New York Journal of Mathematics

New York J. Math. **9** (2003) 295–302.

On unit-regular ideals

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Abstract. In this paper we introduce the notion of unit-regular ideals for unital rings, which is a natural generalization of unit-regular rings. It is shown that every square matrix over unit-regular ideals admits a diagonal reduction. We also prove that a regular ideal of a unital ring is unit-regular if and only if pseudo-similarity via the ideal is similarity.

Let I be an ideal of a unital ring R. We say that I is regular in case for every $x \in I$ there exists $y \in I$ such that $x = xyx$. Following Goodearl [7], a unital ring R is unit-regular provided that for every $x \in R$ there exists $u \in U(R)$ such that $x = xux$. Unit-regular rings play an important role in the structure theory of regular rings. In this paper we introduce the notion of unit-regular ideals for unital rings, which is a natural generalization of unit-regular rings. We say that an ideal I of a unital ring R is unit-regular in case for every $x \in I$, there exists $u \in U(R)$ such that $x = xux$.

Let D be a division ring, V a countably generated infinite dimensional vector space over D. Let $I = \{x \in \mathrm{End}_D V \mid \dim_D(xV) < \infty\}$. Clearly, I is an ideal of $\mathrm{End}_D V$. Given any $x \in I$, we have right D-module split exact sequences $0 \to \mathrm{Ker}\, x \to V \to xV \to 0$ and $0 \to xV \to V \to V/xV \to 0$. Then $V \cong xV \oplus \mathrm{Ker}\, x \cong V/xV \oplus xV$; hence, $\dim_D(\mathrm{Ker}\, x) = \dim_D(V/xV) = \infty$ because $\dim_D(xV) < \infty$. By [5, Corollary], $x \in \mathrm{End}_D(V)$ is unit-regular. Therefore I is a unit-regular ideal of $\mathrm{End}_D(V)$, while $\mathrm{End}_D(V)$ is not a unit-regular ring by [5, Corollary]. This shows that the notion of unit-regular ideal is a nontrivial generalization of unit-regularity for regular rings.

An $m \times n$ matrix A over a unital ring R is called to admit a diagonal reduction if there exist $P \in \mathrm{GL}_m(R)$ and $Q \in \mathrm{GL}_n(R)$ such that PAQ is a diagonal matrix. It is well-known that every square matrix over unit-regular rings admits a diagonal reduction by invertible matrices (cf. [9, Theorem 3]). But Henriksen's method can not be extend to unit-regular ideals. P. Ara et al. have extended this result to separative exchange rings (cf. [1, Theorem 2.4]). Let D be a division ring, V an infinite dimensional vector space over D. Set $R = \mathrm{End}_D(V)$. Then R is one-sided unit-regular, so it is a separative regular ring. Given any $A \in M_n(R)$, by [1, Theorem 2.5], A admits a diagonal reduction. So we can find $U, V \in \mathrm{GL}_n(R)$ such that $UAV = \mathrm{diag}(r_1, \ldots, r_n)$. Assume now that all $r_i \in R$ are idempotents.

Received January 21, 2003.

Mathematics Subject Classification. 16E50, 16U99.

Key words and phrases. unit-regular ideal, diagonal reduction, pseudo-similarity.

ISSN 1076-9803/03

Let $E = \text{diag}(r_1, \dots, r_n)$. Then $A = U^{-1}EV^{-1}$, whence $AVUA = A$. That is, $M_n(R)$ is unit-regular. This shows that R is unit-regular, a contradiction. This infers that there exists some square matrix over R which doesn't admit a diagonal reduction with idempotent entries. In other words, we may not reduce some square matrices over unit-regular ideals to diagonal matrices with idempotent entries by Ara's technique. In this paper, we will prove that every square matrix over unit-regular ideals admits a diagonal reduction with idempotent entries. We also prove that a regular ideal of a unital ring is unit-regular if and only if pseudo-similarity via the ideal is similarity, which give a nontrivial generalization of [8, Theorem].

Throughout, all rings are associative with identity and all modules are right modules. $U(R)$ denotes the set of all units of R and $\text{GL}_n(R)$ denotes the general linear group of R. The notation $\text{FP}(I)$ stands for the set of all finitely generated projective right R-modules P such that $P = PI$.

