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Ext Classes and Embeddings for C^*-Algebras of Graphs with Sinks

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Abstract. We consider directed graphs E obtained by adding a sink to a fixed graph G. We associate an element of $\mathrm{Ext}(C^*(G))$ to each such E, and show that the classes of two such graphs are equal in $\mathrm{Ext}(C^*(G))$ if and only if the associated C^*-algebra of one can be embedded as a full corner in the C^*-algebra of the other in a particular way. If every loop in G has an exit, then we are able to use this result to generalize some known classification theorems for C^*-algebras of graphs with sinks.

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1. Introduction

The Cuntz-Krieger algebras $\mathcal{O}_A$ are C^*-algebras generated by a family of partial isometries whose relations are defined by a finite matrix A with entries in $\{0, 1\}$ and no zero rows. In 1982 Watatani [10] noted that one can view $\mathcal{O}_A$ as the C^*-algebra of a finite directed graph G with vertex adjacency matrix A, and the condition that A has no zero rows implies that G has no sinks.

In the late 1990's analogues of these C^*-algebras were considered for possibly infinite graphs which are allowed to contain sinks [4, 5]. Since that time there has been much interest in these graph algebras. By allowing graphs which are infinite and may contain sinks, the class of graph algebras has been extended to include many C^*-algebras besides the Cuntz-Krieger algebras. At the same time, graph algebras remain tractable C^*-algebras to study. Like the Cuntz-Krieger algebras,

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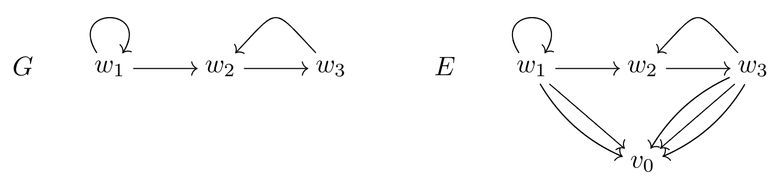
their basic structure is understandable and many of their invariants (such as K-theory or Ext) can be readily computed [2]. Furthermore, it has been found that many results about Cuntz-Krieger algebras hold for graph algebras with only minor modifications.

In addition, the graph approach has the advantage that it provides a convenient tool for visualization. If G is a graph and $C^*(G)$ is its associated C^*-algebra, then many facts about $C^*(G)$ can be translated into properties of G that can be determined by observation. Thus C^*-algebraic questions may be translated into (often easier to deal with) graph questions. Although a similar thing can be done for Cuntz-Krieger algebras — properties of $\mathcal{O}_A$ can be translated into properties of the matrix A — many of these results take a nicer form if one works with graphs instead of matrices.

Since sinks were specifically excluded from the original Cuntz-Krieger treatments as well as some of the earlier graph algebra work, there is now some interest in investigating the effect of sinks on the structure of graph algebras. This interest is further motivated by a desire to understand the Exel-Laca algebras of [3], which may be thought of as Cuntz-Krieger algebras of infinite matrices. It was shown in [6] that Exel-Laca algebras can be realized as direct limits of C^*-algebras of finite graphs with sinks. Therefore, it is reasonable to believe that results regarding C^*-algebras of graphs with sinks could prove useful in the study of Exel-Laca algebras.

Some progress in the study of C^*-algebras of graphs with sinks was made in [8] where the authors looked at a fixed graph G and considered 1-*sink extensions* of G. Loosely speaking, a 1-sink extension (E, v_0) of G is a graph E which is formed by adding a single sink v_0 to G. We say a 1-sink extension is *essential* if every vertex of G can reach the sink v_0. Here is an example of a graph G and an essential 1-sink extension E of G.



As in [8], we may associate an invariant called the *Wojciech vector* to a 1-sink extension. This vector is the element $\omega_E \in \prod_{G^0} \mathbb{N}$ whose w^{th} entry is the number of paths in $E^1 \backslash G^1$ from w to the sink v_0. For instance, the Wojciech vector in the above example is $\omega_E = \left(\begin{smallmatrix} 2 \\ 0 \\ 3 \end{smallmatrix}\right)$.

It was shown in [8] that for any 1-sink extension E of G there is an exact sequence

$$(1) \qquad 0 \longrightarrow I_{v_0} \xrightarrow{\ i\ } C^*(E) \xrightarrow{\ \pi_E\ } C^*(G) \longrightarrow 0.$$

Here I_{v_0} denotes the ideal generated by the projection p_{v_0} corresponding to the sink v_0. If E_1 and E_2 are 1-sink extensions, then we say that $C^*(E_2)$ may be $C^*(G)$-*embedded* into $C^*(E_1)$ if $C^*(E_2)$ is isomorphic to a full corner of $C^*(E_1)$ via an isomorphism which commutes with the π_{E_i}'s.

It was shown in [8] that $C^*(G)$-embeddability of 1-sink extensions is determined by the class of the Wojciech vector in $\operatorname{coker}(A_G - I)$, where A_G is the vertex matrix of G. Specifically, it was shown in [8, Theorem 2.3] that if G is a graph with no sinks or sources, (E_1, v_1) and (E_2, v_2) are two essential 1-sink extensions of G whose

Wojciech vectors have only a finite number of nonzero entries, and ω_{E_1} and ω_{E_2} are in the same class in $\operatorname{coker}(A_G - I)$, then there exists a 1-sink extension F of G such that $C^*(F)$ may be $C^*(G)$-embedded in both $C^*(E_1)$ and $C^*(E_2)$. In addition, a version of this result was proven for non-essential 1-sink extensions [8, Proposition 3.3] and a partial converse for both results was obtained in [8, Corollary 5.4]. In this paper we show that when every loop in G has an exit, much stronger results hold.

We shall see in §3 that if (E, v_0) is a 1-sink extension of G, then (except in degenerate cases) we will have $I_{v_0} \cong \mathcal{K}$. Thus we see from (1) that $C^*(E)$ is an extension of $C^*(G)$ by the compact operators. Hence, E determines an element in $\operatorname{Ext}(C^*(G))$. In §3 we prove the following.

Theorem. *Let G be a row-finite graph and (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. Then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if E_1 and E_2 determine the same element in $\operatorname{Ext}(C^*(G))$.*

It was shown in [9] that if G is a graph in which every loop has an exit, then $\operatorname{Ext}(C^*(G)) \cong \operatorname{coker}(A_G - I)$. Using the isomorphism constructed there we are able to translate the above result into a statement about the Wojciech vectors. Specifically we prove the following.

Theorem. *Let G be a row-finite graph in which every loop has an exit. If (E_1, v_1) and (E_2, v_2) are essential 1-sink extensions of G, then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if $[\omega_{E_1}] = [\omega_{E_2}]$ in $\operatorname{coker}(A_G - I)$.*

Provided that one is willing to allow all the loops in G to have exits, this result is an improvement over [8, Theorem 2.3] in the following respects. First of all, G is allowed to have sources and there are no conditions on the Wojciech vectors of E_1 and E_2. Second, we see that the graph F in the statement of [8, Theorem 2.3] can actually be chosen to be either E_1 or E_2. And finally, we see that the equality of the Wojciech vectors in $\operatorname{coker}(A_G - I)$ is not only sufficient but necessary. In §5 we obtain a version of this theorem for non-essential extensions.

This paper is organized as follows. We begin in §2 with some preliminaries regarding graph algebras. We also give precise definitions of 1-sink extensions, the Wojciech vector, and $C^*(G)$-embeddability. In §3 we show how to associate an element of $\operatorname{Ext}(C^*(G))$ to a (not necessarily essential) 1-sink extension. We then prove that if E_1 and E_2 are 1-sink extensions of G, then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if E_1 and E_2 determine the same element in $\operatorname{Ext}(C^*(G))$. In §4 we recall the definition of the isomorphism $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ from [9] and we prove that for essential 1-sink extensions, $C^*(G)$-embeddability may be characterized in terms of the Wojciech vector. In §5 we discuss non-essential extensions and again use the isomorphism ω to obtain a characterization of $C^*(G)$-embeddability for arbitrary 1-sink extensions. We conclude with an example and some observations.

2. Preliminaries

A (directed) graph $G = (G^0, G^1, r, s)$ consists of a countable set G^0 of vertices, a countable set G^1 of edges, and maps $r, s : G^1 \to G^0$ which identify the range and source of each edge. A vertex $v \in G^0$ is called a *sink* if $s^{-1}(v) = \emptyset$ and a *source* if $r^{-1}(v) = \emptyset$. We say that a graph is *row-finite* if each vertex emits only finitely many edges; that is, $s^{-1}(v)$ is finite for all $v \in G^0$. All of our graphs will be assumed to be row-finite.

If G is a row-finite directed graph, a *Cuntz-Krieger G-family* in a C^*-algebra is a set of mutually orthogonal projections $\{p_v : v \in G^0\}$ together with a set of partial isometries $\{s_e : e \in G^1\}$ which satisfy the *Cuntz-Krieger relations*

$$s_e^* s_e = p_{r(e)} \text{ for } e \in E^1 \quad\text{and}\quad p_v = \sum_{\{e : s(e)=v\}} s_e s_e^* \text{ whenever } v \in G^0 \text{ is not a sink.}$$

Then $C^*(G)$ is defined to be the C^*-algebra generated by a universal Cuntz-Krieger G-family [4, Theorem 1.2].

A *path* in a graph G is a finite sequence of edges $\alpha := \alpha_1 \alpha_2 \ldots \alpha_n$ for which $r(\alpha_i) = s(\alpha_{i+1})$ for $1 \leq i \leq n-1$, and we say that such a path has length $|\alpha| = n$. For $v, w \in G^0$ we write $v \geq w$ to mean that there exists a path with source v and range w. For $K, L \subseteq G^0$ we write $K \geq L$ to mean that for each $v \in K$ there exists $w \in L$ such that $v \geq w$. A *loop* is a path whose range and source are equal, and for a given loop $x := x_1 x_2 \ldots x_n$ we say that x is based at $s(x_1) = r(x_n)$. An *exit* for a loop x is an edge e for which $s(e) = s(x_i)$ for some i and $e \neq x_i$. A graph is said to satisfy Condition (L) if every loop in G has an exit.

We call a loop *simple* if it returns to its base point exactly once; that is $s(x_1) \neq r(x_i)$ for $1 \leq i < n$. A graph is said to satisfy Condition (K) if no vertex in the graph is the base of exactly one simple loop; that is, every vertex is either the base of no loops or the base of more than one simple loop. Note that Condition (K) implies Condition (L).

A 1-*sink extension* of G is a row-finite graph E which contains G as a subgraph and satisfies:

1. $H := E^0 \setminus G^0$ is finite, contains no sources, and contains exactly 1 sink.
2. There are no loops in E whose vertices lie in H.
3. If $e \in E^1 \setminus G^1$, then $r(e) \in H$.
4. If w is a sink in G, then w is a sink in E.

When we say (E, v_0) is a 1-sink extension of G, we mean that v_0 is the sink outside G^0. An edge e with $r(e) \in H$ and $s(e) \in G^0$ is called a *boundary edge* and the sources of the boundary edges are called *boundary vertices*. We write B_E^1 for the set of boundary edges and B_E^0 for the set of boundary vertices. If $w \in G^0$ we denote by $Z(w, v_0)$ the set of paths α from w to v_0 which leave G immediately in the sense that $r(\alpha_1) \notin G^0$. The *Wojciech vector* of E is the element ω_E of $\prod_{G^0} \mathbb{N}$ given by

$$\omega_E(w) := \#Z(w, v_0) \text{ for } w \in G^0.$$

If (E, v_0) is a 1-sink extension of G, then there exists a surjection $\pi_E : C^*(E) \to C^*(G)$ for which

$$0 \longrightarrow I_{v_0} \xrightarrow{\ i\ } C^*(E) \xrightarrow{\ \pi_E\ } C^*(G) \longrightarrow 0$$

is a short exact sequence [8, Corollary 1.3]. Here I_{v_0} denotes the ideal in $C^*(E)$ generated by the projection p_{v_0} corresponding to the sink v_0. We say that (E, v_0) is an *essential* 1-sink extension if $G^0 \geq v_0$. It follows from [8, Lemma 2.2] that E is an essential 1-sink extension if and only if I_{v_0} is an essential ideal in $C^*(E)$. Also note that if there exists an essential 1-sink extension of G, then G cannot have any sinks.

