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Power Weakly Mixing Infinite Transformations

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Abstract. We construct a rank one infinite measure preserving transformation T such that for all sequences of nonzero integers $\{k_1, \ldots, k_r\}$, $T^{k_1} \times \ldots \times T^{k_r}$ is ergodic.

Contents

1. Introduction

It is well known that for the case of finite measure preserving transformations, if T is weakly mixing then $T^{k_1} \times \ldots \times T^{k_r}$ is ergodic for any sequence of nonzero integers $\{k_1, \ldots, k_r\}$. Kakutani and Parry proved in [KP] that there exist infinite (measure preserving) transformations such that $T \times \cdots \times T$ (r terms) is ergodic but $T \times \cdots \times T$ ($r+1$ terms) is not; in this case the transformation is said to have *ergodic index* r. T is said to have *infinite ergodic index* if $T \times \cdots \times T$ (r terms) is ergodic for all $r > 0$. In [KP], they also constructed infinite Markov shifts of infinite ergodic index. A finite or infinite measure preserving transformation T is *weakly mixing* if for all finite measure preserving ergodic transformations S, the product $T \times S$ is ergodic. For the case of infinite transformations T, it was shown in [ALW] that ergodicity of $T \times T$ implies weak mixing but that there exist infinite weak mixing transformations with $T \times T$ not conservative, hence not ergodic. It was shown in [AFS] that an infinite transformation T may be weakly mixing with $T \times T$ conservative but still not ergodic, and that there exist rank one infinite transformations of infinite ergodic index.

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In this paper we introduce a condition stronger than infinite ergodic index. Define a transformation T to be *power weakly mixing* if for all finite sequences of nonzero integers $\{k_1, \ldots, k_r\}$,

$$T^{k_1} \times \ldots \times T^{k_r}$$

is ergodic. Clearly, any power weakly mixing transformation has infinite ergodic index. Also, T is weakly mixing, but it follows from [ALW] that there exists a conservative ergodic infinite measure preserving transformation R such that $T \times R$ is not conservative, hence not ergodic. Finally, we mention that it has been shown recently that infinite ergodic index does not imply power weak mixing [AFS2].

In Section 2 we prove some preliminaries on approximation and in Section 3 we construct a rank one infinite measure preserving transformation which is power weakly mixing. We refer to [AFS] for terms not defined here.

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2. Approximation Properties

In this section we prove an approximation lemma for transformations defined by cutting and stacking [F]. This idea has been used earlier in e.g., [AFS] to show that a specific transformation has infinite ergodic index. However, here we present it in greater generality that permits other applications such as in [AFS2]. Thus we first describe cutting and stacking constructions [F].

Let X be a finite or infinite interval of real numbers and μ be Lebesgue measure. A *column* $\mathcal{C}$ consists of a collection of pairwise disjoint intervals in X of the same measure denoted by $B^{(0)}, B^{(1)}, \ldots, B^{(h-1)}$, where $h > 0$. The elements of $\mathcal{C}$ are called *levels* and h is the *height* of $\mathcal{C}$. The column $\mathcal{C}$ partially defines a transformation T on levels $B^{(i)}$, $i = 0, \ldots, h-2$, by the (unique orientation preserving) translation that takes interval $B^{(i)}$ to interval $B^{(i+1)}$. Thus we shall write $B^{(i)}$ as $T^i B^{(0)}$, $i = 0, \ldots, h-1$.

A *cutting and stacking* construction for a measure preserving transformation $T : X \to X$ consists of a sequence of columns

$$\mathcal{C}_n = \{B_n, TB_n, \ldots, T^{h_n - 1} B_n\}$$

of height h_n such that:

i) $\mathcal{C}_{n+1}$ is obtained by cutting $\mathcal{C}_n$ into c_n equal-measure subcolumns (or *copies* of C_n), putting a number of *spacers* (new levels of the same measure as any of the levels in the c_n subcolumns) above each subcolumn, and stacking from left to right (i.e., the top (or top spacer if it exists) of each subcolumn is mapped by translation to the bottom subinterval of the adjacent column to its right). In this way $\mathcal{C}_{n+1}$ consists of c_n copies of $\mathcal{C}_n$, possibly separated by spacers. We assume $c_n \geq 2$.

