The essential spectrum consists exactly of the points $z \in \mathbb{C}$ for which there exist a $\tau \in \mathbb{T}_+ \cup \{-1, 1\}$ and $\mu \in [0,1]$ such that $b_{\tau,\mu}(\phi - z) = 0$. □

We remark in this connection that the essential range of $\phi \in PC$, i.e., the spectrum of ϕ as an element of $L^\infty(\mathbb{T})$, is just $\{\ \phi(\tau \pm 0)\ |\ \tau \in \mathbb{T}\ \}$. Hence the essential range of ϕ is always contained in the essential spectrum of $M(\phi)$. In general this inclusion is proper.

In what follows we give a geometric description of the spectrum of $M(\phi)$. We will assume for simplicity that ϕ has only finitely many jumps and is piecewise smooth. The general case could be treated with the same ideas, but would require more detailed explanations.

First of all, the values of $\phi(t)$ as t runs along $\mathbb{T}$ (in positive orientation) describe a certain curve. This curve may be closed if ϕ is continuous, or may consist of several components if ϕ has jumps (see Figure 1). Moreover, this curve has a natural orientation induced from $t \in \mathbb{T}$.

Now we fill in certain additional pieces as follows. If the function ϕ is discontinuous at $\tau \in \{-1, 1\}$, then we add the half circle $\mathcal{C}_\tau(\phi(\tau-0); \phi(\tau+0))$. If ϕ is discontinuous at $\tau \in \mathbb{T} \setminus \{-1,1\}$, but is continuous at $\bar\tau$, then we fill in the line segment $\mathcal{L}(\phi(\tau-0); \phi(\tau+0))$. Finally, if ϕ has jumps at both $\tau, \bar\tau \in \mathbb{T}\setminus\{-1,1\}$, then we add $\mathcal{H}(\phi(\tau-0), \phi(\bar\tau-0); \phi(\tau+0), \phi(\bar\tau+0))$. These new pieces constitute themselves oriented curves, where the orientation is now induced from the parameter $\mu \in [0,1]$ that occurs in the definition of these sets.

Gluing together these pieces with the former curve we obtain an oriented curve $\phi^\#$. By Corollary 3.5, the image of $\phi^\#$ is exactly $\operatorname{sp}_{\text{ess}} M(\phi)$. Remark again that the "geometric form" of this curve may be quite different. For instance, it may consist of several closed oriented curves (see Figure 2).

Because $\phi^\#$ possesses an orientation, it is possible to associate to each point $z \notin \operatorname{im}\phi^\#$ a winding number $\operatorname{wind}(\phi^\#, z)$. If $\phi^\#$ consists of several components, this is just the sum of the usual winding numbers with respect to these components.

These considerations now allow us to describe the spectrum of $M(\phi)$:

$$\operatorname{sp} M(\phi) \;=\; \operatorname{im}\phi^\# \cup \Big\{\ z \notin \operatorname{im}\phi^\# \ \Big|\ \operatorname{wind}(\phi^\#, z) \neq 0\ \Big\}. \tag{24}$$

The proof can be carried out by continuously deforming ϕ. We leave the details to the reader.

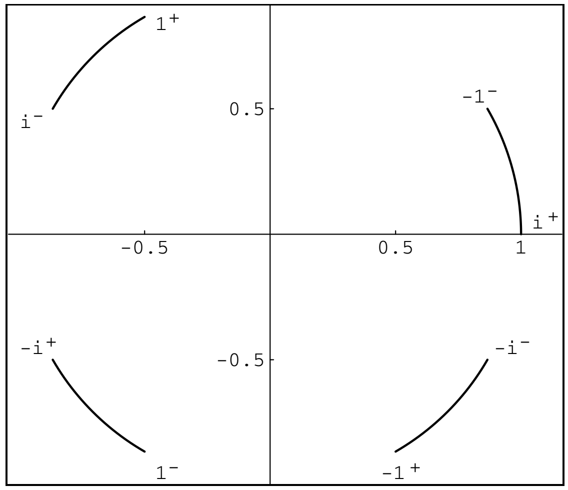

FIGURE 1. Image of a piecewise continuous function with four jumps.