Suppose (E_1, v_1) and (E_2, v_2) are 1-sink extensions of G. We say that $C^*(E_2)$ is $C^*(G)$-*embeddable into* $C^*(E_1)$ if there is an isomorphism ϕ of $C^*(E_2) = C^*(s_e, p_v)$ onto a full corner in $C^*(E_1) = C^*(t_f, q_w)$ such that $\phi(p_{v_2}) = q_{v_1}$ and $\pi_E \circ \phi = \pi_F : C^*(F) \to C^*(G)$. We call ϕ a $C^*(G)$-*embedding*. Notice that if $C^*(E_2)$ is $C^*(G)$-embeddable into $C^*(E_1)$, then $C^*(E_2)$ is Morita equivalent to $C^*(E_1)$ in a way which respects the common quotient $C^*(G)$.

If G is a graph, the *vertex matrix* of G is the $G^0 \times G^0$ matrix A_G whose entries are given by $A_G(v, w) := \#\{e \in G^1 : s(e) = v \text{ and } r(e) = w\}$, and the *edge matrix* of G is the $G^1 \times G^1$ matrix B_G whose entries are given by

$$B_G(e, f) := \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{otherwise.} \end{cases}$$

We shall frequently be concerned with the maps $A_G - I : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ and $B_G - I : \prod_{G^1} \mathbb{Z} \to \prod_{G^1} \mathbb{Z}$ given by left multiplication.

Throughout we shall let $\mathcal{H}$ denote a separable infinite-dimensional Hilbert space, $\mathcal{K}$ the compact operators on $\mathcal{H}$, $\mathcal{B}$ the bounded operators on $\mathcal{H}$, and $\mathcal{Q} := \mathcal{B}/\mathcal{K}$ the associated Calkin algebra. We shall also let $i : \mathcal{K} \to \mathcal{B}$ denote the inclusion map and $\pi : \mathcal{B} \to \mathcal{Q}$ the projection map. If A is a C^*-algebra, then an *extension* of A (by the compact operators) is a homomorphism $\tau : A \to \mathcal{Q}$. An extension is said to be *essential* if it is a monomorphism.

3. $C^*(G)$-embeddability and CK-equivalence

In order to see how $\operatorname{Ext}(C^*(G))$ and $C^*(G)$-embeddability are related, we will follow the approach in [9, §3] and view Ext as the CK-equivalence classes of essential extensions.

Definition 3.1. If τ_1 and τ_2 are two (not necessarily essential) extensions of A by $\mathcal{K}$, then τ_1 and τ_2 are *CK-equivalent* if there exists either an isometry or coisometry $W \in \mathcal{B}$ for which

$$\tau_1 = \operatorname{Ad}(\pi(W)) \circ \tau_2 \quad \text{and} \quad \tau_2 = \operatorname{Ad}(\pi(W^*)) \circ \tau_1.$$

Remark 3.2. In light of [9, Corollary 5.15] we see that the above definition is equivalent to the one given in [9, Definition 3.1]. Also note that CK-equivalence is not obviously an equivalence relation. However, for certain classes of extensions (such as essential extensions) it has been shown to be an equivalence relation [9, Remark 3.2].

Recall that if E is a 1-sink extension of G with sink v_0, then it follows from [4, Corollary 2.2] that $I_{v_0} \cong \mathcal{K}(\ell^2(E^*(v_0)))$ where $E^*(v_0) = \{\alpha \in E^* : r(\alpha) = v_0\}$. Thus $I_{v_0} \cong \mathcal{K}$ when $E^*(v_0)$ contains infinitely many elements, and $I_{v_0} \cong M_n(\mathbb{C})$ when $E^*(v_0)$ contains a finite number of elements. If G has no sources, then it is easy to see that $E^*(v_0)$ must have infinitely many elements, and it was shown in [9, Lemma 6.6] that if E is an essential 1-sink extension of G, then $E^*(v_0)$ will also have infinitely many elements. Consequently, in each of these cases we will have $I_{v_0} \cong \mathcal{K}$. Furthermore, one can see from the proof of [4, Corollary 2.2] that p_{v_0} is a minimal projection in I_{v_0}.

Definition 3.3. Let G be a row-finite graph and let (E, v_0) be a 1-sink extension of G. If $I_{v_0} \cong \mathcal{K}$, (i.e., $E^*(v_0)$ has infinitely many elements), then choose any isomorphism $i_E : \mathcal{K} \to I_{v_0}$, and define the *extension associated to* E to be (the strong equivalence class of) the Busby invariant $\tau : C^*(G) \to \mathcal{Q}$ associated to the short exact sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{i_E} C^*(E) \xrightarrow{\pi_E} C^*(G) \longrightarrow 0.$$

If $I_{v_0} \cong M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ (i.e., $E^*(v_0)$ has finitely many elements), then the *extension associated to* E is defined to be (the strong equivalence class of) the zero map $\tau : C^*(G) \to \mathcal{Q}$. That is, $\tau : C^*(G) \to \mathcal{Q}$ and $\tau(x) = 0$ for all $x \in C^*(G)$.

Note that the extension associated to E is always a map from $C^*(G)$ into $\mathcal{Q}$. Also note that the above definition is well-defined in the case when $I_{v_0} \cong \mathcal{K}$. That is, two different choices of i_E will produce extensions with strongly equivalent Busby invariants (see problem 3E(c) of [11] for more details). Also, since p_{v_0} is a minimal projection, $i_E^{-1}(p_{v_0})$ will always be a rank 1 projection.

Our goal in the remainder of this section is to prove the following theorem and its corollary.

Theorem 3.4. *Let G be a row-finite graph, and let E_1 and E_2 be* 1*-sink extensions of G. Then the extensions associated to E_1 and E_2 are CK-equivalent if and only if one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other.*

Corollary 3.5. *Let G be a row-finite graph, and let E_1 and E_2 be essential* 1*-sink extensions of G. Then the extensions associated to E_1 and E_2 are equal in* $\operatorname{Ext}(C^*(G))$ *if and only if one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other.*

Remark 3.6. Note that we are not assuming that each of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other, only that one of them can.

Proof of Corollary 3.5 . Because E_1 and E_2 are essential it follows from [9, Lemma 6.6] that $I_{v_1} \cong I_{v_2} \cong \mathcal{K}$. Furthermore, [9, Lemma 3.2] shows that two essential extensions are equal in Ext if and only if they are CK-equivalent. □

Lemma 3.7. *Let P and Q be rank* 1 *projections in $\mathcal{B}$. Then there exists a unitary $U \in \mathcal{B}$ such that $P = U^*QU$ and $I - U$ has finite rank.*

Proof. Straightforward. □

Lemma 3.8. *Let G be a row-finite graph, and let (E, v_0) be a* 1*-sink extension of G. If the extension associated to E is the zero map, then there is an isomorphism $\Psi : C^*(E) \to C^*(G) \oplus I_{v_0}$ which makes the diagram*

$$\begin{array}{ccc} C^*(E) & \xrightarrow{\Psi} & C^*(G) \oplus I_{v_0} \\ & \searrow^{\pi_E} & \downarrow{\scriptstyle p_1} \\ & & C^*(G) \end{array}$$

commute. Here p_1 is the projection $(a, b) \mapsto a$.

Proof. Since the extension associated to E is zero, one of two things must occur. If $I_{v_0} \cong \mathcal{K}$, then τ is the Busby invariant of $0 \to I_{v_0} \xrightarrow{i} C^*(E) \xrightarrow{\pi_E} C^*(G) \to 0$. If $I_{v_0} \cong M_n(\mathbb{C})$, then since $M_n(\mathbb{C})$ is unital it follows that $\mathcal{Q}(I_{v_0}) \cong \mathcal{M}(M_n(\mathbb{C}))/M_n(\mathbb{C}) = 0$ and the Busby invariant of $0 \to I_{v_0} \xrightarrow{i} C^*(E) \xrightarrow{\pi_E} C^*(G) \to 0$ must be the zero map. In either case, the Busby invariant of the extension $0 \to I_{v_0} \xrightarrow{i} C^*(E) \xrightarrow{\pi_E} C^*(G) \to 0$ is zero. From [11, Proposition 3.2.15] it follows that $C^*(E) \cong C^*(G) \oplus I_{v_0}$ via the map $\Psi(x) := (\pi_E(x), \sigma(x))$, where $\sigma : C^*(E) \to I_{v_0}$ denotes the (unique) map for which $\sigma \circ i$ is the identity. The fact that $p_1 \circ \Psi = \pi_E$ then follows by checking each on generators of $C^*(E)$. □

Proof of Sufficiency in Theorem 3.4. Let E_1 and E_2 are 1-sink extensions of G whose associated extensions are CK-equivalent. Also let v_1 and v_2 denote the sinks of E_1 and E_2 and τ_1 and τ_2 be the extensions associated to E_1 and E_2. Consider the following cases.

Case 1: Either $E^*(v_1)$ is finite or $E^*(v_2)$ is finite.

Without loss of generality let us assume that $E^*(v_1)$ is finite and the number of elements in $E^*(v_1)$ is less than or equal to the number of elements in $E^*(v_2)$. Then $I_{v_1} \cong M_n(\mathbb{C})$ for some finite n, and because $I_{v_2} \cong \mathcal{K}(\ell^2(E^*(v_2)))$ we see that either $I_{v_2} \cong \mathcal{K}$ or $I_{v_2} \cong M_m(\mathbb{C})$ for some finite $m \geq n$. In either case we may choose an imbedding $\phi : I_{v_1} \to I_{v_2}$ which maps onto a full corner of I_{v_2}. (Note that since I_{v_2} is simple we need only choose ϕ to map onto a corner, and then that corner is automatically full.) Furthermore, since p_{v_1} and q_{v_2} are rank 1 projections, we may choose ϕ in such a way that $\phi(p_{v_1}) = q_{v_2}$. We now define $\Phi : C^*(G) \oplus I_{v_1} \to C^*(G) \oplus I_{v_2}$ by $\Phi((a,b)) = (a, \phi(b))$. We see that Φ maps $C^*(G) \oplus I_{v_1}$ onto a full corner of $C^*(G) \oplus I_{v_2}$ and that Φ makes the diagram

$$\begin{array}{ccc} C^*(G) \oplus I_{v_1} & \xrightarrow{\quad\Phi\quad} & C^*(G) \oplus I_{v_2} \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & C^*(G) & \end{array}$$

commute, where p_1 is the projection $(a,b) \mapsto a$. Now since $\tau_1 = 0$ and τ_2 is CK-equivalent to τ_2, it follows that $\tau_2 = 0$. Thus Lemma 3.8, the existence of Φ, and the above commutative diagram imply that $C^*(E_1)$ is $C^*(G)$-embeddable into $C^*(E_2)$.

Case 2: Both $E^*(v_1)$ and $E^*(v_2)$ are infinite.

Then $I_{v_1} \cong I_{v_2} \cong \mathcal{K}$. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E_1-family in $C^*(E_1)$, and let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E_2-family in $C^*(E_2)$. For $k \in \{1, 2\}$, choose isomorphisms $i_{E_k} : \mathcal{K} \to I_{v_k}$ so that the Busby invariant τ_k of

$$0 \longrightarrow \mathcal{K} \xrightarrow{i_{E_k}} C^*(E_k) \xrightarrow{\pi_{E_k}} C^*(G) \longrightarrow 0$$

is an extension associated to E_k. By hypothesis τ_1 and τ_2 are CK-equivalent. Therefore, after interchanging the roles of E_1 and E_2 if necessary, we may assume that there exists an isometry $W \in \mathcal{B}$ for which $\tau_1 = \mathrm{Ad}(\pi(W)) \circ \tau_2$ and $\tau_2 = \mathrm{Ad}(\pi(W^*)) \circ \tau_1$.