ii) B_n is a union of elements from $\{B_{n+1}, TB_{n+1}, \ldots, T^{h_{n+1} - h_n} B_{n+1}\}$.

iii) $\bigcup_n \mathcal{C}_n$ generates the Borel sets, i.e., for all subsets A in X, $\mu(A) > 0$, and for all $\epsilon > 0$, there exists C, a union of elements from $\mathcal{C}_n$, for some n, such that $\mu(A \triangle C) < \epsilon$.

Suppose $I = T^j B_\ell$ is in $\mathcal{C}_\ell$, for $j = 0, \ldots, h_\ell - 1$. For any $n > \ell$, I is the union of some elements in $\mathcal{C}_n = \{B_n, TB_n, \ldots, T^{h_n-1}B_n\}$. We call the elements in this union *sublevels* or *copies* of I.

Given a real number $0 < \epsilon < 1$, and a subset A of X with $\mu(A) > 0$, we say that a subset I of X is $(1-\epsilon)$-*full* of A provided

$$\mu(I \cap A) > (1-\epsilon)\mu(I).$$

A set I in the product space $\Pi_{i=1}^r X$ is a *rectangle* if I can be written as the Cartesian product of levels in some column $\mathcal{C}_k$. We let ν be the product measure μ^r. Rectangles I are defined to be $(1-\epsilon)$-full of a set A in a similar way as before.

Lemma 2.1 (Double Approximation Lemma)**.** *Suppose A is a subset of the product space $\Pi_{i=1}^r X$ with $\nu(A) > 0$. Let $I = I_1 \times \ldots \times I_r$ be a rectangle in $\mathcal{C}_\ell$ that is $(1-\epsilon)$-full of A. For $n > \ell$, let $P_n = c_\ell \cdots c_{n-1}$, let V_n index the P_n copies of C_ℓ in C_n, and let $V = V(n,r) = V_n \times \cdots \times V_n$ (r times). Then for any δ, $0 < \delta < 1$, and for any τ, $0 < \tau < 100(1-\epsilon)$, there exists an integer N such that for all $n > N$, there is a set V'' of size at least τ percent of V such that for all $v = (v_1, \ldots, v_r) \in V''$, I_v is $(1-\delta)$-full of A and each I_v is of the form $I_v = I_1'' \times \ldots \times I_r''$ where I_m'' is a sublevel of I_m in the v_m-copy of $\mathcal{C}_\ell$, $m = 1, \ldots, r$.*

Proof. For convenience, let A denote $I \cap A$ and let t denote $\frac{\tau}{100}$. Then $\nu(I \triangle A) < \epsilon\nu(I)$. We have that $V_n = \{1, \ldots, P_n\}$ and $V = \{(v_1, \ldots, v_r) | v_i \in V_n\}$. Then $I = \cup_{v \in V} I_v$.

Choose $c > \frac{\delta+1}{1-t-\epsilon} > 0$. Next pick $N > \ell$ sufficiently large so that for any $n \geq N$ there exists V' a subset of V such that $I' = \cup_{v \in V'} I_v$ satisfies

$$\nu(I' \triangle A) < \frac{\delta}{c}\nu(I).$$

Thus,

$$\begin{aligned}\nu(I' \triangle I) &< \frac{\delta}{c}\nu(I) + \epsilon\nu(I)\\ &= (\frac{\delta}{c} + \epsilon)\nu(I).\end{aligned}$$

Now let $V'' = \{v \in V' | \nu(I_v \setminus A) < \delta\nu(I_v)\}$ and set $I'' = \cup_{v \in V''} I_v$, the union of the $(1-\delta)$-full I_v subintervals. Then

$$\begin{aligned}\delta\nu(I' \triangle I'') &= \sum_{v \in V' \triangle V''} \delta\nu(I_v)\\ &\leq \sum_{v \in V' \triangle V''} \nu(I_v \setminus A)\\ &\leq \nu(I' \triangle A).\end{aligned}$$