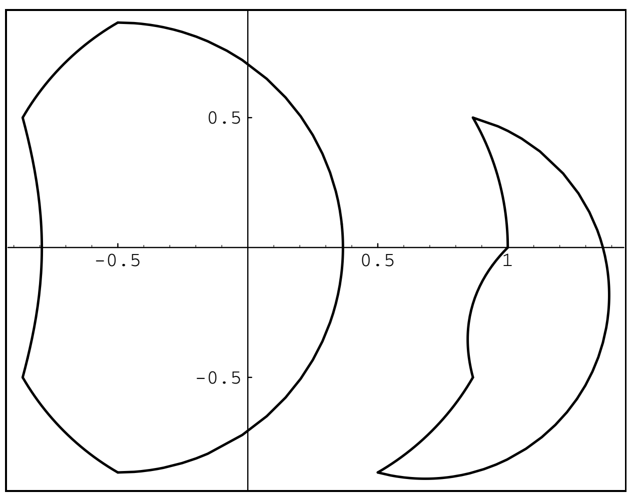

FIGURE 2. Essential spectrum of $M(\phi)$ with ϕ corresponding to the function in Figure 1.

The figures above give an example of the image of a piecewise continuous curve and its (essential) spectrum. The example comes from a product of four of the standard t_β functions, with jumps at $1, -1, i$ and $-i$ and values of β equal to $1/3, -3/4, 5/12$ and $1/3$ respectively and a normalizing factor of $-e^{-i\pi/24}$ for picture purposes only. In this example, the spectrum consists of the area bounded by the curve in Figure 2. As one can see the spectrum as well as the essential spectrum of this operator is not connected, contrasted to the Toeplitz case.

4. Stability of the finite section method for PC symbols

If one considers the finite truncations of the infinite matrix $M(\phi)$, one obtains the sequence $\{M_n(\phi)\}_{n=1}^{\infty}$ of $n \times n$ matrices, the so-called finite sections. This sequence is said to be stable if the matrices are invertible for sufficiently large n and if their inverses (considered as operators in $\mathcal{L}(\ell_2)$) are uniformly bounded.

The problem of stability occurs naturally when examining, for instance, the question whether the inverses of $M_n(\phi)$ converge strongly on ℓ_2 to the inverse of $M(\phi)$. This question is obviously of interest in numerical analysis, but it also emerges in the investigation of the asymptotic behavior of the determinants of $M_n(\phi)$ (see [2]).

Employing Banach algebra methods, necessary and sufficient conditions for the stability of sequences of the form $\{T_n(\phi) + H_n(\psi)\}_{n=1}^{\infty}$ with $\phi, \psi \in PC$ were established by Roch and Silbermann in [7, Theorem 5.3]. These conditions say that the sequence is stable if and only if the operators $T(\phi) + H(\psi)$ and $T(\widetilde{\phi})$ are invertible and if certain operators A_τ (associated to each point $\tau \in \mathbb{T}_+ \cup \{-1, 1\}$) are invertible. Spitkovsky and Tashbaev pointed out that the invertibility of the operators A_τ can be reduced to the factorization of certain 2×2 matrix functions, and above all they solved the corresponding factorization problem [8].

Invertibility criteria for Toeplitz operators with piecewise continuous symbols are well known [3]. But the invertibility of Toeplitz + Hankel operators in the general case remains still unsolved.

We proceed with recalling the stability results [7, 8] for the special case $\phi = \psi$.

Proposition 4.1. *Let $\phi \in PC$. Then the sequence $\{M_n(\phi)\}_{n=1}^{\infty}$ is stable if and only if the operators $M(\phi)$ and $T(\widetilde{\phi})$ are invertible and the following conditions are satisfied:*

(i) $|\sigma_\tau(\phi)| < 1/2$, $|\sigma_{\bar{\tau}}(\phi)| < 1/2$ *and* $|\sigma_\tau(\phi) + \sigma_{\bar{\tau}}(\phi)| < 1/2$ *for each* $\tau \in \mathbb{T}_+$;
(ii) $-1/2 < \sigma_1(\phi) < 1/4$ *and* $-1/4 < \sigma_{-1}(\phi) < 1/2$.

Here $\sigma_\tau(\phi) := (2\pi)^{-1} \arg\left(\phi(\tau - 0)/\phi(\tau + 0)\right)$ *where the argument is chosen in* $(-\pi, \pi]$.