For $k \in \{1, 2\}$, let $PB_k := \{(T, a) \in \mathcal{B} \oplus C^*(G) : \pi(T) = \tau_k(a)\}$ be the pullback C^*-algebra along π and τ_k. It follows from [11, Proposition 3.2.11] that $PB_k \cong$

$C^*(E_k)$. Now for $k \in \{1, 2\}$, let σ_k be the unique map which makes the diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{i_{E_k}} & C^*(E_k) \\ & \searrow^{i} & \downarrow{\scriptstyle \sigma_k} \\ & & \mathcal{B} \end{array}$$

commute. Then $\sigma_1(p_{v_1})$ and $\sigma_2(q_{v_2})$ are rank 1 projections in $\mathcal{B}$. Choose a unit vector $x \in (\ker W^*)^\perp$. By Lemma 3.7 there exists a unitary $U_1 \in \mathcal{B}$ such that $U_1\sigma_2(q_{v_2})U_1^*$ is the projection onto $\operatorname{span}\{x\}$, and for which $I - U_1$ is compact. Therefore, by the way in which x was chosen $WU_1\sigma_2(q_{v_2})U_1^*W^*$ is a rank 1 projection. We may then use Lemma 3.7 again to produce a unitary $U_2 \in \mathcal{B}$ for which $U_2(WU_1\sigma_2(q_{v_2})U_1^*W^*)U_2^* = \sigma_1(p_{v_1})$, and $I - U_2$ is compact.

Let $V := U_2WU_1$. Then V is an isometry, and we may define a map $\Phi : PB_2 \to PB_1$ by $\Phi((T, a)) = (VTV^*, a)$. Since $V^*V = I$ it follows that Φ is a homomorphism, and since U_1 and U_2 differ from I by a compact operator, we see that $\pi(V) = \pi(W)$. Therefore

$$\pi(VTV^*) = \pi(W)\pi(T)\pi(W^*) = \pi(W)\tau_2(a)\pi(W^*) = \tau_1(a)$$

so $(VTV^*, a) \in PB_1$, and Φ does in fact take values in PB_1.

For $k \in \{1, 2\}$, let $p_k : PB_k \to C^*(G)$ be the projection $p_k((T, a)) = a$. Then the diagram

$$\begin{array}{ccc} PB_2 & \xrightarrow{\Phi} & PB_1 \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & C^*(G) & \end{array}$$

commutes and $\Phi((\sigma_2(q_{v_2}), 0)) = (\sigma_1(p_{v_1}), 0)$. Also, for $k \in \{1, 2\}$, let Ψ_k be the standard isomorphism from $C^*(E_k)$ to PB_k given by $\Psi_k(x) = (\sigma_1(x), \pi_{E_k}(x))$ [11, Proposition 3.2.11]. Then for each $k \in \{1, 2\}$, the diagram

$$\begin{array}{ccc} C^*(E_k) & \xrightarrow{\Psi_k} & PB_k \\ & {\scriptstyle \pi_{E_k}}\searrow \quad \swarrow{\scriptstyle p_k} & \\ & C^*(G) & \end{array}$$

commutes and we have that $\Psi_1(p_{v_1}) = (\sigma_1(p_{v_1}), 0)$ and $\Psi_2(q_{v_2}) = (\sigma_2(q_{v_2}), 0)$. If we define $\phi : C^*(E_2) \to C^*(E_1)$ by $\phi := \Psi_1^{-1} \circ \Phi \circ \Psi_2$, then the diagram

(2)
$$\begin{array}{ccc} C^*(E_2) & \xrightarrow{\phi} & C^*(E_1) \\ & {\scriptstyle \pi_{E_2}}\searrow \quad \swarrow{\scriptstyle \pi_{E_1}} & \\ & C^*(G) & \end{array}$$

commutes and $\phi(q_{v_2}) = p_{v_1}$.

We shall now show that ϕ embeds $C^*(E_2)$ onto a full corner of $C^*(E_1)$. We begin by showing that Φ embeds PB_2 onto a corner of PB_1. To see that Φ is injective, note that since V is an isometry

$$\begin{aligned}\|VTV^*\|^2 &= \|(VTV^*)(VTV^*)^*\| = \|VTV^*VT^*V^*\| = \|VTT^*V^*\| \\ &= \|(VT)(VT)^*\| = \|VT\|^2 = \|T\|^2.\end{aligned}$$

Therefore $\|VTV^*\| = \|T\|$, and

$$\|\Phi((T,a))\| = \|(VTV^*,a)\| = \max\{\|VTV^*\|, \|a\|\} = \max\{\|T\|, \|a\|\} = \|(T,a)\|.$$

Next we shall show that the image of Φ is a corner in PB_1. Let $P := VV^*$ be the range projection of V. We shall define a map $L_P : PB_1 \to PB_1$ by $L_P((T,a)) = (PT, a)$. To see that L_P actually takes values in PB_1 recall that U_1 and U_2 differ from I by a compact operator and therefore $\pi(V) = \pi(W)$. We then have that

$$\begin{aligned}\pi(PT) &= \pi(VV^*)\pi(T) = \pi(WW^*)\tau_1(a) = \pi(WW^*)\pi(W)\tau_2(a)\pi(W^*) \\ &= \pi(W)\tau_2(a)\pi(W^*) = \tau_1(a).\end{aligned}$$

Hence $(PT,a) \in PB_1$. In a similar way we may define $R_P : PB_1 \to PB_1$ by $R_P((T,a)) = (TP, a)$. Since P is a projection, we see that L_P and R_P are bounded linear maps. One can also check that (L_P, R_P) is a double centralizer and therefore defines an element $\mathcal{P} := (L_P, R_P) \in \mathcal{M}(PB_1)$. Because P is a projection, $\mathcal{P}$ must also be a projection. Also for any $(T,a) \in PB_1$ we have that $\mathcal{P}(T,a) = (PT,a)$ and $(T,a)\mathcal{P} = (TP,a)$.

Now for all $(T,a) \in PB_2$ we have

$$\begin{aligned}\Phi((T,a)) &= (VTV^*, a) = (VV^*VTV^*VV^*, a) \\ &= (PVTV^*P, a) = \mathcal{P}(VTV^*,a)\mathcal{P} = \mathcal{P}\Phi((T,a))\mathcal{P}\end{aligned}$$

and therefore Φ maps PB_2 into the corner $\mathcal{P}(PB_1)\mathcal{P}$. We shall now show that Φ actually maps onto this corner. If $(T,a) \in \mathcal{P}(PB_1)\mathcal{P}$, then

$$\pi(V^*TV) = \pi(W)^*\pi(T)\pi(W) = \pi(W)^*\tau_1(a)\pi(W) = \tau_2(a)$$

and so $(VTV^*, a) \in PB_2$. But then $\Phi((V^*TV, a)) = (VV^*TVV^*, a) = (PTP, a) = \mathcal{P}(T,a)\mathcal{P} = (T,a)$. Thus Φ embeds PB_2 onto the corner $\mathcal{P}(PB_1)\mathcal{P}$.

Because Ψ_1 and Ψ_2 are isomorphisms, it follows that ϕ embeds $C^*(E_2)$ onto a corner of $C^*(E_1)$. We shall now show that this corner must be full. This will follow from the commutativity of diagram (2). Let I be any ideal in $C^*(E_1)$ with the property that $\operatorname{im}\phi \subseteq I$. Since $\phi(q_{v_2}) = p_{v_1}$ it follows that $p_{v_1} \in \operatorname{im}\phi \subseteq I$. Therefore, $I_{v_1} \subseteq I$. Furthermore, for any $w \in G^0$ we have by commutativity that $\pi_{E_1}(p_w - \phi(q_w)) = 0$. Therefore $p_w - \phi(q_w) \in \ker \pi_{E_1} = I_{v_1}$, and it follows that $p_w - \phi(q_w) \in I_{v_1} \subseteq I$. Since $\phi(q_w) \in \operatorname{im}\phi \subseteq I$, this implies that $p_w \in I$ for all $w \in G^0$. Thus $p_w \in I$ for all $w \in G^0 \cup \{v_1\}$. If we let $H := \{v \in E_1^0 : p_v \in I\}$, then it follows from [1, Lemma 4.2] that H is a saturated hereditary subset of $C^*(E_1)$. Since we see from above that H contains $G^0 \cup \{v_1\}$, and since E_1 is a 1-sink extension of G, it follows that $H = E_1^0$. Therefore $I_H = C^*(E_1)$ and since $I_H \subseteq I$ it follows that $I = C^*(E_1)$. Hence $\operatorname{im}\phi$ is a full corner in $C^*(E_1)$. □

Proof of Necessity in Theorem 3.4. Let E_1 and E_2 be 1-sink extensions of G and suppose that $C^*(E_2)$ is $C^*(G)$-embeddable into $C^*(E_1)$. Let v_1 and v_2 denote the sinks of E_1 and E_2, respectively. For $k \in \{1,2\}$ let $E_k^*(v_k) := \{\alpha \in E_k^* : r(\alpha) = v_k\}$, and let $\phi : C^*(E_2) \to C^*(E_1)$ be a $C^*(G)$-embedding. Consider the following cases.

Case 1: $E_1^*(v_1)$ is finite.

Then $I_{v_1} \cong M_n(\mathbb{C})$ for some finite n. Since $\phi(I_{v_2}) \subseteq I_{v_1}$, and $I_{v_2} \cong \mathcal{K}(\ell^2(E_2^*(v_2)))$, a dimension argument implies that $E_2^*(v_2)$ must be finite. Thus if τ_1 and τ_2 are the extensions associated to E_1 and E_2, we have that $\tau_1 = \tau_2 = 0$ so that τ_1 and τ_2 are CK-equivalent.

Case 2: $E_1^*(v_1)$ is infinite.

Then $I_{v_1} \cong \mathcal{K}$. Choose any isomorphism $i_{E_1} : \mathcal{K} \to I_{v_1}$, and let $\sigma : C^*(E_1) \to \mathcal{B}$ be the (unique) map which makes the diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{i_{E_1}} & C^*(E_1) \\ & \underset{i}{\searrow} & \downarrow{\scriptstyle \sigma} \\ & & \mathcal{B} \end{array}$$

commute. If we let τ_1 be the corresponding Busby invariant, then τ_1 is the extension associated to E_1.

Furthermore, we know that $I_{v_2} \cong \mathcal{K}(H)$, where H is a Hilbert space which is finite-dimensional if $E_2^*(v_2)$ is finite and infinite-dimensional if $E_2^*(v_2)$ is infinite. Choose an isomorphism $i_{E_2} : \mathcal{K}(H) \to I_{v_2}$. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(H) & \xrightarrow{i_{E_2}} & C^*(E_2) & \xrightarrow{\pi_{E_2}} & C^*(G) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & \mathcal{K} & \xrightarrow{i_{E_1}} & C^*(E_1) & \xrightarrow{\pi_{E_1}} & C^*(G) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau_1} & & \\ 0 & \longrightarrow & \mathcal{K} & \xrightarrow{i} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{Q} & \longrightarrow & 0 \end{array} \tag{3}$$

commutes and has exact rows.

Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E_2-family in $C^*(E_2)$ and $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E_1-family in $C^*(E_1)$.