So

$$\begin{aligned}\nu(I'' \triangle I) &\leq \frac{1}{\delta}\nu(I' \triangle A) + \nu(I' \triangle I)\\ &< \frac{1}{c}\nu(I) + (\frac{\delta}{c} + \epsilon)\nu(I)\\ &< (1-t)\nu(I).\end{aligned}$$

Therefore, more than τ percent of the subrectangles contained in I are in I'' and are thus $(1-\delta)$-full of A. $\square$

3. A Power Weakly Mixing T

In this section we construct a rank one infinite measure preserving transformation T that is power weakly mixing; then we mention a family of such transformations for which the same proof applies. We start by defining inductively a sequence of columns $\{\mathcal{C}_n\}$. Let $\mathcal{C}_0$ have base $B_0 = [0, 1)$ and height $h_0 = 1$. Given a column $\mathcal{C}_\ell$ with base $B_\ell = [0, \frac{1}{4^{\ell-1}})$ and height h_ℓ, $\mathcal{C}_{\ell+1}$ is formed by cutting $\mathcal{C}_\ell$ vertically three times so that B_ℓ is cut into the intervals $B_{\ell,1} = [0, \frac{1}{4^\ell})$, $B_{\ell,2} = [\frac{1}{4^\ell}, \frac{1}{2}(\frac{1}{4^{\ell-1}}))$, $B_{\ell,3} = [\frac{1}{2}(\frac{1}{4^{\ell-1}}), \frac{3}{4}(\frac{1}{4^{\ell-1}}))$, $B_{\ell,4} = [\frac{3}{4}(\frac{1}{4^{\ell-1}}), \frac{1}{4^{\ell-1}})$. We then add a column of spacers h_ℓ high to the top of the subcolumn whose base is $B_{\ell,2}$. Next we add one spacer to the top of the subcolumn whose base is $B_{\ell,4}$; this is called the *staircase spacer* of $\mathcal{C}_\ell$. Then stack from left to right, i.e., the top level on the left is sent to the bottom level on the right by the translation map. The resulting column $\mathcal{C}_{\ell+1}$ now has base $B_{\ell+1} = [0, \frac{1}{4^\ell})$ and height $h_{\ell+1} = 5h_\ell + 1$. The union of the columns is $X = [0, \infty)$. This defines a conservative ergodic rank one infinite measure preserving transformation T.

Any column $\mathcal{C}_n = \{B_n, \ldots, T^{h_n-1}B_n\}$ has four subcolumns

$$\mathcal{C}_{n,i} = \{B_{n,i}, \ldots, T^{h_n-1}B_{n,i}\}$$

for $i = 1, \ldots, 4$. Using the new column of spacers that is added to $\mathcal{C}_n$, and ignoring the top level of $\mathcal{C}_{n+1}$, we can think of $\mathcal{C}_{n+1}$ as divided into five *sections*, numbered from bottom to top. In this way each level of $\mathcal{C}_n$ has a sublevel in the first, second, fourth and fifth section of $\mathcal{C}_{n+1}$.

Lemma 3.1. *Given subsets A and B of $\Pi_{i=1}^r X$ of positive measure there exist rectangles $I = I_1 \times \cdots \times I_r$ and $J = J_1 \times \cdots \times J_r$ with $I_1, \ldots, I_r, J_1, \ldots, J_r$ in a column $\mathcal{C}_\ell$ such that for all $m = 1, \ldots, r$, I_m may be chosen in the fifth section and J_m in the second section of $\mathcal{C}_\ell$, or I_m in the second section and J_m in the fifth section of $\mathcal{C}_\ell$, and with I and J $\frac{3}{4}$-full of sets A and B respectively.*