Lemma 1. *Let I be a regular ideal of a unital ring R. Then the following are equivalent:*

(1) *I is unit-regular.*
(2) *If $aR + bR = R$ with $a \in I$, then there exists $y \in R$ such that $a + by \in U(R)$.*
(3) *If $Ra + Rb = R$ with $a \in I$, then there exists $z \in R$ such that $a + zb \in U(R)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $aR + bR = R$ with $a \in I$. Then $ax + bz = 1$ for some $x, z \in R$. Since $a \in I$, we have $u \in U(R)$ such that $a = aua$. Set $au = e$. Then $e \in R$ is an idempotent. Furthermore, we have $eu^{-1}x + bz = 1$; hence $e + bz(1-e) = 1 - eu^{-1}x(1-e) \in U(R)$. Let $y = z(1-e)u^{-1}$. We see that $a + by = \big(1 - eu^{-1}x(1-e)\big)u^{-1} \in U(R)$, as asserted.

(2) $\Rightarrow$ (1) Given any $x \in I$, we have $y \in R$ such that $x = xyx$. From $xy + (1-xy) = 1$, we have $z \in R$ such that $x + (1-xy)z \in U(R)$. By [6, Lemma 3.1], we have $s \in R$ such that $y + s(1-xy) = u \in U(R)$. Therefore $x = xyx = x\big(y + s(1-xy)\big)x = xux$, as required.

(1) $\Leftrightarrow$ (3) By symmetry, we get the result. $\square$

For any $\alpha, \beta, a, b \in R$, we set

$$[\alpha, \beta] = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad B_{12}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad B_{21}(b) = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

In [4, Proposition 2], the first author and F. Li showed that ideal-stable range conditions are invariant under matrix extensions. Now we give an analogue for unit-regular ideals.

Theorem 2. *Let I be a unit-regular ideal of a unital ring R. Then $M_n(I)$ is a unit-regular ideal of $M_n(R)$.*

Proof. Let I be a unit-regular ideal of a unital ring R. By [2, Lemma 2], $M_n(I)$ is a regular ideal of $M_n(R)$. Suppose that $AX + B = I_n$ with $A = (a_{ij}) \in M_n(I)$ and $X = (x_{ij}), B = (b_{ij}) \in M_n(R)$. Then $\begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} = \begin{pmatrix} X & XA - I_n \\ I_n & A \end{pmatrix}^{-1} \in \text{GL}_2\big(M_n(R)\big)$. Since $a_{11}R + \cdots + a_{1n}R + b_{11}R + \cdots + b_{1n}R = R$ with $a_{11} \in I$, by Lemma 1, we can find $y_2, \dots, y_n, z_1, \dots, z_n \in R$ such that

$$a_{11} + a_{12}y_2 + \cdots + a_{1n}y_n + b_{11}z_1 + \cdots + b_{1n}z_n = u_1 \in U(R).$$

Thus

$$\begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0_{1\times(2n-1)}} \\ y_2 & \\ \vdots & \\ y_n & \mathbf{I_{2n-1}} \\ z_1 & \\ \vdots & \\ z_n & \end{pmatrix} = \begin{pmatrix} u_1 & a_{12} & \dots & a_{1n} & b_{11} & \dots & b_{1n} \\ a'_{21} & a_{22} & \dots & a_{2n} & b_{21} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ ** & a_{n2} & \dots & a_{nn} & b_{n1} & \dots & b_{nn} \\ ** & 0 & \dots & 0 & x_{11} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ ** & 0 & \dots & -1 & x_{n1} & \dots & x_{nn} \end{pmatrix};$$

hence,

$$\begin{pmatrix} * & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} \begin{pmatrix} * & 0 \\ ** & I_n \end{pmatrix} = \begin{pmatrix} u_1 & a'_{12} & \dots & a'_{1n} & b'_{11} & \dots & b'_{1n} \\ 0 & a'_{22} & \dots & a'_{2n} & b'_{21} & \dots & b'_{2n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ 0 & a'_{n2} & \dots & a'_{nn} & b'_{n1} & \dots & b'_{nn} \\ ** & 0 & \dots & 0 & x_{11} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ ** & 0 & \dots & -1 & x_{n1} & \dots & x_{nn} \end{pmatrix},$$

where $a'_{22} = a_{22} - a'_{21}u_1^{-1}a_{12} \in I$. Analogously, we claim that