We shall now define a bounded linear transformation $U : H \to \mathcal{H}$. Since $i_{E_2}^{-1}(p_{v_2})$ is a rank 1 projection, we may write $i_{E_2}^{-1}(p_{v_2}) = e \otimes e$, where e is a unit vector in $\operatorname{im} i_{E_2}^{-1}(p_{v_2})$. Likewise, we may write $i_{E_1}^{-1}(q_{v_1}) = f \otimes f$ for some unit vector $f \in \operatorname{im} i_{E_1}^{-1}(q_{v_1})$. For convenience of notation write $\beta := \sigma \circ \phi \circ i_{E_2}$. Note that $\phi(p_{v_2}) = q_{v_1}$ implies that $\beta(e \otimes e) = f \otimes f$. Now for any $h \in H$ define

$$U(h) := \beta(h \otimes e)(f).$$

Then U is a linear transformation and

$$\begin{aligned} \langle U(h), U(k) \rangle &= \langle \beta(h \otimes e)(f), \beta(k \otimes e)(f) \rangle = \langle \beta(k \otimes e)^* \beta(h \otimes e)(f), f \rangle \\ &= \langle \beta(\langle h, k \rangle (e \otimes e))(f), f \rangle = \langle h, k \rangle \langle \beta(e \otimes e)(f), f \rangle \\ &= \langle h, k \rangle \langle (f \otimes f)(f), f \rangle = \langle h, k \rangle \langle f, f \rangle = \langle h, k \rangle. \end{aligned}$$

Therefore U is an isometry.

Now since ϕ embeds $C^*(E_2)$ onto a full corner of $C^*(E_1)$, it follows that there exists a projection $p \in \mathcal{M}(C^*(E_1))$ such that $\operatorname{im} \phi = pC^*(E_1)p$. Because σ is a nondegenerate representation (since $\sigma(I_{v_1}) = \mathcal{K}$), it extends to a representation

$\overline{\sigma} : \mathcal{M}(C^*(E_1)) \to \mathcal{B}$ by [7, Corollary 2.51]. Let $P := \overline{\sigma}(p)$. We shall show that $\operatorname{im} P \subseteq \operatorname{im} U$. Let $g \in \operatorname{im} P$. Also let f be as before. Then $g \otimes f \in \mathcal{K}$ and

$$\sigma(pi_{E_1}(g \otimes f)p) = \overline{\sigma}(p)\sigma(i_{E_1}(g \otimes f))\overline{\sigma}(p) = P(g \otimes f)P = \sigma(i_{E_1}(g \otimes f)).$$

Now since $pi_{E_1}(g \otimes f)p \in pC^*(E_1)p = \operatorname{im} \phi$, there exists $a \in C^*(E_2)$ such that $\phi(a) = pi_{E_1}(g \otimes f)p$. In addition, since $\pi_{E_1} : C^*(E_1) \to C^*(G)$ is surjective, it extends to a homomorphism $\overline{\pi}_{E_1} : \mathcal{M}(C^*(E_1)) \to \mathcal{M}(C^*(G))$ by [7, Corollary 2.51]. By commutativity and exactness we then have that

$$\pi_{E_2}(a) = \pi_{E_1}(\phi(a)) = \pi_{E_1}(pi_{E_1}(g \otimes f)p) = \overline{\pi}_{E_1}(p)\pi_{E_1}(i_{E_1}(g \otimes f))\overline{\pi}_{E_1}(p) = 0.$$

Thus $a \in \operatorname{im} i_{E_2}$ by exactness, and we have that $a = i_{E_2}(T)$ for some $T \in \mathcal{K}(H)$. Let $h := T(e)$. Then

$$\begin{aligned} U(T(e)) &= \beta(T(e) \otimes e)(f) = \beta(T \circ (e \otimes e))(f) = \beta(T)\beta(e \otimes e)(f) \\ &= \beta(T)(f \otimes f)(f) = \sigma(pi_{E_1}(g \otimes f)p)(f) = \sigma(i_{E_1}(g \otimes f))(f) \\ &= (g \otimes f)(f) = \langle f, f \rangle g = g. \end{aligned}$$

Thus $g \in \operatorname{im} U$ and $\operatorname{im} P \subseteq \operatorname{im} U$.

Now if H is a finite-dimensional space, it follows that $\operatorname{im} U$ is finite-dimensional. Since $\operatorname{im} P \subseteq \operatorname{im} U$, this implies that P has finite rank and hence $\pi(P) = 0$. Now if $x \in C^*(G)$, then since π_{E_2} is surjective there exists an element $a \in C^*(E_2)$ for which $\pi_{E_2}(a) = x$. Since $\pi_{E_1}(\phi(a)) = \pi_{E_2}(a) = x$, it follows that $\tau_1(x) = \pi(\sigma(\phi(a)))$. But since $\phi(a) \in \operatorname{im} \phi = pC^*(E_1)p$ we have that $\phi(a) = p\phi(a)$ and thus $\tau_1(x) = \pi(\overline{\sigma}(p)\sigma(\phi(p)) = 0$. Since x was arbitrary this implies that $\tau_1 = 0$. Furthermore, since H is finite-dimensional, the extension associated to E_2 is $\tau_2 = 0$. Thus τ_1 and τ_2 are CK-equivalent.

Therefore, all that remains is to consider the case when H is infinite-dimensional. In this case $H = \mathcal{H}$ and $\mathcal{K}(H) = \mathcal{K}$. Furthermore, if S is any element of $\mathcal{K}$, then for all $h \in \mathcal{H}$ we have that

$$(\beta(S) \circ U)(h) = \beta(S)(\beta(h \otimes e)(f)) = \beta(Sh \otimes e)(f) = U(Sh).$$

Since U is an isometry this implies that $U^*\beta(S)U = S$ for all $S \in \mathcal{K}$. Therefore, $\operatorname{Ad}(U^*) \circ \beta$ is the inclusion map $i : \mathcal{K} \to \mathcal{B}$. Since $\operatorname{Ad}(U^*) \circ \beta = \operatorname{Ad}(U^*) \circ \sigma \circ \phi \circ i_{E_2}$, this implies that $\operatorname{Ad}(U^*) \circ \sigma \circ \phi$ is the unique map which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{i_{E_2}} & C^*(E_2) \\ & \searrow^{i} & \downarrow{\scriptstyle \operatorname{Ad}(U^*) \circ \sigma \circ \phi} \\ & & \mathcal{B}. \end{array}$$

Therefore, if τ_2 is (the Busby invariant of) the extension associated to $C^*(E_2)$, then by definition τ_2 is equal to the following. For any $x \in C^*(G)$ choose an $a \in C^*(E_2)$ for which $\pi_{E_2}(a) = x$. Then $\tau_2(x) := \pi(\operatorname{Ad}(U^*) \circ \sigma \circ \phi(a))$. Using the commutativity of diagram (3), this implies that

$$\begin{aligned} \tau_2(x) &= \operatorname{Ad}(\pi(U^*)) \circ \pi(\sigma(\phi(a))) = \operatorname{Ad}(\pi(U^*)) \circ \tau_1(\pi_{E_1}(\phi(a))) \\ &= \operatorname{Ad}(\pi(U^*)) \circ \tau_1(\pi_{E_2}(a)) = \operatorname{Ad}(\pi(U^*)) \circ \tau_1(x). \end{aligned}$$

So for all $x \in C^*(G)$ we have that

$$\tau_2(x) = \pi(U^*)\tau_1(x)\pi(U). \tag{4}$$

Now if a is any element of $C^*(E_2)$, then $\phi(a) \in pC^*(E_1)p$. Thus $\phi(a) = p\phi(a)$ and

$$\sigma(\phi(a)) = \sigma(p\phi(a)) = \overline{\sigma}(p)\sigma(\phi(a)) = P\sigma(\phi(a)).$$

Hence $\operatorname{im}\sigma(\phi(a)) \subseteq \operatorname{im} P \subseteq \operatorname{im} U$, and we have that

$$UU^*\sigma\phi(a) = \sigma\phi(a) \qquad \text{for all } a \in C^*(E_2).$$

Furthermore, for any $x \in C^*(G)$, we may choose an $a \in C^*(E_2)$ for which $\pi_{E_2}(a) = x$, and using the commutativity of diagram (3) we then have that

$$\begin{aligned} UU^*\sigma\phi(a) &= \sigma\phi(a) \\ \pi(UU^*)\pi\sigma\phi(a) &= \pi\sigma\phi(a) \\ \pi(UU^*)\tau_1\pi_{E_1}\phi(a) &= \tau_1\pi_{E_1}\phi(a) \\ \pi(UU^*)\tau_1\pi_{E_2}(a) &= \tau_1\pi_{E_2}(a) \\ \pi(UU^*)\tau_1(x) &= \tau_1(x). \end{aligned}$$

In addition, this implies that for any $x \in C^*(G)$ we have that $\pi(UU^*)\tau_1(x^*) = \tau_1(x^*)$, and taking adjoints this gives that

$$\tau_1(x)\pi(UU^*) = \tau_1(x) \qquad \text{for all } x \in C^*(G).$$

Thus for all $x \in C^*(G)$ we have

$$\tau_1(x) = \pi(UU^*)\tau_1(x)\pi(UU^*) = \pi(U)\big(\pi(U^*)\tau_1(x)\pi(U)\big)\pi(U^*) = \pi(U)\tau_2(x)\pi(U^*).$$

This, combined with Equation (4), implies that $\tau_1 = \operatorname{Ad}(\pi(U)) \circ \tau_2$ and $\tau_2 = \operatorname{Ad}(\pi(U^*)) \circ \tau_1$. Since U is an isometry, τ_1 and τ_2 are CK-equivalent. □

4. $C^*(G)$-embeddability for essential 1-sink extensions

In the previous section it was shown that if E_1 and E_2 are two 1-sink extensions of G, then one of the $C^*(E_i)$'s can be $C^*(G)$-embedded into the other if and only if their associated extensions are CK-equivalent. While this gives a characterization of $C^*(G)$-embeddability, it is somewhat unsatisfying due to the fact that CK-equivalence of the Busby invariants is not an easily checkable condition. We shall use the Wojciech map defined in [9] to translate this result into a statement about the Wojciech vectors of E_1 and E_2. We shall do this for essential 1-sink extensions in this section, and in the next section we shall consider non-essential 1-sink extensions.

We begin by recalling the definition of the Wojciech map. If $E \in Q$ is a projection, and X is an element of $\mathcal{Q}$ such that EXE is invertible in $E\mathcal{Q}E$, then we denote by $\operatorname{ind}_E(X)$ the Fredholm index of $E'X'E'$ in $\operatorname{im} E'$, where E' is any projection in $\mathcal{B}$ for which $\pi(E') = E$ and X is any element of $\mathcal{B}$ such that $\pi(X') = X$.

Let G be a row-finite graph with no sinks which satisfies Condition (L), and let $\{s_e, p_v\}$ be the generating Cuntz-Krieger G-family for $C^*(G)$. If $\tau : C^*(G) \to \mathcal{Q}$ is an essential extension of $C^*(G)$, define $E_e := \tau(s_e s_e^*)$ for all $e \in G^1$. If $t : C^*(G) \to \mathcal{Q}$ is another essential extension of $C^*(G)$ with the property that $t(s_e s_e^*) = E_e$ for all $e \in G^1$, then we define a vector $d_{\tau,t} \in \prod_{G^1} \mathbb{Z}$ by

$$d_{\tau,t} := -\operatorname{ind}_{E_e} \tau(s_e)t(s_e^*).$$

We then define the *Cuntz-Krieger* map $d : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(B_G - I)$ by

$$d(\tau) := [d_{\tau,t}],$$

where t is any degenerate essential extension of $C^*(G)$ with the property that $t(s_e s_e^*) = \tau(s_e s_e^*)$ for all $e \in G^1$.

Furthermore, we define the *source matrix* of G to be the $G^0 \times G^1$ matrix S_G defined by

$$S_G(v,e) = \begin{cases} 1 & \text{if } s(e) = v \\ 0 & \text{otherwise.} \end{cases}$$

It follows from [9, Lemma 6.2] that $S_G : \prod_{G^1} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ induces an isomorphism $\overline{S_G} : \operatorname{coker}(B_G - I) \to \operatorname{coker}(A_G - I)$, and we define the *Wojciech map* $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ by

$$\omega(\tau) = \overline{S_G} \circ d.$$

It was shown in [9, Theorem 6.16] that both the Cuntz-Krieger map and the Wojciech map are isomorphisms.

Theorem 4.1. *Let G be a row-finite graph which satisfies Condition* (L). *Also let E_1 and E_2 be essential* 1*-sink extensions of G. Then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if*

$$[\omega_{E_1}] = [\omega_{E_2}] \textit{ in } \operatorname{coker}(A_G - I),$$

where ω_{E_i} is the Wojciech vector of E_i and $A_G - I : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$.