Proof. As said before, one has to analyze the invertibility of certain operators A_τ with $\tau \in \mathbb{T}_+ \cup \{-1, 1\}$. We refer to [8, Theorem 4] and to the proof given there. It is stated that the operators A_τ are invertible if and only if $\phi(\tau \pm 0) \neq 0$ for each $\tau \in \mathbb{T}$ (which is already a consequence of the invertibility of $M(\phi)$ or of $T(\widetilde{\phi})$) and if the following conditions A) and B) are fulfilled:

A) For $\tau \in \mathbb{T}_+$, the values $\rho(\tau) = \phi(\tau - 0)/\phi(\tau + 0)$ and $\rho(\bar{\tau}) = \phi(\bar{\tau} - 0)/\phi(\bar{\tau} + 0)$ are not negative, and if ζ_1 and ζ_2 are the zeros of the quadratic polynomial

$$\zeta^2 - \left(\frac{\phi(\tau - 0)\phi(\bar{\tau} - 0)}{\phi(\tau + 0)\phi(\bar{\tau} + 0)} + 1 \right) \zeta + \frac{\phi(\tau - 0)\phi(\bar{\tau} - 0)}{\phi(\tau + 0)\phi(\bar{\tau} + 0)},$$

then ζ_1 and ζ_2 are not negative, and

$$0 = \arg \rho(\tau) + \arg \rho(\bar{\tau}) - \arg \zeta_1 - \arg \zeta_2,$$

where the arguments are chosen in $(-\pi, \pi)$. However, the zeros of the quadratic polynomial are just $\zeta_1 = 1$ and $\zeta_2 = \rho(\tau)\rho(\bar{\tau})$. Now it is easily seen that this is equivalent to (i).

B) For $\tau \in \{-1,1\}$, zero does not lie in the closed half disk that has the boundary

$$\mathcal{L}\Big(\phi(\tau-0);\phi(\tau+0)\Big) \cup \mathcal{C}_\tau\Big(\phi(\tau-0);\phi(\tau+0)\Big).$$

Again this is equivalent to (ii). □

Note that the preceding result completely solves the stability problem for sequences $\{M_n(\phi)\}_{n=1}^\infty$ with $\phi \in PC$. We now want to point out what stability means for functions represented in the form (21).

Theorem 4.2. *Suppose that $\phi \in PC$ has finitely many jumps. Then $\{M_n(\phi)\}_{n=1}^\infty$ is stable if and only if ϕ can be written in the form*

$$\phi(e^{i\theta}) \;=\; b(e^{i\theta})t_{\beta^+}(e^{i\theta})t_{\beta^-}(e^{i(\theta-\pi)})\prod_{r=1}^{R} t_{\beta_r^+}(e^{i(\theta-\theta_r)})t_{\beta_r^-}(e^{i(\theta+\theta_r)}) \tag{25}$$

where $\theta_1,\dots,\theta_R \in (0,\pi)$ are pairwise distinct points, b is a continuous nonvanishing function with winding number zero, and

(i) $|\mathrm{Re}\,\beta_r^+| < 1/2$, $|\mathrm{Re}\,\beta_r^-| < 1/2$ *and* $|\mathrm{Re}\,(\beta_r^+ + \beta_r^-)| < 1/2$ *for each* $1 \le r \le R$;
(ii) $-1/2 < \mathrm{Re}\,\beta^+ < 1/4$ *and* $-1/4 < \mathrm{Re}\,\beta^- < 1/2$.

Proof. First of all note that $T(\widetilde{\phi})$ is invertible if and only if so is $T(\phi)$. This, in turn, is equivalent to the condition that ϕ can be written in the form (25) such that the moduli of the real parts of the β's are less than $1/2$ and that the winding number of b is zero [3].

Now assume that $M_n(\phi)$ is stable. It is not too hard to see that the conditions (i) and (ii) of Proposition 4.1 imply that ϕ can be represented in the form (25) by choosing values of the β's corresponding to the appropriate $\sigma_\tau(\phi)$. The winding number condition on b follows from the invertibility of $M(\phi)$ or $T(\phi)$. Note that the reverse implication is trivial. □

Comparing the preceding theorem with Theorem 3.4, we can draw the immediate conclusion that — in contrast to the (scalar) Toeplitz case — the invertibility of $M(\phi)$ does in general not guarantee the stability of the sequence $\{M_n(\phi)\}_{n=1}^\infty$ of finite sections.