$$\begin{aligned} &[*,*] \begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} B_{21}(*)[*,*] \\ &= \begin{pmatrix} u_1 & a_{12}^{(n)} & \dots & a_{1n}^{(n)} & b_{11}^{(n)} & \dots & b_{1n}^{(n)} \\ 0 & u_2 & \dots & a_{2n}^{(n)} & b_{21}^{(n)} & \dots & b_{2n}^{(n)} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_n & b_{n1}^{(n)} & \dots & b_{nn}^{(n)} \\ ** & * & \dots & * & x_{11} & \dots & x_{1n} \\ \vdots & \vdots & \ddots & & \vdots & \ddots & \vdots \\ ** & * & \dots & * & x_{n1} & \dots & x_{nn} \end{pmatrix} = [*,*]B_{21}(*)B_{12}(*), \end{aligned}$$

where $u_1, u_2, \dots, u_n \in U(R)$. So $\begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} = [*,*]B_{21}(*)B_{12}(*)B_{21}(*)$; and then $\begin{pmatrix} A & B \\ -I_n & X \end{pmatrix} B_{21}(Y) = [*,*]B_{21}(*)B_{12}(*)$ for a $Y \in M_n(R)$. This implies that $A + BY \in \mathrm{GL}_n(R)$. It follows by Lemma 1 that $M_n(I)$ is unit-regular. □

Corollary 3. *Let I be a unit-regular ideal of a unital ring R. Then every square matrix over I is a product of an idempotent matrix and an invertible matrix.*

Proof. Let $A \in M_n(I)$. In view of Theorem 2, there exists $U \in \mathrm{GL}_n(R)$ such that $A = AUA$. Set $E = AU$. Then $E = E^2$ and $A = EU^{-1}$, as asserted. □

Lemma 4. *Let I be a unit-regular ideal of a unital ring R. Suppose that $a, b \in I$. Then the following hold:*

(1) *If $aR = bR$, then there exists $u \in U(R)$ such that $a = bu$.*
(2) *If $Ra = Rb$, then there exists $u \in U(R)$ such that $a = ub$.*

Proof. Suppose that $aR = bR$ with $a, b \in I$. Then we have $x, y \in R$ such that $ax = b$ and $a = by$. Assume that $a = aa'a$. Replacing $a'ax$ with x, we may assume that $x \in I$. Likewise, we may assume that $y \in I$. Obviously, $b = ax = byx$. From $yx + (1 - yx) = 1$, we have $z \in R$ such that $y + (1 - yx)z = u \in U(R)$ by Lemma 1. As a result, we get $a = by = b\big(y + (1 - yx)z\big) = bu$. The second statement is proved by the symmetry. □

Theorem 5. *Let I be a regular ideal of a unital ring R. Then the following are equivalent:*

(1) *I is unit-regular.*
(2) *If $aR \cong bR$ with $a, b \in I$, then there exist $u, v \in U(R)$ such that $a = ubv$.*

Proof. (1) ⇒ (2) Suppose that $\psi : aR \cong bR$ with $a, b \in I$. Clearly, $\psi(a)R = bR$. Because of the regularity of I, we have an idempotent $e \in R$ such that $bR = eR$. Hence $\psi(a)R = eR$. This infers that $\psi(a) \in R$ is regular as well. So we can find $c \in R$ such that $\psi(a) = \psi(a)c\psi(a) = \psi\big(ac\psi(a)\big)$. It follows that $a = ac\psi(a) \in R\psi(a)$, whence $Ra \subseteq R\psi(a)$. Inasmuch as $a \in I$ is regular, we have $a = ada$ for some $d \in R$. This implies that $\psi(a) = \psi(ada) = \psi(a)da \in Ra$; hence, $R\psi(a) \subseteq Ra$. So we see that $Ra = R\psi(a)$. Clearly, $\psi(a) \in I$. In view of Lemma 4, there exist $u, v \in U(R)$ such that $\psi(a) = ua$ and $b = \psi(a)v$. Therefore we conclude that $b = uav$.