Proof. Choose rectangles $I' = I'_1 \times \cdots \times I'_r$ and $J' = J'_1 \times \cdots \times J'_r$, with I'_m and J'_m in column $\mathcal{C}_{\ell-1}$, such that I' and J' are $(1 - \frac{1}{4^{r+1}})$-full of A and B respectively. Now look at the copies of $\mathcal{C}_{\ell-1}$ in $\mathcal{C}_\ell$ placed as second and fifth subcolumns in the above order. To have I_m above J_m, let I_m be the top copy of I'_m in $\mathcal{C}_\ell$ and let J_m be the bottom copy of J'_m in $\mathcal{C}_\ell$. To have I_m below J_m make an analogous choice. Let $I = I_1 \times \cdots \times I_r$ and $J = J_1 \times \cdots \times J_r$. One verifies that I and J are $\frac{3}{4}$-full of A and B. $\square$

Given a level I in $\mathcal{C}_n$ and an integer $k > 0$, we will be interested in studying $T^{kh_n}I$ (a translation of I through $\mathcal{C}_n$ k times). The intersection of $T^{kh_n}I$ with

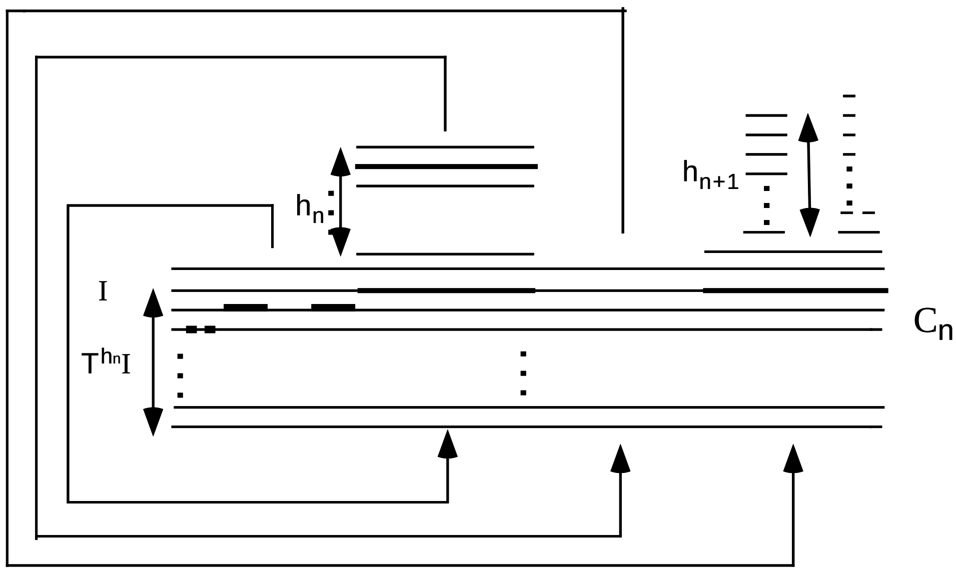

FIGURE 1. A $\mathcal{C}_n$ column.

the levels of $\mathcal{C}_n$ is called the *k-crescent* of I in $\mathcal{C}_n$. The part of $T^{kh_n}I$ that is in $\mathcal{C}_{n,i}$ will be called the (k,i)*-subcrescent* of I in $\mathcal{C}_n$, for $i = 1,\dots,4$, or simply a k-subcrescent. To simplify our estimates we will be mainly concerned with the $(k,1)$- and $(k,2)$-subcrescents. Figure 1 illustrates the 1-crescent in C_n of the interval in the top level of the fifth section of C_n.

Lemma 3.2. *Let I be a level in the fifth section of column $\mathcal{C}_n$, $n > 0$, let J be any level in the second section of $\mathcal{C}_n$, and let d be the distance that J is below I, $h_n/5 < d < h_n$. If $k = 0,1,\dots,4$ then $T^{kh_n}I$ contains a copy of I in the fourth or fifth section of C_{n+1} that is at distance d from a copy of J in C_{n+1}. Furthermore, for $k = 1,2,\dots,4$ we have*

$$\mu(T^{kh_n}I \cap J) > \frac{1}{8^d}\mu(J).$$

Proof. The statement of the lemma is clear when $k = 0$. Next we observe that $T^{h_n}I$ contains a copy of I in the fourth subcolumn of C_n that becomes a full level in the fifth section of C_{n+1}; this will still be at distance d from a copy of J in the fifth section of C_{n+1}. At the same time, the $(1,1)$-subcrescent of I in C_n starts one level below I (in C_n) and consists, on each level at distance j below I, $1 \le j < h_n$, of two intervals, contained in $T^{h_n}I$, each of length $(\frac{1}{16})^j$. Thus $\mu(T^{h_n}I \cap J) > \frac{1}{8^d}\mu(J)$.