Proof. Let τ_1 and τ_2 be the extensions associated to E_1 and E_2, respectively. It follows from Corollary 3.5 that τ_1 and τ_2 are in the same equivalence class in $\operatorname{Ext}(C^*(G))$ if and only if one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other. Since E_1 and E_2 are essential 1-sink extensions of G, the graph G contains no sinks. By [9, Theorem 6.16] the Wojciech map $\omega : \operatorname{Ext}(C^*(G)) \to \operatorname{coker}(A_G - I)$ is an isomorphism, and by [9, Proposition 6.11] the value of the Wojciech map on τ_i is the class $[\omega_{E_i}]$ in $\operatorname{coker}(A_G - I)$. □

5. $C^*(G)$-embeddability for non-essential 1-sink extensions

Recall from [1, §6] that a *maximal tail* in a graph E is a nonempty subset of E^0 which is cofinal under $\geq$, is backwards hereditary ($v \geq w$ and $w \in \gamma$ imply $v \in \gamma$), and contains no sinks (for each $w \in \gamma$, there exists $e \in E^1$ with $s(e) = w$ and $r(e) \in \gamma$). The set of all maximal tails of G is denoted by χ_G.

Also recall from [8, §3] that if (E, v_0) is a 1-sink extension of G, then the *closure* of v_0 is the set

$$\overline{v_0} := \bigcup \{\gamma : \gamma \text{ is a maximal tail in } G \text{ and } \gamma \geq v_0\}.$$

Notice first that the extension is essential if and only if $\overline{v_0} = G^0$. Also notice that the closure is a subset of G^0 rather than E^0. It has been defined in this way so that one may compare the closures in different extensions.

As in [8], we mention briefly how this notion of closure is related to the closure of sets in $\operatorname{Prim} C^*(E)$, as described in [1, §6]. For each sink v, let $\lambda_v := \{w \in E^0 : w \geq v\}$, and let

$$\Lambda_E := \chi_E \cup \{\lambda_v : v \text{ is a sink in } E\}.$$

The set Λ_E has a topology in which the closure of a subset S is $\{\lambda : \lambda \geq \bigcup_{\chi \in S} \chi\}$, and it is proved in [1, Corollary 6.5] that when E satisfies Condition (K) of [5], $\lambda \mapsto I(E^0 \setminus \lambda)$ is a homeomorphism of Λ_E onto $\operatorname{Prim} C^*(E)$. If (E, v_0) is a 1-sink extension of G, then the only loops in E are those in G, so E satisfies Condition (K) whenever G does. A subset of G^0 is a maximal tail in E if and only if it is a maximal tail in G, and because every sink in G is a sink in E, we deduce that $\Lambda_E = \Lambda_G \cup \{\lambda_{v_0}\}$.

We now return to the problem of proving an analogue of Theorem 4.1 for non-essential extensions.

Lemma 5.1. *Let G be a graph which satisfies Condition* (K)*, and let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. If $C^*(E_2)$ is $C^*(G)$-embeddable into $C^*(E_1)$, then $\overline{v_1} = \overline{v_2}$.*

Proof. Let $\phi : C^*(E_2) \to C^*(E_1)$ be a $C^*(G)$-embedding. Also let $p \in \mathcal{M}(C^*(E_1))$ be the projection which determines the full corner $\operatorname{im} \phi$. Now for $i \in \{1, 2\}$ we have that $\Lambda_{E_i} = \Lambda_G \cup \{\lambda_{v_i}\}$ is homeomorphic to $\operatorname{Prim} C^*(E_i)$ via the map $\lambda \mapsto I_{H_\lambda}$, where $H_\lambda := E_i^0 \backslash \lambda$ by [1, Corollary 6.5]. Furthermore, since ϕ embeds $C^*(E_2)$ onto a full corner of $C^*(E_1)$ it follows that $C^*(E_2)$ is Morita equivalent to $C^*(E_1)$ and the Rieffel correspondence is a homeomorphism between $\operatorname{Prim} C^*(E_2)$ and $\operatorname{Prim} C^*(E_1)$, which in this case is given by $I \mapsto \phi^{-1}(pIp)$ [7, Proposition 3.24]. Composing the homeomorphisms which we have described, we obtain a homeomorphism from $h : \Lambda_{E_2} \to \Lambda_{E_1}$, where $h(\lambda)$ is the unique element of Λ_{E_1} for which $\phi(I_{H_\lambda}) = pI_{H_{h(\lambda)}}p$.

We shall now show that this homeomorphism h is equal to the map h described in [8, Lemma 3.2]; that is h restricts to the identity on Λ_G. Let $\lambda \in \Lambda_G \subseteq \Lambda_{E_2}$. We begin by showing that $h(\lambda) \in \Lambda_G$. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger E_2-family, and let $\{t_f, q_w\}$ be the canonical Cuntz-Krieger E_1-family. Since $\lambda \in \Lambda_G$ it follows that $v_2 \notin \lambda$. Therefore $v_2 \in H_\lambda$ and $p_{v_2} \in I_{H_\lambda}$. Consequently, $\phi(p_{v_2}) \in \phi(I_{H_\lambda})$, and since $\phi(p_{v_2}) = q_{v_1}$ and $pI_{H_{h(\lambda)}}p = \phi(I_{H_\lambda})$ it follows that $q_{v_1} \in pI_{H_{h(\lambda)}}p \subseteq I_{H_{h(\lambda)}}$. Thus $v_1 \in H_{h(\lambda)}$ and $v_1 \notin h(\lambda)$. It follows that $h(\lambda) \neq \lambda_{v_1}$, and hence $h(\lambda) \in \Lambda_G$.

We shall now proceed to show that $h(\lambda) = \lambda$. Since $h(\lambda) \in \Lambda_G$ it follows that $H_{v_1} \subseteq H_{h(\lambda)}$. Thus $\ker \pi_{E_1} = I_{H_{v_1}} \subseteq I_{H_{h(\lambda)}}$. Now let $w \in \lambda$. If we let $\{u_g, r_x\}$ be the canonical Cuntz-Krieger G-family, then since $w \in G^0$ we have that $\pi_{E_2}(p_w) = r_w$. It then follows that

$$\pi_{E_1}(\phi(p_w) - q_w) = \pi_{E_1}(\phi(p_w)) - \pi_{E_1}(q_w) = \pi_{E_2}(p_w) - \pi_{E_1}(q_w) = r_w - r_w = 0.$$

Thus $\phi(p_w) - q_w \in \ker \pi_{E_1} \subseteq I_{H_{h(\lambda)}}$. We shall now show that $q_w \notin I_{H_{h(\lambda)}}$. To do this we suppose that $q_w \in I_{H_{h(\lambda)}}$ and arrive at a contradiction. If $q_w \in I_{H_{h(\lambda)}}$, then we would have that $\phi(p_w) \in I_{H_{h(\lambda)}}$. Thus $p\phi(p_w)p \in pI_{H_{h(\lambda)}}p$ and $p\phi(p_w)p \in \phi(I_{H_\lambda})$. Now $\phi(p_w) \in \phi(C^*(E_2))$ and $\phi(C^*(E_2)) = pC^*(E_1)p$. Hence $p\phi(p_w)p = \phi(p_w)$ and we have that $\phi(p_w) \in \phi(I_{H_\lambda})$. Since ϕ is injective this implies that $q_w \in I_{H_\lambda}$ and $w \in H_\lambda$ and $w \notin \lambda$ which is a contradiction. Therefore we must have that $q_w \notin I_{H_{h(\lambda)}}$ and $w \notin H_{h(\lambda)}$ and $w \in h(\lambda)$. Hence $\lambda \subseteq h(\lambda)$.

To show inclusion in the other direction let $w \in h(\lambda)$. Then $w \in H_{h(\lambda)}$ and $q_w \notin I_{H_{h(\lambda)}}$. As above, it is the case that $\phi(p_w) - q_w \in I_{H_{h(\lambda)}}$. Therefore, $\phi(p_w) \notin I_{H_{h(\lambda)}}$ and since $pI_{H_{h(\lambda)}}p \subseteq I_{H_{h(\lambda)}}$ it follows that $\phi(p_w) \notin pI_{H_{h(\lambda)}}p$ or $\phi(p_w) \notin \phi(I_{H_\lambda})$. Thus $p_w \notin I_{H_\lambda}$ and $w \notin H_\lambda$ and $w \in \lambda$. Hence $h(\lambda) \subseteq \lambda$.

Thus $\lambda = h(\lambda)$ for any $\lambda \in \Lambda_G$, and the map $h : \Lambda_{E_2} \to \Lambda_{E_1}$ restricts to the identity on Λ_G. Since this map is a bijection it must therefore take λ_{v_2} to λ_{v_1}. Therefore h is precisely the map described in [8, Lemma 3.2], and it follows from [8, Lemma 3.2] that $\overline{v_1} = \overline{v_2}$. □

Definition 5.2. Let G be a row-finite graph which satisfies Condition (K). If (E, v_0) is a 1-sink extension of G we define

$$H_E := G^0 \backslash \overline{v_0}.$$

We call H_E the *inessential part of* E.

Lemma 5.3. *Let G be a row-finite graph which satisfies Condition* (K) *and let (E, v_0) be a* 1*-sink extension of G. Then H_E is a saturated hereditary subset of G^0.*

Proof. Let $v \in H_E$ and $e \in G^1$ with $s(e) = v$. If $r(e) \notin H_E$, then $r(e) \in \overline{v_0}$ and hence $r(e) \in \gamma$ for some $\gamma \in \chi_G$ with the property that $\gamma \geq v_0$. Since maximal tails are backwards hereditary this implies that $v = s(e) \in \gamma$. Hence $v \in \overline{v_0}$ and $v \notin H_E$ which is a contradiction. Thus we must have $r(e) \in H_E$ and H_E is hereditary.

Suppose that $v \notin H_E$. Then $v \in \overline{v_0}$ and $v \in \gamma$ for some $\gamma \in \chi_G$ with the property that $\gamma \geq v_0$. Since maximal tails contain no sinks there exists an edge $e \in G^1$ with $s(e) = v$ and $r(e) \in \gamma$. Thus $r(e) \in \overline{v_0}$ and $r(e) \notin H_E$. Hence H_E is saturated. □

Remark 5.4. Recall that if A is a C^*-algebra, then there is a lattice structure on the set of ideals of A given by $I \wedge J := I \cap J$ and $I \vee J :=$ the smallest ideal containing $I \cup J$. Furthermore, if G is a graph then the set of saturated hereditary subsets of G^0 also has a lattice structure given by $H_1 \wedge H_2 := H_1 \cap H_2$ and $H_1 \vee H_2 :=$ the smallest saturated hereditary subset containing $H_1 \cup H_2$. If G is a row-finite graph satisfying Condition (K), then it is shown in [1, Theorem 4.1] that the map $H \mapsto I_H$, where I_H is the ideal in $C^*(G)$ generated by $\{p_v : v \in H\}$, is a lattice isomorphism from the lattice of saturated hereditary subsets of G^0 onto the lattice of ideals of $C^*(G)$. We shall make use of this isomorphism in the following lemmas in order to calculate $\ker \tau$ for an extension $\tau : C^*(G) \to \mathcal{Q}$.

Lemma 5.5. *Let $0 \to \mathcal{K} \overset{i_E}{\to} E \overset{\pi_E}{\to} A \to 0$ be a short exact sequence, and let σ and τ be the unique maps which make the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \xrightarrow{i_E} & E & \xrightarrow{\pi_E} & A & \longrightarrow & 0 \\
 & & \| & & \downarrow \sigma & & \downarrow \tau & & \\
0 & \longrightarrow & \mathcal{K} & \xrightarrow{i} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

commute. Then $\ker(\pi \circ \sigma) = i_E(\mathcal{K}) \vee \ker \sigma$ *and* $\ker \tau = \pi_E(i_E(\mathcal{K}) \vee \ker \sigma)$.