(2) ⇒ (1) Given any $x \in I$, there exists $y \in R$ such that $x = xyx$. Set $e = xy$. Then we have $xR = eR$ with $x, e \in I$, so there are $u, v \in U(R)$ such that $x = uev$. We easily check that $x = x(v^{-1}u^{-1})x$, as required. □

Lemma 6. *Let I be a regular ideal of a unital ring R. If $P \in \mathrm{FP}(I)$, then there exist idempotents $e_1, \dots, e_n \in I$ such that $P \cong e_1R \oplus \cdots \oplus e_nR$.*

Proof. Suppose that $P \in \mathrm{FP}(I)$. Then we have a right R-module Q such that $P \oplus Q \cong nR$ for some $n \in \mathbb{N}$. Let $e : nR \to P$ be the projection onto P. Then $P \cong e(nR)$, whence $\mathrm{End}_R(P) \cong eM_n(R)e$. Inasmuch as $P = PI$, we have $e(nR) = e(nR)I \subseteq nI$. Set $e = (\alpha_1, \dots, \alpha_n) \in M_n(R)$. We have $e(1, 0, \dots, 0)^T \in nI$. Hence $\alpha_1 \in nI$. Likewise, we have $\alpha_2, \dots, \alpha_n \in nI$. Therefore $e \in M_n(I)$. Since I is a regular ideal of R, by [2, Lemma 2], $M_n(I)$ is also regular. One directly checks that $\mathrm{End}_R(P)$ is a regular ring, hence an exchange ring. Thus P has the finitely exchange property. Set $M = P \oplus Q$. Then we have $M = P \oplus Q = \bigoplus_{i=1}^{n} R_i$ with all $R_i \cong R$. By the finite exchange property of P, we have $Q_i (1 \le i \le n)$ such that $M = P \oplus \left(\bigoplus_{i=1}^{n} Q_i\right)$, where all Q_i are direct summands of R_i respectively. Assume that $Q_i \oplus P_i = R_i$ for all i. Then $P \oplus \left(\bigoplus_{i=1}^{n} Q_i\right) = \left(\bigoplus_{i=1}^{n} P_i\right) \oplus \left(\bigoplus_{i=1}^{n} Q_i\right)$. Hence $P \cong P_1 \oplus \cdots \oplus P_n$, where P_i is isomorphic to a direct summand of R as a right R-module for all i. So we have idempotents e_i such that $P_i \cong e_i R$. Clearly, $e_i R$ is a finitely generated projective right R-module. It follows from $P = PI$ that $P \bigotimes_R (R/I) = 0$; hence, $P_i \bigotimes_R (R/I) = 0$. That is, $(e_i R) \bigotimes_R (R/I) = 0$, so $e_i R = e_i RI \subseteq I$. Furthermore, we have $e_i \in I$ for all i. Therefore $P \cong e_1 R \oplus \cdots \oplus e_n R$ with all $e_i \in I$. □

Theorem 7. *Let I be a unit-regular ideal of a unital ring R. Then for any $A \in M_n(I)$, there exist invertible matrices $P, Q \in M_n(R)$ such that*

$$PAQ = \mathrm{diag}(e_1, \dots, e_n)$$

for some idempotents $e_1, \dots, e_n \in I$.

Proof. Since I is a unit-regular ideal of R, $M_n(I)$ is a unit-regular ideal of $M_n(R)$ by Theorem 2. Given any $A \in M_n(I)$, we have $B \in \mathrm{GL}_n(R)$ such that $A = ABA$. Set $E = AB$. Then $E = E^2 \in M_n(I)$ and $AM_n(R) = EM_n(R)$. Clearly, $ER^n \in \mathrm{FP}(I)$. From Lemma 6, we can find idempotents $e_1, \dots, e_n \in I$ such that $ER^n \cong e_1 R \oplus \cdots \oplus e_n R \cong \mathrm{diag}(e_1, \dots, e_n) R^n$ as right R-modules. Hence $ER^{n\times 1} \cong \mathrm{diag}(e_1, \dots, e_n) R^{n\times 1}$, where $R^{n\times 1}$ consisting of all n-column vectors over R is a right R-module and a left $M_n(R)$-module. Let $R^{1\times n} = \{(x_1, \dots, x_n) \mid x_i \in R\}$. Then $R^{1\times n}$ is a left R-module and a right $M_n(R)$-module. One checks that $(ER^{n\times 1}) \bigotimes_R R^{1\times n} \cong \left(\mathrm{diag}(e_1, \dots, e_n) R^{n\times 1}\right) \bigotimes_R R^{1\times n}$. In addition, $R^{n\times 1} \bigotimes R^{1\times n} \cong M_n(R)$ as right $M_n(R)$-modules. Thus,

$$AM_n(R) = EM_n(R) \cong \mathrm{diag}(e_1, \dots, e_n) M_n(R).$$

According to Theorem 5, we have invertible matrices $P, Q \in M_n(R)$ such that $PAQ = \mathrm{diag}(e_1, \dots, e_n)$, as asserted. □