Another point is that we had to use in the proof only the invertibility of either $M(\phi)$ or $T(\phi)$. This has its explanation in the following. "If $\phi \in PC$ is invertible and satisfies conditions (i) and (ii) of Proposition 4.1, then both $M(\phi)$ and $T(\phi)$ are Fredholm and $\mathrm{ind}\,M(\phi) = \mathrm{ind}\,T(\phi)$. In particular, under these assumptions, $M(\phi)$ is invertible if and only if so is $T(\phi)$." We omit a proof of these assertions, which can be carried out by similar arguments as in the proof of Theorem 3.3.

In contrast, it is possible that both $M(\phi)$ and $T(\phi)$ are invertible but conditions (i) and (ii) of Proposition 4.1 are not fulfilled (hence $\{M_n(\phi)\}_{n=1}^\infty$ is not stable). We give two concrete examples of functions and note that representation (21) is not unique. The first example is

$$\begin{aligned}\phi(e^{i\theta}) \;&=\; t_{\beta^+}(e^{i\theta})t_{\beta^-}(e^{i(\theta-\pi)})\\ &=\; -t_{\beta^+-1}(e^{i\theta})t_{\beta^-+1}(e^{i(\theta-\pi)})\end{aligned} \tag{26}$$

where $\operatorname{Re}\beta^+ \in (1/4, 1/2)$ and $\operatorname{Re}\beta^- \in (-1/2, -1/4)$. The second example is

$$\begin{aligned}\phi(e^{i\theta}) &= t_{\beta_1^+}(e^{i(\theta-\theta_1)})t_{\beta_1^-}(e^{i(\theta+\theta_1)})t_{\beta_2^+}(e^{i(\theta-\theta_2)})t_{\beta_2^-}(e^{i(\theta+\theta_2)}) \\ &= e^{i(\theta_2-\theta_1)}t_{\beta_1^+-1}(e^{i(\theta-\theta_1)})t_{\beta_1^-}(e^{i(\theta+\theta_1)})t_{\beta_2^++1}(e^{i(\theta-\theta_2)})t_{\beta_2^-}(e^{i(\theta+\theta_2)}) \qquad (27)\end{aligned}$$

where $\theta_1, \theta_2 \in (0, \pi)$, $\theta_1 \neq \theta_2$, $\operatorname{Re}\beta_1^+ \in (0, 1/2)$, $\operatorname{Re}\beta_1^- \in (0, 1/2)$, $\operatorname{Re}(\beta_1^+ + \beta_1^-) \in (1/2, 1)$, $\operatorname{Re}\beta_2^+ \in (-1/2, 0)$, $\operatorname{Re}\beta_2^- \in (-1/2, 0)$ and $\operatorname{Re}(\beta_2^+ + \beta_2^-) \in (-1, -1/2)$. In both examples, the first and second representation guarantee the invertibility of $T(\phi)$ and $M(\phi)$, respectively.

5. Stability for approximate identities of PC symbols

Let $\Lambda \subseteq [0, \infty)$ be an unbounded index set. Given a (generalized) sequence $\{\phi_\lambda\}_{\lambda\in\Lambda}$ of functions in $L^\infty(\mathbb{T})$, one can consider the sequence $\{M(\phi_\lambda)\}_{\lambda\in\Lambda}$ of operators. This sequence is said to be stable if

(i) There exists a $\lambda_0 \in [0, \infty)$ such that the operators $M(\phi_\lambda)$ are invertible for each $\lambda \in [\lambda_0, \infty) \cap \Lambda$;

(ii) $\sup_{\lambda\in[\lambda_0,\infty)\cap\Lambda} \|M(\phi_\lambda)^{-1}\|_{\mathcal{L}(\ell_2)}$ is finite.

It is, of course, hopeless to examine the stability problem for arbitrary such sequences. We will therefore restrict ourselves to certain classes that will be described below.