Proof. Since $\ker(\pi \circ \sigma)$ is an ideal which contains $i_E(K)$ and $\ker \sigma$, it follows that $i_E(\mathcal{K}) \vee \ker \sigma \subseteq \ker(\pi \circ \sigma)$.

Conversely, if $x \in \ker(\pi \circ \sigma)$ then $\pi(\sigma(x)) = 0$ and $\sigma(x) \in \mathcal{K} = \sigma(i_E(\mathcal{K}))$. Thus $\sigma(x) = \sigma(a)$ for some $a \in i_E(\mathcal{K})$. Hence $x - a \in \ker \sigma$ and $x \in i_E(\mathcal{K}) \vee \ker \sigma$. Thus $\ker(\pi \circ \sigma) = i_E(\mathcal{K}) \vee \ker \sigma$.

In addition, the commutativity of the above diagram implies that $\pi^{-1}(\ker \tau) = \ker(\pi \circ \tau)$. Since π_E is surjective it follows that $\ker \tau = \pi_E(\ker(\pi \circ \sigma))$ and from the previous paragraph $\ker \tau = \pi_E(i_E(\mathcal{K}) \vee \ker \sigma)$. □

For Lemmas 5.6 and 5.7 fix a row-finite graph G which satisfies Condition (K). Also let (E, v_0) be a fixed 1-sink extension of G which has the property that $E^*(v_0) := \{\alpha \in E^* : r(\alpha) = v_0\}$ contains infinitely many elements. Then $I_{v_0} \cong \mathcal{K}$, and we may choose an isomorphism $i_E : \mathcal{K} \to I_{v_0}$ and let σ and τ be the (unique) maps which make the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K} & \xrightarrow{i_E} & C^*(E) & \xrightarrow{\pi_E} & C^*(G) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} & & \\
0 & \longrightarrow & \mathcal{K} & \xrightarrow{i} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{Q} & \longrightarrow & 0
\end{array}$$

commute. In particular, note that τ is the extension associated to E.

Lemma 5.6. *If σ is as above, then $\ker \sigma = I_{H'}$ where $H' := \{v \in E^0 : v \not\geq v_0\}$.*

Proof. Since G satisfies Condition (K) and E is a 1-sink extension of G, it follows that E also satisfies Condition (K). Thus $\ker \sigma = I_H$ for some saturated hereditary subset $H \subseteq E^0$. Let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$. Now because $\sigma(q_{v_0})$ is a rank 1 projection, it follows that $q_{v_0} \notin \ker \sigma = I_H$ and thus $v_0 \notin H$. Since H is hereditary this implies that for any $w \in H$ we must have $w \not\geq v_0$. Hence $H \subseteq H'$.

Now let $F := E/H$; that is, F is the graph given by $F^0 := E^0 \backslash H$ and $F^1 := \{e \in E^1 : r(e) \notin H\}$. Then by [1, Theorem 4.1] we see that $C^*(F) \cong C^*(E)/I_H = C^*(E)/\ker \sigma$. Thus we may factor σ as $\overline{\sigma} \circ p$ to get the commutative diagram

$$\begin{array}{ccccc}
\mathcal{K} & \xrightarrow{i_E} & C^*(E) & \xrightarrow{p} & C^*(F) \\
\| & & \downarrow{\scriptstyle \sigma} & \swarrow{\scriptstyle \overline{\sigma}} & \\
\mathcal{K} & \xrightarrow{i} & \mathcal{B} & &
\end{array}$$

where p is the standard projection and $\overline{\sigma}$ is the monomorphism induced by σ. From the commutativity of this diagram it follows that $p \circ i_E : \mathcal{K} \to C^*(F)$ is injective. Let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger F-family in $C^*(F)$. Also let I_{v_0} be the ideal in $C^*(E)$ generated by q_{v_0}, and let J_{v_0} be the ideal in $C^*(F)$ generated by p_{v_0}. Using [4, Corollary 2.2] and the fact that any path in E with range v_0 is also a path in F, we have that

$$\begin{aligned}
p(i_E(\mathcal{K})) &= p(I_{v_0}) \\
&= p(\overline{\operatorname{span}}\{t_\alpha t_\beta^* : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) = v_0\}) \\
&= \overline{\operatorname{span}}\{p(t_\alpha t_\beta^*) : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta) = v_0\} \\
&= \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F^* \text{ and } r(\alpha) = r(\beta) = v_0\} \\
&= J_{v_0}.
\end{aligned}$$

From the commutativity of the above diagram it follows that $\overline{\sigma}$ is the (unique) map which makes the diagram

$$\begin{array}{ccc}
\mathcal{K} & \xrightarrow{p \circ i_E} & C^*(F) \\
& {\scriptstyle i}\searrow & \downarrow{\scriptstyle \overline{\sigma}} \\
& & \mathcal{B}
\end{array}$$

commute. Since $\overline{\sigma}$ is injective, $p(i_E(\mathcal{K})) = J_{v_0}$ is an essential ideal in $C^*(F)$ by [11, Proposition 2.2.14].

Now suppose that there exists $w \in F^0$ with $w \not\geq v_0$ in F. Then for every $\alpha \in F^*$ with $r(\alpha) = v_0$ we must have that $s(\alpha) \neq w$. Hence $p_w s_\alpha = 0$. Since $J_{v_0} = \overline{\text{span}}\{s_\alpha s_\beta : \alpha, \beta \in F^* \text{ and } r(\alpha) = r(\beta) = v_0\}$ it follows that $p_w J_{v_0} = 0$. Since $p_w \neq 0$ this would imply that J_{v_0} is not an essential ideal. Hence we must have that $w \not\geq v_0$ for all $w \in F^0$.

Furthermore, if $\alpha \in F^*$ is a path with $s(\alpha) = w$ and $r(\alpha) = v_0$, then $\alpha \in E^*$. So if $w \not\geq v_0$ in E, then we must have that $w \not\geq v_0$ in F. Consequently, if $w \in H'$, then $w \not\geq v_0$ in E, and we cannot have $w \in F^0$ because there is a path in F from every element of F^0 to v_0, and hence a path in E from every element of F^0 to v_0. Thus $w \notin F^0 := E^0 \backslash H$, and $w \in H$. Hence $H' \subseteq H$. □

Lemma 5.7. *Let G and (E, v_0) be as before. If H_E is the inessential part of E, $H' := \{v \in E^0 : v \not\geq v_0\}$, and $H_{v_0} := E^0 \backslash G^0$; then in E we have that*

$$H' \vee H_{v_0} = H_E \cup H_{v_0}.$$

Proof. We shall first show that $H_E \cup H_{v_0}$ is a saturated hereditary subset of E^0. To see that it is hereditary, let $v \in H_E \cup H_{v_0}$. If $e \in E^1$ with $s(e) = v$, then one of two things must occur. If $e \in G^1$, then $s(e) = v$ must be in G^0 and hence $v \in H_E$. Since we know from Lemma 5.3 that H_E is a saturated hereditary subset of G, it follows that $r(e) \in H_E \subseteq H_E \cup H_{v_0}$. On the other hand, if $e \notin G^1$, then $r(e) \notin G^0$, and hence $r(e) \in H_{v_0} \subseteq H_E \cup H_{v_0}$. Thus $H_E \cup H_{v_0}$ is hereditary.

To see that $H_E \cup H_{v_0}$ is saturated, let $v \notin H_E \cup H_{v_0}$. Then $v \in \overline{v_0}$ and $v \in \gamma$ for some $\gamma \in \chi_G$ with the property that $\gamma \geq v_0$. Since maximal tails contain no sinks, there exists $e \in G^1$ with $s(e) = v$ and $r(e) \in \gamma$. But then $r(e) \in \overline{v_0}$ and $r(e) \notin H_E$. Since $e \in G^1$ this implies that $r(e) \notin H_E \cup H_{v_0}$. Thus $H_E \cup H_{v_0}$ is saturated.

Now since $H' \subset H_E$ we see that $H_E \cup H_{v_0}$ is a saturated hereditary subset which contains $H' \cup H_{v_0}$. Thus $H' \vee H_{v_0} \subseteq H_E \cup H_{v_0}$.

Conversely, suppose that $v \in H_E \cup H_{v_0}$. If S is any saturated hereditary subset of E which contains $H' \cup H_{v_0}$, then for every vertex $w \notin S$ we know that w cannot be a sink, because if it were we would have $w \not\geq v_0$. Thus we may find an edge $e \in G^1$ with $s(e) = w$ and $r(e) \notin S$. Furthermore, since $H' \cup H_{v_0} \subseteq S$ we must also have that $r(e) \geq v_0$. Thus if $v \notin S$, we may produce an infinite path α in G with $s(\alpha) = v$ and $s(\alpha_i) \geq v_0$ for all $i \in \mathbb{N}$. If we let $\gamma := \{w \in G^0 : w \geq s(\alpha_i) \text{ for some } i \in \mathbb{N}\}$, then $\gamma \in \chi_G$ and $\gamma \geq v_0$. Hence $v \in \overline{v_0}$ and $v \notin H_E \cup H_{v_0}$ which is a contradiction. Thus we must have $v \in S$ for all saturated hereditary subsets S containing $H' \cup H_{v_0}$. Hence $v \in H' \vee H_{v_0}$ and $H_E \cup H_{v_0} \subseteq H' \vee H_{v_0}$. □

Lemma 5.8. *Let G be a row-finite graph which satisfies Condition* (K)*. Also let (E, v_0) be a* 1*-sink extension of G. If τ is the extension associated to E, then*

$$\ker \tau = I_{H_E}.$$

Proof. Consider the following two cases.

Case 1: The set $E^*(v_0)$ contains finitely many elements.

Then from the definition of the extension associated to E, we have that $\tau = 0$. However, if $E^*(v_0)$ has only finitely many elements then $\gamma \not\geq v_0$ for all $\gamma \in \chi_G$. Hence $H_E = G^0$ and $I_{H_E} = C^*(G)$.

Case 2: The set $E^*(v_0)$ contains infinitely many elements.

Then $I_{v_0} \cong \mathcal{K}$, and from Lemma 5.5 we have that $\ker \tau = \pi_E(I_{v_0} \vee \ker \sigma)$. Also Lemma 5.6 implies that $\ker \sigma = I_{H'}$. Since $I_{v_0} = I_{H_{v_0}}$, we see that from Lemma 5.7 that $I_{v_0} \vee \ker \sigma = I_{H_{v_0}} \vee I_{H'} = I_{H_{v_0} \vee H'} = I_{H_E \cup H_{v_0}}$.

Now if we let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger G-family in $C^*(G)$ and $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$, then

$$\ker \tau = \pi_E(I_{H_E \cup H_{v_0}}) = \pi_E(\langle \{q_v : v \in H_E \cup H_{v_0}\} \rangle) = \langle \{p_v : v \in H_E\} \rangle = I_{H_E}.$$

□

Lemma 5.9. *Let G be a row-finite graph which satisfies Condition* (K)*, and let (E, v_0) be a* 1*-sink extension of G. If $w \in H_E$, then*

$$\#\{\alpha \in E^* : s(\alpha) = w \text{ and } r(\alpha) = v_0\} < \infty.$$

Proof. Suppose that there were infinitely many such paths. Then since G is row-finite there must exist an edge $e_1 \in G^1$ with $s(e_1) = w$ and with the property that there exist infinitely many $\alpha \in E^*$ for which $s(\alpha) = r(e_1)$ and $r(\alpha) = v_0$. Likewise, there exists an edge $e_2 \in G^1$ with $s(e_2) = r(e_1)$ and with the property that there are infinitely many $\alpha \in E^*$ for which $s(\alpha) = r(e_2)$ and $r(\alpha) = v_0$. Continuing in this fashion we produce an infinite path $e_1 e_2 e_3 \ldots$ with the property that $r(e_i) \geq v_0$ for all $i \in \mathbb{N}$. If we let $\gamma := \{v \in G^0 : v \geq s(e_i) \text{ for some } i \in \mathbb{N}\}$, then $\gamma \in \chi_G$ and $\gamma \geq v_0$. Since $w \in \gamma$, it follows that $w \in \overline{v_0}$ and $w \notin H_E := E^0 \backslash \overline{v_0}$, which is a contradiction. □

Definition 5.10. Let G be a row-finite graph which satisfies Condition (K), and let (E, v_0) be a 1-sink extension of G. Then $n_E \in \prod_{H_E} \mathbb{Z}$ is the vector whose entries are given by

$$n_E(v) = \#\{\alpha \in E^* : s(\alpha) = v \text{ and } r(\alpha) = v_0\} \qquad \text{for } v \in H_E.$$

Note that the previous Lemma shows that $n_E(v) < \infty$ for all $v \in H_E$.