Let I be an ideal of a unital ring R. We say that I has stable range one provided that $aR + bR = R$ with $a \in 1 + I, b \in R$ implies that $a + by \in U(R)$ for a $y \in R$. It is well known that I having stable range one depends only on the ring structure of I and not on the ambient ring R. Let I and J be regular ideals of a unital ring R. If I has stable range one, then $I + J$ is unit-regular if and only if so is J.

Corollary 8. *Let R be a regular, right self-injective ring, and let $A \in M_n(R)$. If $AM_n(R)$ is directly finite, then there exist invertible matrices $P, Q \in M_n(R)$ such that $PAQ = \mathrm{diag}(e_1, \dots, e_n)$ for some idempotents $e_1, \dots, e_n \in R$.*

Proof. Let $I = \{x \in R \mid xR$ is a directly finite right R–module $\}$. In view of [7, Corollary 9.21], I is an ideal of R. Given any idempotent $e \in I$, we know from [7, Corollary 9.3 and Theorem 9.17] that eRe is unit-regular; hence, I has stable range one. This infers that I is unit-regular. Inasmuch as $AM_n(R)$ is directly finite, we deduce that $A \in M_n(I)$. Therefore we complete the proof by Theorem 7. □

Let R be a unital ring, and let $A \in M_n(R)$. If $M_n(R)AM_n(R)$ is a unit-regular ideal of $M_n(R)$, we claim that there exist invertible matrices $P, Q \in M_n(R)$ such that $PAQ = \text{diag}(e_1, \dots, e_n)$ for some idempotents $e_1, \dots, e_n \in R$. Since $M_n(R)AM_n(R)$ is a unit-regular ideal of $M_n(R)$, we have an ideal J of R such that $M_n(J) = M_n(R)AM_n(R)$. Hence $A \in M_n(J)$. Clearly, J is a regular ideal of R; hence, A is a regular matrix over J. Analogously to Theorem 7, the result follows. We say that a is pseudo-similar to b via I provided that there exist $x, y, z \in I$ such that $xay = b, zbx = a$ and $xyx = xzx = x$. We denote it by $a \widetilde{\sim} b$ via I. Note that if $eR \cong fR$ for idempotents $e, f \in I$ then $e \widetilde{\sim} f$ via I, where I is an ideal of R.

Lemma 9. *Let I be an ideal of a unital ring R. Then the following are equivalent:*

(1) *$a \widetilde{\sim} b$ via I.*
(2) *There exist some $x, y \in I$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$.*

Proof. (2)⇒(1) is trivial.

(1)⇒(2) As $a \widetilde{\sim} b$ via I, there are $x, y, z \in I$ such that $b = xay, zbx = a$ and $x = xyx = xzx$. Then $xa(yxy) = xzbx(yxy) = xzb(xyx)y = xzbxy = xay = b$. Analogously, $(zxz)bx = a$. By replacing y with yxy and z with zxz, we can assume $y = yxy$ and $z = zxz$. Furthermore, we directly check that $xazxy = xzbxzxy = xzbxy = xay = b, zxybx = zxyxayx = zxayx = zbx = a, zxy = zxyxzxy$ and $x = xzxyx$, thus yielding the result. □

Theorem 10. *Let I be a regular ideal of a unital ring R. Then the following are equivalent:*

(1) *I is unit-regular.*
(2) *Whenever $a \widetilde{\sim} b$ via I, there exists $u \in U(R)$ such that $a = ubu^{-1}$.*

Proof. (1) ⇒ (2) Suppose that $a \widetilde{\sim} b$ via I. According to Lemma 9, there exist $x, y \in I$ such that $a = xby, b = yax, x = xyx$ and $y = yxy$. Since I is unit-regular, we have $v \in U(R)$ such that $y = yvy$. Let $u = (1 - xy - vy)v(1 - yx - yv)$. It is easy to verify that $(1 - xy - vy)^2 = 1 = (1 - yx - yv)^2$; hence, $u \in U(R)$. In addition, we have $au = a(1 - xy - vy)v(1 - yx - yv) = -av(1 - yx - yv) = -av + ax + av = ax$. Likewise, we have $xb = ub$. Clearly, $ax = xbyx = xyaxyx = xyax = xb$. Therefore $au = ub$, as required.