For $T^{2h_n}I$ we observe that there is a copy of I is in the third subcolumn of C_n hence fourth section of C_{n+1}; in this case we choose a copy of J in the same section thus at distance d. There is also a $(2,1)$-subcrescent of I in the first subcolumn of C_n. $T^{3h_n}I$ and $T^{4h_n}I$ also contain a copy of I in the fourth subcolumn of C_n. Also, $T^{3h_n}I$ contains a $(3,2)$-subcrescent in the second column of C_n that comes from the $(2,1)$-subcrescent mentioned above. $T^{4h_n}I$ also contains a $(4,1)$-subcrescent in the first subcolumn of C_n. ☐

Lemma 3.3. *Let I be a level in the fifth section of column $\mathcal{C}_n$, let J be any level in the second section of $\mathcal{C}_n$, with d the distance that J is below I, $h_n/5 < d < h_n$. Let k be any integer with $0 < k < d/25$. Then*

$$\mu(T^{kh_n} I \cap J) > \frac{1}{8^{k+d}} \mu(J).$$

Proof. Write

$$k = \sum_{j=0}^{k'} k_j 5^j$$

where $k_j = 0, 1, \ldots, 4$, and k' is the largest positive integer such that $k_{k'} \neq 0$. Note that $k' = \lfloor \log_5 k \rfloor$.

Now

$$kh_n = \sum_{j=0}^{k'} k_j 5^j h_n.$$

Using that $h_{n+1} = 5h_n + 1$ we obtain

$$5^j h_n = h_{n+j} - \frac{5}{4}(5^j - 1).$$

Thus

$$kh_n = \sum_{j=0}^{k'} (k_j h_{n+j} - \frac{5}{4} k_j (5^j - 1)).$$

Now

$$\begin{aligned}
\sum_{j=0}^{k'} \frac{5}{4} k_j (5^j - 1) &\leq \frac{5}{4} \sum_{j=0}^{k'} 4(5^j) \\
&= 5(\frac{5^{k'+1} - 1}{4}) \\
&< 5 \cdot 5^{k'+1} \\
&\leq 25k.
\end{aligned}$$

Now let

$$\begin{aligned}
a_0 &= k_{k'} h_{n+k'}, \\
a_1 &= \sum_{j=0}^{k'-1} k_j h_{n+j}, \\
a_2 &= \sum_{j=0}^{k'} \frac{5}{4} k_j (5^j - 1).
\end{aligned}$$

Then $T^{kh_n} I = T^{(a_0 + a_1 - a_2)} I$. We first consider $T^{a_1} I$. For each $j = 0, \ldots, k' - 1$, apply the first part of Lemma 3.2 to finally obtain a full level in column $C_{n+k'}$ that is contained in $T^{a_1} I$ and at distance d above a copy of J in $C_{n+k'}$. By the second part of Lemma 3.2, $T^{a_0}(T^{a_1}I)$ contains a subcrescent of I that intersects the copy of J. Thus

$$\mu(T^{a_0}(T^{a_1}I)\cap J) > \frac{1}{8^{k+d}}\mu(J).$$

Finally $T^{-a_2}(T^{a_0}(T^{a_1}I))$ moves the subcrescent down at most $25k$ levels. From the choice of k we know that $25k < d$, and therefore the subcrescent of I that has been translated down at most $25k$ levels still intersects the copy of J, with a lower bound for the intersection still valid as calculated above. $\square$

Theorem 3.1. *For any sequence of nonzero integers $\{k_1,\ldots,k_r\}$, the transformation $T^{k_1}\times\cdots\times T^{k_r}$ is ergodic.*