Lemma 5.11. *Let G be a row-finite graph which satisfies Condition* (K)*, and let (E, v_0) be a* 1*-sink extension of G. If $v \in H_E$ and $n_E(v) > 0$, then $A_G(v, v) = 0$; that is, there does not exist an edge $e \in G^1$ with $s(e) = r(e) = v$.*

Proof. If there was such an edge $e \in G^1$, then $\gamma = \{w \in G^0 : w \geq v\}$ would be a maximal tail and since $n_E(v) > 0$ it would follow that $\gamma \geq v_0$. Since $v \in \gamma$ this implies that $v \in \overline{v_0}$ which contradicts the fact that $v \in H_E := G^0 \backslash \overline{v_0}$. □

Lemma 5.12. *Let G be a row-finite graph which satisfies Condition* (K)*, and let (E, v_0) be a* 1*-sink extension of G. Also let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$. If $e \in G^1$ and $r(e) \in H_E$, then*

$$\operatorname{rank} \sigma(t_e) = n_E(r(e)).$$

Proof. If $n_E(r(e)) = 0$, then $r(e) \not\geq v_0$ and by Lemma 5.6 we have $\sigma(q_{r(e)}) = 0$. Since $\sigma(t_e)$ is a partial isometry $\operatorname{rank} \sigma(t_e) = \operatorname{rank} \sigma(t_e^* t_e) = \operatorname{rank} \sigma(q_{r(e)}) = 0$. Therefore we need only consider the case when $n_E(r(e)) > 0$.

Let B_E^1 denote the boundary edges of E. Also let $k_e := \max\{|\alpha| : \alpha \in E^*, s(\alpha) = r(e), \text{ and } r(\alpha) \in B_E^1\}$. By Lemma 5.9 we see that k_e is finite. We shall prove the claim by induction on k_e.

Base Case: $k_e = 0$. Let $e_1, e_2, \dots, e_n$ be the boundary edges of E which have source $r(e)$. Then it follows from [9, Lemma 6.9] that for $1 \leq i \leq n$

$$\operatorname{rank} \sigma(t_{e_i}) = \#Z(r(e_i), v_0)$$

where $Z(r(e_i), v_0)$ is the set of paths from $r(e_i)$ to v_0. Also if $f \in G^1$ is an edge with $s(f) = r(e)$, then because $n_E(r(e)) > 0$ Lemma 5.11 implies that $r(f) \neq r(e)$. Furthermore, since $k_e = 0$ we must have that $r(f) \ngeq v_0$. Therefore, just as before we must have $\operatorname{rank} \sigma(t_f) = 0$. Now since the projections $\{t_f t_f^* : f \in E^1 \text{ and } s(f) = r(e)\}$ are mutually orthogonal, we see that

$$\begin{aligned}
\operatorname{rank} \sigma(t_e) &= \operatorname{rank} \sigma(t_e^* t_e) \\
&= \operatorname{rank} \sum_{\substack{f \in E^1 \\ s(f) = r(e)}} \sigma(t_f t_f^*) \\
&= \operatorname{rank} \sigma(t_{e_1}) + \ldots + \operatorname{rank} \sigma(t_{e_n}) + \sum_{\substack{f \in G^1 \\ s(f) = r(e)}} \operatorname{rank} \sigma(t_f t_f^*) \\
&= \#Z(r(e_1), v_0) + \ldots + \#Z(r(e_n), v_0) \\
&= n_E(r(e)).
\end{aligned}$$

Inductive Step: Assume that the claim holds for all edges f with $k_f \leq m$. We shall now show that the claim holds for edges $e \in G^1$ with $k_e = m + 1$. Let $e_1, e_2, \dots, e_n$ be the exits of E with source $r(e)$. As above we have that $\operatorname{rank} \sigma(t_{e_i}) = \#Z(r(e_i), v_0)$ for all $1 \leq i \leq n$. Now if $f \in G^1$ is any edge with $s(f) = r(e)$, then Lemma 5.11 implies that $r(f) \neq r(e)$. Thus we must have that $k_f \leq k_e - 1$, and by the induction hypothesis $\operatorname{rank} \sigma(t_f) = n_E(r(f))$. Furthermore, since the projections $\{t_f t_f^* : f \in E^1 \text{ and } s(f) = r(e)\}$ are mutually orthogonal, we see that

$$\begin{aligned}
\operatorname{rank} \sigma(t_e) &= \operatorname{rank} \sigma(t_e^* t_e) \\
&= \operatorname{rank} \sum_{\substack{f \in E^1 \\ s(f) = r(e)}} \sigma(t_f t_f^*) \\
&= \operatorname{rank} \sigma(t_{e_1}) + \ldots + \operatorname{rank} \sigma(t_{e_n}) + \sum_{\substack{f \in G^1 \\ s(f) = r(e)}} \operatorname{rank} \sigma(t_f t_f^*) \\
&= \#Z(r(e_1), v_0) + \ldots + \#Z(r(e_n), v_0) + \sum_{\substack{f \in G^1 \\ s(f) = r(e)}} n_E(r(f)) \\
&= n_E(r(e)).
\end{aligned}$$

□

Let G be a row-finite graph which satisfies Condition (K) and let (E, v_0) be a 1-sink extension of G. If $H_E := G^0 \backslash \overline{v_0}$ is the inessential part of E, then since H_E is a saturated hereditary subset of G we may form the graph $F := G/H_E$ given by $F^0 := G^0 \backslash H_E$ and $F^1 := \{e \in G^1 : r(e) \notin H_E\}$. With respect to the decomposition $G^0 = \overline{v_0} \cup H_E$ the vertex matrix A_G of G will then have the form

$$A_G = \begin{pmatrix} A_F & X \\ 0 & C \end{pmatrix}$$

where A_F is the vertex matrix of the graph F.

Furthermore, if $\tau : C^*(G) \to \mathcal{Q}$ is the Busby invariant of the extension associated to E, then by Lemma 5.8 we know that $\ker \tau = I_{H_E}$. Hence $C^*(G)/\ker \tau \cong C^*(F)$ by [1, Theorem 4.1] and we may factor τ as $\overline{\tau} \circ p$

$$\begin{array}{ccc} C^*(G) & \xrightarrow{p} & C^*(F) \\ {\scriptstyle \tau}\downarrow & \swarrow{\scriptstyle \overline{\tau}} & \\ \mathcal{Q} & & \end{array}$$

where p is the standard projection and $\overline{\tau}$ is the monomorphism induced by τ. Note that since $\overline{\tau}$ is injective it is an essential extension of $C^*(F)$. Furthermore, with respect to the decomposition $G^0 = \overline{v_0} \cup H_E$ the Wojciech vector of E will have the form $\omega_E = \begin{pmatrix} \omega_E^1 \\ \omega_E^2 \end{pmatrix}$.

Lemma 5.13. *If* $d : \operatorname{Ext}(C^*(F)) \to \operatorname{coker}(B_F - I)$ *is the Cuntz-Krieger map, then*

$$d(\overline{\tau}) = [x]$$

where $[x]$ *denotes the class in* $\operatorname{coker}(B_F - I)$ *of the vector* $x \in \prod_{F^1} \mathbb{Z}$ *given by* $x(e) := \omega_E^1(r(e)) + (Xn_E)(r(e))$ *for all* $e \in F^1$.

Proof. Notice that because of the way H_E was defined, F will have no sinks. Also note that the diagram

$$\begin{array}{ccccccc} \mathcal{K} & \xrightarrow{i_E} & C^*(E) & \xrightarrow{\pi_E} & C^*(G) & \xrightarrow{p} & C^*(F) \\ \| & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} & \swarrow{\scriptstyle \overline{\tau}} & \\ \mathcal{K} & \xrightarrow{i} & \mathcal{B} & \xrightarrow{\pi} & \mathcal{Q} & & \end{array}$$

commutes. Let $\{t_e, q_v\}$ be the canonical Cuntz-Krieger E-family in $C^*(E)$. For each $e \in F^1$ let

$$H_e := \operatorname{im} \sigma(t_e t_e^*)$$

and for each $v \in F^0$ let

$$H_v := \bigoplus_{\substack{e \in F^1 \\ s(e) = v}} H_e.$$

Also for each $v \in F^0$ define P_v to be the projection onto H_v and for each $e \in F^1$ define S_e to be the partial isometry with initial space $H_{r(e)}$ and final space H_e. Then $\{S_e, P_v\}$ is a Cuntz-Krieger F-family in $\mathcal{B}$. If we let $\{s_e, p_v\}$ be the canonical Cuntz-Krieger F-family in $C^*(F)$, then by the universal property of $C^*(F)$ there exists a homomorphism $\tilde{t} : C^*(F) \to \mathcal{B}$ such that $\tilde{t}(s_e) = S_e$ and $\tilde{t}(p_v) = P_v$. Let $t := \pi \circ \tilde{t}$. Since G satisfies Condition (K), it follows that the quotient $F := G/H_E$ also satisfies Condition (K). Because $t(p_v) \neq 0$ for all $v \in F$ this implies that $\ker t = 0$ and t is an essential extension of $C^*(F)$.

Because $\sigma(t_e)$ is a lift of $\overline{\tau}(s_e)$ for all $e \in F^1$ we see that $\operatorname{ind}_{E_e} \overline{\tau}(s_e) t(s_e)$ equals the Fredholm index of $\sigma(t_e) S_e^*$ in H_e. Since S_e^* is a partial isometry with initial space H_e and final space $H_{r(e)} \subseteq \operatorname{im} \sigma(q_{r(e)})$, and since $\sigma(t_e)$ is a partial isometry with initial space $\operatorname{im} \sigma(q_{r(e)})$ and final space H_e, it follows that

$$\dim(\ker(\sigma(t_e) S_e^*)) = 0.$$

Also, $\sigma(t_e^*)$ is a partial isometry with initial space H_e and final space $\operatorname{im}\sigma(q_{r(e)})$, and S_e is a partial isometry with initial space $H_{r(e)}$ and final space H_e. Because $q_{r(e)} = \sum_{\{f\in E^1 : s(f)=r(e)\}} t_f t_f^*$ we see that

$$\operatorname{im}\sigma(q_{r(e)}) = H_{r(e)} \oplus \bigoplus_{\substack{f\in E^1\setminus F^1\\ s(f)=r(e)}} \operatorname{im}\sigma(t_f t_f^*)$$

Thus

$$\dim(\ker(S_e\sigma(t_e^*))) = \sum_{\substack{f\in E^1\setminus F^1\\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f t_f^*) = \sum_{\substack{f\in E^1\setminus F^1\\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f).$$

Now if f is any boundary edge of E, then $\operatorname{rank}\sigma(t_f) = \#Z((r(f), v_0)$ by [9, Lemma 6.9], where $Z((r(f), v_0)$ is the set of paths in F from $r(f)$ to v_0. Also if $f \in G^1$ is an edge with $r(f) \in H_E$, then $\operatorname{rank}\sigma(t_f) = n_E(r(f))$ by Lemma 5.12. Therefore,

$$\begin{aligned}\dim(\ker(S_e\sigma(t_e^*))) &= \sum_{\substack{f \text{ is a boundary edge}\\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f) + \sum_{\substack{r(f)\in H_E\\ s(f)=r(e)}} \operatorname{rank}\sigma(t_f)\\ &= \sum_{\substack{f \text{ is a boundary edge}\\ s(f)=r(e)}} \#Z(r(f), v_0) + \sum_{\substack{r(f)\in H_E\\ s(f)=r(e)}} n_E(r(f))\\ &= \omega_E(r(e)) + \sum_{w\in H_E} X(r(e), w) n_E(w).\end{aligned}$$

Thus

$$\begin{aligned}d_{\overline{\tau},t}(e) &= -\operatorname{ind}_{E_e} \overline{\tau}(s_e)t(s_e^*)\\ &= \omega_E(r(e)) + \sum_{w\in H_E} X(r(e), w) n_E(w)\\ &= \omega_E^1(r(e)) + (Xn_E)(r(e)).\end{aligned}$$

□

Lemma 5.14. *If* $\omega : \operatorname{Ext}(C^*(F)) \to \operatorname{coker}(A_F - I)$ *is the Wojciech map, then*

$$\omega(\overline{\tau}) = [\omega_E^1 + Xn_E]$$

where $[\omega_E^1 + Xn_E]$ *denotes the class of the vector* $\omega_E^1 + Xn_E$ *in* $\operatorname{coker}(A_F - I)$.