Let K be a function in $L^1(\mathbb{R})$ which satisfies

$$\int_{-\infty}^{\infty} K(x)\, dx = 1. \qquad (28)$$

For $\lambda \in [0, \infty)$ we introduce the linear bounded mappings $k_\lambda : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})$ defined by

$$(k_\lambda\phi)(e^{ix}) = \int_{-\infty}^{\infty} \phi(e^{i(x-y)})\lambda K(\lambda y)\, dy. \qquad (29)$$

The mapping k_λ is called the approximate identity generated by K. A typical example of an approximate identity is the harmonic extension h_μ defined by

$$h_\mu : \sum_{n=-\infty}^{\infty} e^{inx}\phi_n \mapsto \sum_{n=-\infty}^{\infty} \mu^{|n|}e^{inx}\phi_n, \qquad 0 \leq \mu < 1. \qquad (30)$$

The use of the index μ instead of λ should not cause confusion. The relationship is constituted through $h_\mu\phi = k_\lambda\phi$ with $K(x) = 1/\bigl(\pi(1 + x^2)\bigr)$ and $\lambda = -1/\log\mu$. Other examples of approximate identities are the Fejer-Cesaro means and the moving average [1]. We remark that stability of Toeplitz operators $\{T(k_\lambda\phi)\}_\lambda$ for quite general types of approximating identities was first examined by Böttcher and Silbermann [4] (see also [3, Chap. 3]).

For a given approximate identity k_λ generated by K we define a function f by

$$f\left(\frac{1+ix}{1-ix}\right) = \int_{-\infty}^{x} K(y)\, dy, \qquad x \in \mathbb{R}. \qquad (31)$$

Note that $f \in PC_{-1}$, where PC_{-1} denotes the set of functions in PC which are continuous on $\mathbb{T} \setminus \{-1\}$. In particular $f(-1+0) = 0$ and $f(-1-0) = 1$. Remark that in most cases of interest (e.g., if $K(x) \geq 0$), the spectrum of f in $L^\infty(\mathbb{T})$ is just $[0, 1]$.

For an invertible function $\phi \in PC_{-1}$, the Cauchy index is defined as follows:

$$\operatorname{ind}\phi = \frac{1}{2\pi}\left[\arg\phi(e^{i\theta})\right]_{\theta=-\pi}^{\pi} \tag{32}$$

where $\arg\phi$ is chosen in PC_{-1}. Given a function $\phi \in PC$ and $\tau \in \mathbb{T}$ we introduce for a fixed approximate identity with corresponding function f, the function

$$\phi_\tau(e^{i\theta}) = \phi(\tau+0)f(e^{i\theta}) + \phi(\tau-0)(1-f(e^{i\theta})). \tag{33}$$

Note that $\phi_\tau \in PC_{-1}$ and $\phi_\tau(-1\pm 0) = \phi(\tau \mp 0)$.

It is the goal of what follows to investigate sequences $\{M(k_\lambda\phi)\}_{\lambda\in\Lambda}$ with $\phi \in PC$. In [5], various sequences of convolution type operators with such generating functions have been examined in view of stability, including sequences of the form $\{T(k_\lambda\phi) + H(k_\lambda\psi)\}_{\lambda\in\Lambda}$. The stability criterion established there says that the latter sequence is stable if and only if certain associated operators are invertible. Unfortunately, these invertibility problems could not be handled in general. We will indicate next how the special case $\phi = \psi$ can be treated.

Theorem 5.1. *Let k_λ be an approximate identity and let $\phi \in PC$. Then the sequence $\{M(k_\lambda\phi)\}_{\lambda\in\Lambda}$ is stable if and only if $M(\phi)$ is invertible, for each $\tau \in \mathbb{T}$ the functions $\phi_\tau \in PC_{-1}$ are invertible and the following is satisfied*:

(i) $|\operatorname{ind}\phi_\tau + \operatorname{ind}\phi_{\bar\tau}| < 1/2$ *for each* $\tau \in \mathbb{T}_+$;

(ii) $-1/4 < \operatorname{ind}\phi_1 < 3/4$ *and* $-3/4 < \operatorname{ind}\phi_{-1} < 1/4$.