Proof. Let $K=\max\{|k_i|\}$. Let A and B be in $\Pi_{i=1}^r X$ with $\nu(A)>0$ and $\nu(B)>0$. Find rectangles $I=I_1\times\cdots\times I_r$ and $J=J_1\times\cdots\times J_r$ such that

$$\nu(A\cap I) > \frac{3}{4}\nu(I),$$

$$\nu(B\cap J) > \frac{3}{4}\nu(J),$$

and $I_m, J_m, m=1,\ldots,r$ are all in the same column $\mathcal{C}_\ell$, and if k_m is positive choose I_m and J_m in the fifth and second sections of C_ℓ respectively, and if k_m is negative choose I_m and J_m in the second and fifth sections of C_ℓ respectively. We may further assume that $125K < h_\ell$. Let d_i be the distance between I_i and J_i for all i, and put $d=\max\{d_i\}$. Since $d > h_\ell/5$ it follows that $25K < d$.

Choose δ so that

$$0<\delta<(\frac{1}{8^{K+d}})^r.$$

Apply the Double Approximation Lemma twice (with $1-\epsilon=\frac{3}{4}$) to find $I'=I_1'\times\cdots\times I_r'$ and $J'=J_1'\times\cdots\times J_r'$ such that I' and J' are $(1-\frac{\delta}{2})$-full of A and B respectively, $I_1',\ldots,I_r',J_1',\ldots,J_r'$ are all in some column $\mathcal{C}_n$, $n>\ell$, and for each i, I_i' and J_i' are in the same $\mathcal{C}_\ell$-copy in $\mathcal{C}_n$ (this follows from the fact that for each application of the lemma there are at least $\tau=74(<100\cdot\frac{3}{4})$-percent of the copies satisfying the conditions and thus one can choose a common copy for A and B). Thus the distance between I_i' and J_i' is still d_i.

Let $H=h_n$. Then for all positive k_i, by Lemma 3.3,

$$\mu(T^{k_iH}I_i'\cap J_i') \geq \frac{1}{8^{k_i+d_i}}\mu(I_i') \geq \frac{1}{8^{K+d}}\mu(I_i').$$

By an analogous argument to Lemma 3.3, for all negative k_i,

$$\mu(T^{k_iH}I_i'\cap J_i') = \mu(I_i'\cap T^{|k_i|H}J_i') \geq \frac{1}{8^{k_i+d_i}}\mu(J_i') \geq \frac{1}{8^{K+d}}\mu(I_i').$$

Therefore,

$$\nu[(T^{k_1}\times\cdots\times T^{k_r})^H I'\cap J'] \geq (\frac{1}{8^{K+d}})^r\nu(I').$$

Thus,

$$\begin{aligned}
\nu((T^{k_1} \times \cdots \times T^{k_r})^H A \cap B) \\
&\geq \nu([(T^{k_1} \times \cdots \times T^{k_r})^H I' \cap J'] \setminus [(T^{k_1} \times \cdots \times T^{k_r})^H A \cap B]) \\
&\geq \nu((T^{k_1} \times \cdots \times T^{k_r})^H I' \cap J') - \nu((T^{k_1} \times \cdots \times T^{k_r})^H (I \setminus A)) - \nu(J \setminus B) \\
&\geq (\frac{1}{8^{K+d}})^r \nu(J') - \frac{\delta}{2}\nu(I') - \frac{\delta}{2}\nu(J') > 0.
\end{aligned}$$

Therefore $T^{k_1} \times \cdots \times T^{k_r}$ is ergodic. □

Remark. 1. The same proof will apply to a transformation where at the ℓ^{th} stage column C_ℓ is cut into $c > 1$ equally-spaced subcolumns $\mathcal{C}_{\ell,1}, \ldots, \mathcal{C}_{\ell,c}$, a single (staircase) spacer is put on top of column $\mathcal{C}_{\ell,c}$ and a stack of h_ℓ spacers is put on top of any of the middle subcolumns.

2. There exists a rank one infinite measure preserving transformation S such that S has infinite ergodic index but $S \times S^2$ is not conservative, hence S is not power weakly mixing [AFS2].

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