(2) ⇒ (1) Given any $x \in I$, there exists $y \in R$ such that $x = xyx$. Clearly, $\psi : (xy)R \cong (yx)R$ with idempotents $xy, yx \in I$. Hence $xy \widetilde{\sim} yx$ via I, so we have $u \in U(R)$ such that $1 - xy = u(1 - yx)u^{-1}$. Set $a = (1 - xy)u(1 - yx)$ and $b = (1 - yx)u^{-1}(1 - xy)$. Then $1 - xy = ab$ and $1 - yx = ba$. Thus $\phi : (1 - xy)R \cong (1 - yx)R$. Define $u \in \text{End}_R(R)$ so that u restricts to $\psi : xR = (xy)R \cong (yx)R$ and u restricts to $\phi : (1 - xy)R \cong (1 - yx)R$. It is easy to verify that $x = xux$, as asserted. □

Let $A, B \in M_n(R)$. If $M_n(R)AM_n(R) + M_n(R)BM_n(R)$ is a unit-regular ideal of $M_n(R)$, by Theorem 10 we deduce that $A \widetilde{\sim} B$ if and only if there exists some $U \in \text{GL}_n(R)$ such that $A = UBU^{-1}$.

Corollary 11. *Let I be a regular ideal of a unital ring R. Then the following are equivalent:*

(1) *I is unit-regular.*
(2) *Whenever $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1, A_2 \in \mathrm{FP}(I)$ and $A_1 \cong A_2$, we have $B_1 \cong B_2$.*
(3) *Whenever $aR \cong bR$ with $a, b \in I$, we have $R/aR \cong R/bR$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1, A_2 \in \mathrm{FP}(I)$ and $A_1 \cong A_2$. Then we have idempotents $e, f \in I$ such that $eR \cong fR$, and whence $e \overset{\approx}{\sim} f$ via I. By Theorem 10, there exists $u \in U(R)$ such that $e = ufu^{-1}$. Hence $1-e = u(1-f)u^{-1}$. Set $a = (1-e)u(1-f)$ and $b = (1-f)u^{-1}(1-e)$. Then $1-e = ab$ and $1-f = ba$. Therefore we get $B_1 \cong (1-e)R \cong (1-f)R \cong B_2$.

(2) $\Rightarrow$ (3) Suppose that $aR \cong bR$ with $a, b \in I$. Since I is regular, we have idempotents $e, f \in I$ such that $aR = eR$ and $bR = fR$. Hence $R/aR \cong (1-e)R \cong (1-f)R \cong R/bR$.

(3) $\Rightarrow$ (1) Given idempotents $e, f \in I$ such that $eR \cong fR$, then $(1-e)R \cong R/eR \cong R/fR \cong (1-f)R$. Analogously to Theorem 10, we complete the proof. ☐

Recall that an ideal I of a unital ring is of bounded index if there is a positive integer n such that $x^n = 0$ for any nilpotent $x \in I$.

Corollary 12. *Every regular ideal of bounded index is unit-regular.*

Proof. Let R be a unital ring with a regular ideal I of bounded index. Suppose that $R = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1, A_2 \in \mathrm{FP}(I)$ and $A_1 \cong A_2$. Then we have an idempotent $e \in I$ such that $A_1 \cong eR \cong A_2$. Since $\mathrm{End}_R(eR) \cong eRe$ is a regular ring of bounded index, by [7, Corollary 7.11], it is unit-regular. Therefore we get $B_1 \cong B_2$ from [7, Proposition 4.13]. It follows from Corollary 11 that I is a unit-regular ideal of R. ☐

Acknowledgements. The authors are grateful to the referee for his/her suggestions which led to the new version of Theorem 2. These helped us to improve the presentation considerably. This work was supported by the Natural Science Foundation of Zhejiang Province.

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This paper is available via http://nyjm.albany.edu:8000/j/2003/9-15.html.