Proof. It can be derived from [5, Corollary 3.2] that $\{M(k_\lambda\phi)\}_{\lambda\in\Lambda}$ is stable if and only if the operators $M(\phi)$, $A_\tau = T(\phi_\tau) + \tau H(\phi_\tau)$ for $\tau \in \{-1, 1\}$ and

$$A_{\tau,\bar\tau} = \begin{pmatrix} L(\phi_\tau)P + Q & L(\phi_\tau)Q \\ L(\widetilde{\phi_{\bar\tau}})P & L(\widetilde{\phi_{\bar\tau}})Q + P \end{pmatrix}$$

for $\tau \in \mathbb{T}_+$ are invertible. As to the operator $A_{\tau,\bar\tau}$, it is easily seen that

$$A_{\tau,\bar\tau} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} L(\phi_\tau) - P \\ L(\widetilde{\phi_{\bar\tau}}) - Q \end{pmatrix} \begin{pmatrix} P, & Q \end{pmatrix}.$$

Hence the invertibility of $A_{\tau,\bar\tau}$ is equivalent to the invertibility of the singular integral operator

$$I + \begin{pmatrix} P, & Q \end{pmatrix} \begin{pmatrix} L(\phi_\tau) - P \\ L(\widetilde{\phi_{\bar\tau}}) - Q \end{pmatrix} = PL(\phi_\tau) + QL(\widetilde{\phi_{\bar\tau}}).$$

This, in turn, is equivalent to saying that the functions ϕ_τ and $\phi_{\bar\tau}$ are invertible in PC_{-1} and that $|\operatorname{ind}(\phi_\tau\widetilde{\phi_{\bar\tau}}^{-1})| < 1/2$. Now observe that $\operatorname{ind}(\phi_\tau\widetilde{\phi_{\bar\tau}}^{-1}) = \operatorname{ind}\phi_\tau - \operatorname{ind}\widetilde{\phi_{\bar\tau}} = \operatorname{ind}\phi_\tau + \operatorname{ind}\phi_{\bar\tau}$.

If $\tau = 1$, then $A_1 = M(\phi_1)$. The invertibility can be analyzed by Theorem 3.3. It follows that the function ψ with $\phi_1 = \exp\psi$ appearing there is also in PC_{-1}. Hence $\operatorname{Im}\psi = \arg\phi_1$ up to a constant and by definition $\operatorname{Im}\Delta_{-1}(\psi) = \operatorname{ind}\phi_1$.

If $\tau = -1$, then $A_{-1} = WM(\widehat{\phi}_{-1})W$ where $\widehat{\phi}_{-1}(t) = \phi_{-1}(-t)$, $t \in \mathbb{T}$, is the "rotated" function and $W \in \mathcal{L}(\ell_2)$ is the operator $W : (x_k)_{k\geq 0} \mapsto ((-1)^k x_k)_{k\geq 0}$. So we are left with the invertibility of $M(\widehat{\phi}_{-1})$. We can argue as above, and it follows similarly that $\operatorname{Im}\psi = \arg\widehat{\phi}_{-1}$ up to a constant. Hence $\operatorname{Im}\Delta_1(\psi) = \operatorname{ind}\phi_{-1}$. □

The previous result answers the question about stability for the case under consideration completely, and in general no further essential simplifications are possible. In the special case where the spectrum of f is just $[0,1]$, the situation is different.

Corollary 5.2. *Let k_λ be an approximate identity with* $\mathrm{sp}_{L^\infty(\mathbb{T})} f = [0,1]$ *and let* $\phi \in PC$. *Then the sequence* $\{M(k_\lambda \phi)\}_{\lambda\in\Lambda}$ *is stable if and only if* $M(\phi)$ *is invertible and the conditions* (i) *and* (ii) *of Proposition* 4.1 *are satisfied.*

Proof. We remark that ϕ_τ is invertible if and only if $\phi(\tau \pm 0) \neq 0$ and $|\sigma_\tau(\phi)| < 1/2$. Moreover, $\sigma_\tau(\phi) = -\mathrm{ind}\,\phi_\tau$ by definition. □

The next result discusses the stability condition in terms of the product representation. The proof can be carried out in the same way as the proof of Theorem 4.2.