Proof. By definition $\omega := \overline{S_F} \circ d$. From Lemma 5.13 we see that $d(\overline{\tau}) = [x]$, where $x(e) = \omega_E^1(r(e)) + (Xn_E)(r(e))$ for all $e \in F^1$. Therefore, $\omega(\overline{\tau})$ is equal to the class $[y]$ in $\operatorname{coker}(A_F - I)$ where $y \in \prod_{F^0} \mathbb{Z}$ is the vector given by $y := S_F(x)$. Hence for all $v \in F^0$ we have that

$$y(v) = (S_F(x))(v) = \sum_{\substack{e\in F^1\\ s(e)=v}} x(e) = \sum_{\substack{e\in F^1\\ s(e)=v}} \omega_E^1(r(e)) + (Xn_E)(r(e))$$

and thus for all $v \in F^0$ we have that

$$\begin{aligned} y(v) - &\big(\omega_E^1(v) + (Xn_E)(v)\big) \\ &= \left(\sum_{\substack{e \in F^1 \\ s(e)=v}} \omega_E^1(r(e)) + (Xn_E)(r(e)) \right) - \big(\omega_E^1(v) + (Xn_E)(v)\big) \\ &= \left(\sum_{w \in F^0} A_F(v,w) \big(\omega_E^1(w) + (Xn_E)(w)\big) \right) - \big(\omega_E^1(v) + (Xn_E)(v)\big). \end{aligned}$$

Hence $y - (\omega_E^1 + Xn_E) = (A_F - I)(\omega_E^1 + Xn_E)$, and $\omega(\tau) = [y] = [\omega_E^1 + Xn_E]$ in $\operatorname{coker}(A_F - I)$. □

Remark 5.15. Let G be a row-finite graph which satisfies Condition (K) and let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. If $\overline{v_1} = \overline{v_2}$, then we may let $H := H_{E_1} = H_{E_2}$ and form the graph $F := G/H$ given by $F^0 := G^0 \backslash H$ and $F^1 := \{e \in G^1 : r(e) \notin H\}$). Then with respect to the decomposition $G^0 = (G^0 \backslash H) \cup H$, the vertex matrix of G has the form

$$A_G = \begin{pmatrix} A_F & X \\ 0 & C \end{pmatrix}$$

where A_F is the vertex matrix of F. Also with respect to this decomposition, the Wojciech vectors of E_1 and E_2 have the form $\omega_{E_1} = \begin{pmatrix} \omega_{E_1}^1 \\ \omega_{E_1}^2 \end{pmatrix}$ and $\omega_{E_2} = \begin{pmatrix} \omega_{E_2}^1 \\ \omega_{E_2}^2 \end{pmatrix}$.

For $i \in \{1, 2\}$, let $n_{E_i} \in \prod_H \mathbb{Z}$ denote the vector given by $n_{E_i}(v) = \#\{\alpha \in E_i^* : s(\alpha) = v \text{ and } r(\alpha) = v_i\}$.

Theorem 5.16. *Let G be a row-finite graph which satisfies Condition* (K)*, and let (E_1, v_1) and (E_2, v_2) be 1-sink extensions of G. Using the notation in Remark* 5.15*, we have that one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if*

1. $\overline{v_1} = \overline{v_2}$, *and*
2. $[\omega_{E_1}^1 + Xn_{E_1}] = [\omega_{E_2}^1 + Xn_{E_2}]$ *in* $\operatorname{coker}(A_F - I)$.

Proof. It follows from Lemma 5.1 that if one of the $C^*(E_i)$'s is $C^*(G)$-embeddable in the other, then $\overline{v_1} = \overline{v_2}$. Thus we may let $H := H_{E_1} = H_{E_2}$ and form the graph $F := G/H$ as discussed in Remark 5.15.

If we let τ_1 and τ_2 be the Busby invariants of the extensions associated to E_1 and E_2, then it follows from Lemma 5.8 that $\ker \tau_1 = \ker \tau_2 = I_H$. Thus for each $i \in \{1, 2\}$, we may factor τ_i as $\tau_i = \overline{\tau}_i \circ p$

$$\begin{array}{ccc} C^*(G) & \xrightarrow{\ p\ } & C^*(F) \\ {\scriptstyle \tau_i}\big\downarrow & \swarrow{\scriptstyle \overline{\tau}_i} & \\ \mathcal{Q} & & \end{array}$$

where p is the standard projection and $\overline{\tau}_i$ is the monomorphism induced by τ_i.

It then follows from Theorem 3.4 that one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other if and only if τ_1 and τ_2 are CK-equivalent. Since $\tau_i = \overline{\tau}_i \circ p$ we see that τ_1 and τ_2 are CK-equivalent if and only if $\overline{\tau}_1$ and $\overline{\tau}_2$ are CK-equivalent.

Furthermore, since $\overline{\tau}_1$ and $\overline{\tau}_2$ are essential extensions we see from Corollary 3.5 that $\overline{\tau}_1$ and $\overline{\tau}_2$ are CK-equivalent if and only if $\overline{\tau}_1$ and $\overline{\tau}_2$ are equal in $\operatorname{Ext}(C^*(F))$. If $\omega : \operatorname{Ext}(C^*(F)) \to \operatorname{coker}(A_F - I)$ is the Wojciech map, then this will occur if and only if $\omega(\overline{\tau}_1) = \omega(\overline{\tau}_2)$, and by Lemma 5.14 we see that this happens if and only if $[\omega^1_{E_1} + Xn_{E_1}] = [\omega^1_{E_2} + Xn_{E_2}]$ in $\operatorname{coker}(A_F - I)$. □

Remark 5.17. Note that when E_1 and E_2 are both essential we have $\overline{v_1} = \overline{v_2} = G^0$ and $H = \emptyset$. In this case $F = G$, X is empty, and $\omega^1_{E_i} = \omega_{E_i}$ for $i = 1, 2$. Thus the result for essential extensions in Theorem 4.1 is a special case of the above theorem.

In addition, we see that the above theorem gives a method of determining $C^*(G)$-embeddability from basic calculations with data that can be easily read of from the graphs. To begin, the condition that $\overline{v_1} = \overline{v_2}$ can be checked simply by looking at E_1 and E_2. In addition, the set H, the matrices A_F and X, and the vectors $\omega^1_{E_i}$ and n_{E_i} for $i = 1, 2$ can easily be read off from the graphs G, E_1, and E_2. Finally, determining whether $[\omega^1_{E_1} + Xn_{E_1}] = [\omega^1_{E_2} + Xn_{E_2}]$ in $\operatorname{coker}(A_F - I)$ amounts to ascertaining whether $(\omega^1_{E_1} - \omega^1_{E_2}) + (X(n_{E_1} - n_{E_2})) \in \operatorname{im}(A_F - I)$, a task which reduces to checking whether a system of linear equations has a solution.

We now mention an interesting consequence of the above theorem.

Definition 5.18. Let G be a row-finite graph which satisfies Condition (K), and let (E, v_0) be a 1-sink extension of G. We say that E is *totally inessential* if $\overline{v_0} = \emptyset$; that is, if $\{\gamma \in \chi_G : \gamma \geq v_0\} = \emptyset$.

Corollary 5.19. *Let G be a row-finite graph which satisfies Condition* (K). *If* (E_1, v_1) *and* (E_2, v_2) *are* 1*-sink extensions of G which are totally inessential, then one of the $C^*(E_i)$'s may be $C^*(G)$-embedded into the other.*

Proof. Using the notation established in Remark 5.15 and the proof of Theorem 5.16, we see that if E_1 and E_2 are totally inessential, then $H = G^0$. Hence $F = \emptyset$ and $\overline{\tau}_1 = \overline{\tau}_2 = 0$. Thus $\overline{\tau}_1$ and $\overline{\tau}_2$ are trivially CK-equivalent. Hence τ_1 and τ_2 are CK-equivalent and it follows from Theorem 3.4 that one of the $C^*(E_i)$'s can be $C^*(G)$-embedded into the other.

Alternatively, we see that if E_1 and E_2 are totally inessential, then $F = \emptyset$, and provided that we interpret the condition that $[\omega^1_{E_1} + Xn_{E_1}] = [\omega^1_{E_2} + Xn_{E_2}]$ in $\operatorname{coker}(A_F - I)$ as being vacuously satisfied, the previous theorem implies that one of the $C^*(E_i)$'s can be $C^*(G)$-embedded into the other. □

Remark 5.20. The case when E_1 and E_2 are both essential and the case when E_1 and E_2 are both inessential can be thought of as the degenerate cases of Theorem 5.16. The first occurs when $\overline{v_0} = G^0$ and $H = \emptyset$, and the second occurs when $\overline{v_0} = \emptyset$ and $H = G^0$.

Example 5.21. Let G be the graph

$$\cdots \xrightarrow{e_{-2}} v_{-1} \xrightarrow{e_{-1}} v_0 \xrightarrow{e_0} v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} v_3 \xrightarrow{e_3} \cdots .$$

Note that $C^*(G) \cong \mathcal{K}$. Since G has precisely one maximal tail $\gamma := G^0$, we see that if E is any 1-sink extension of G, then E will either be essential or totally inessential. Furthermore, one can check that $A_G - I : \prod_{G^0} \mathbb{Z} \to \prod_{G^0} \mathbb{Z}$ is surjective. Thus if E_1 and E_2 are two essential 1-sink extensions of G, we will always have that $[\omega_{E_1}] = [\omega_{E_2}]$ in $\operatorname{coker}(A_G - I)$. In light of Theorem 4.1 and Corollary 5.19 we see that if E_1 and E_2 are two 1-sink extensions of G, then one of the $C^*(E_i)$'s can be $C^*(G)$-embedded in to the other if and only if they are both essential or both totally inessential.

We end with an interesting observation. Note that the statement of the result in Theorem 5.16 involves the $\omega^1_{E_i}$ terms from the Wojciech vectors, but does not make use of the $\omega^2_{E_i}$ terms. If for each $i \in \{1,2\}$ we let $B^0_{E_i}$ denote the boundary vertices of E_i, then we see that the nonzero terms of $\omega^2_{E_i}$ are those entries which correspond to the elements of $B^0_{E_i} \cap H$. Furthermore, if $v \in B^0_{E_i} \cap H$ and there is a path from F^0 to v, then the value of $\omega^2_{E_i}(v)$ will affect the value of n_{E_i}. However, if there is no path from F^0 to v, then the value of $\omega^2_{E_i}(v)$ will be irrelevant to the value of n_{E_i}.

Therefore, whether one of the $C^*(E_i)$'s may be $C^*(G)$-embedded onto the other depends on two things: the number of boundary edges at vertices in F^0 (which determine the value of the $\omega^1_{E_i}$'s), and the number of boundary edges at vertices in H which can be reached by F^0 (which determine the value of the n_{E_i}'s). The boundary edges whose sources are elements of H that cannot be reached by F^0 will not matter.

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