Corollary 5.3. *Let k_λ be an approximate identity with* $\mathrm{sp}_{L^\infty(\mathbb{T})} f = [0,1]$. *Assume that* $\phi \in PC$ *has finitely many jumps. Then* $\{M(k_\lambda\phi)\}_{\lambda\in\Lambda}$ *is stable if and only if* ϕ *can be written in the form and with the conditions stated in Theorem* 4.2.

We can conclude that the invertibility of $M(\phi)$ does in general not guarantee the stability of $\{M(k_\lambda\phi)\}_{\lambda\in\Lambda}$. Again this contrasts the Toeplitz case [5, Sect. 3.3].

Now we turn our attention to another type of sequences. They are obtained by considering exponentials of approximate identities of piecewise continuous functions.

Theorem 5.4. *Let k_λ be an approximate identity and let* $\psi \in PC$. *Then the sequence* $\{M(\exp(k_\lambda\psi))\}_{\lambda\in\Lambda}$ *is stable if and only if* ψ *fulfills the conditions* (i) *and* (ii) *stated in Theorem* 3.3.

Proof. This can be verified by combining Theorem 2.2 and Proposition 3.4 of [5] (see also [5, Corollary 3.5] for a related situation). One arrives at similar invertibility conditions as stated in Theorem 5.1 above. The only difference is that one has to replace ϕ by $\exp\psi$ and ϕ_τ by $\exp\psi_\tau$. The argumentation is analogous. Finally note that $\exp\psi_\tau \in PC_{-1}$ and $\mathrm{ind}\,\exp\psi_\tau = \mathrm{Im}\,\Delta_{-1}(\psi_\tau) = -\mathrm{Im}\,\Delta_\tau(\psi)$. Moreover, the conditions (i) and (ii) of Theorem 3.3 ensure the invertibility of $M(\exp\psi)$. □

This theorem (in conjunction with Theorem 3.3) has an interesting consequence. Suppose that $M(\phi)$ with $\phi \in PC$ is invertible. Then one can always find a $\psi \in PC$ with $\phi = \exp\psi$ such that the sequence $\{M(\exp(k_\lambda\psi))\}_{\lambda\in\Lambda}$ is stable. One just has to choose ψ with the conditions stated in Theorem 3.3. Observe that if ψ (the logarithm of ϕ) is not chosen "properly", then the sequence fails to be stable. The constructed sequence is a so-called approximating sequence for the operator $M(\phi)$ in the sense that $M(\exp(k_\lambda\psi)) \to M(\phi)$ strongly on ℓ_2 as $\lambda \to \infty$.

A special case of the previous theorem, which is of interest in applications [2], is considered next. Let $t_{\beta,\mu}$ with $0 \le \mu < 1$ and $\beta \in \mathbb{C}$ be the (smooth) function defined by

$$t_{\beta,\mu}(e^{i\theta}) = \left(1-\mu e^{-i\theta}\right)^{-\beta}\left(1-\mu e^{i\theta}\right)^{\beta}. \tag{34}$$

Corollary 5.5. *The sequence* $\{M(\phi_\mu)\}_{0\le\mu<1}$ *defined by*

$$\phi_\mu(e^{i\theta}) = b(e^{i\theta})t_{\beta^+,\mu}(e^{i\theta})t_{\beta^-,\mu}(e^{i(\theta-\pi)})\prod_{r=1}^{R} t_{\beta_r^+,\mu}(e^{i(\theta-\theta_r)})t_{\beta_r^-,\mu}(e^{i(\theta+\theta_r)}) \tag{35}$$

where $\theta_1, \ldots, \theta_R \in (0, \pi)$ *are pairwise distinct points and* b *is a continuous nonvanishing function with winding number zero, is stable* $(as\ \mu \to 1-0)$ *if and only if the parameters satisfy the conditions* (i) *and* (ii) *of Theorem* 3.4.

Proof. The functions $t_{\beta,\mu}$ are the exponentials of the harmonic extensions h_μ of the piecewise linear functions $\log t_\beta(e^{i\theta}) = i\beta(\theta - \pi)$, $0 < \theta < 2\pi$. Hence the assertion can be deduced for functions (35) with b replaced by $b_\mu := \exp(h_\mu \log b)$. Because $b_\mu \to b$ in the norm of $L^\infty(\mathbb{T})$, the actual and the modified sequence of operators differ only by a sequence tending to zero in the operator norm. □

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