New York Journal of Mathematics

New York J. Math. **10** (2004) 175–194.

Rank-one group actions with simple mixing $\mathbb{Z}$-subactions

Blair Madore

Abstract. Let G be a countable Abelian group with $\mathbb{Z}^d$ as a subgroup so that $G/\mathbb{Z}^d$ is a locally finite group. (An Abelian group is locally finite if every element has finite order.) We can construct a rank one action of G so that the $\mathbb{Z}$-subaction is 2-simple, 2-mixing and only commutes with the other transformations in the action of G.

Applications of this construction include a transformation with square roots of all orders but no infinite square root chain, a transformation with countably many nonisomorphic square roots, a new proof of an old theorem of Baxter and Akcoglu on roots of transformations, and a simple map with no prime factors. The last example, originally constructed by del Junco, was the inspiration for this work.

Contents

Received March 18, 2004.

Mathematics Subject Classification. 28D05, 37A25.

Key words and phrases. Ergodic, Rank-one, Rank one, Group Action, Simple, Mixing, Subaction, Measure Preserving Group Action.

The author was supported by NSERC, the University of Toronto, and the State University of New York — College at Potsdam.

ISSN 1076-9803/04

1. Introduction and results

Ornstein's rank-one mixing argument [Orn67] has been refined over the years, and its ideas are used often in the literature to construct interesting examples. Notably, del Junco [Jun98] constructed a measure preserving action of $\mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$. Using an Ornstein style argument along with a joinings argument, he showed the transformation T corresponding to the $\mathbb{Z}$-subaction was weak mixing, simple and commuted only with the other transformations in the action. del Junco was then able to argue that T was a simple map with no prime factors. This paper provides an extension of del Junco's construction to a certain class of abelian groups.

Recall that if every element of an abelian group G has finite order then G is a locally finite group.

Theorem 1 (Main Result)**.** *Let G be a countable abelian group with subgroup $\mathbb{Z}^d$ ($d \geq 1$), such that $G/\mathbb{Z}^d$ is a locally finite group. Then there exists a rank-one action of G so that the transformation T corresponding to $(1, 0, 0, \ldots, 0)$ in $\mathbb{Z}^d$ is mixing, simple, and only commutes with the other transformations in the group, i.e., $C(T) = G$.*

We note that, in particular, the theorem is valid for groups $G = \mathbb{Z}^d \oplus H$ where H is a locally finite group, possibly finite, or even the trivial group. The theorem is proved in Section 3.

The main theorem allows the construction of simple transformations T with centralizer $C(T)$ prescribed in advance. Since T is simple, this gives us some control over the roots and factors of T. We'll detail some examples, new and previously known, that can be constructed in this way. First, and most significantly we answer a question posed by King in [Kin00].

1.1. A transformation with square roots of all orders but no infinite square root chain. Let S and T be measure preserving transformations on the same space. If $S^2 = T$ we say S is a square root of T and write $T \to S$. If $S^{2^n} = T$ (T has a 2^nth root, S) we can find a square root chain for T of length n:

$$T \to S^{2^{n-1}} \to S^{2^{n-2}} \to \cdots \to S.$$

J. King has been investigating the problem of embedding the generic transformation into actions of the rationals [Kin00]. A significant obstruction to embedding the generic transformation in an action of the dyadic rationals is the necessity of existence of an infinite square root chain,

$$T \to T^{\frac{1}{2}} \to T^{\frac{1}{4}} \to T^{\frac{1}{8}} \to \cdots .$$

In [Kin00] King asked: "Is there a transformation with square roots of all orders but no infinite square root chain?" We answer this question affirmatively using an appropriate group.

Definition 1 (Carry group of r)**.** Let $r = \{r_i\}_{i=1}^{\infty}$ be any countable sequence of natural numbers. Define G, the carry group of r, to consist of the elements of the Cartesian product

$$\mathbb{Z} \times \mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \cdots,$$

where all but finitely many entries are zero, together with an operation $+_G$ defined by

$$(a_{\mathbb{Z}}, a_1, a_2, \dots) +_G (b_{\mathbb{Z}}, b_1, b_2, \dots) := \left(a_{\mathbb{Z}} + b_{\mathbb{Z}} + \sum_{i=1}^{\infty} \left\lfloor \frac{a_i + b_i}{r_i} \right\rfloor, a_1 +_{r_1} b_1, a_2 +_{r_2} b_2, \dots \right).$$

That is, addition in the $\mathbb{Z}$ coordinate and addition modulo r_i in the $\mathbb{Z}_{r_i}$ coordinate with a possible carry of 1 into the $\mathbb{Z}$ coordinate.

Let G be the carry group of $r = (2, 4, 8, 16, \dots, 2^n, \dots)$, a group discussed by King in [Kin00]. Note that $G/\mathbb{Z}$ is a locally finite group. Apply Theorem 1 to this group to obtain a transformation T, with G as its centralizer. T is the transformation in the action corresponding to the element $(1, 0, 0, \dots)$. The transformation T has square roots of all orders, since the group element $(0, 0, \dots, 0, 1, 0, \dots)$, where the 1 is in the $\mathbb{Z}_{2^n}$ coordinate, is a 2^nth root of $(1, 0, 0, \dots)$. There are, however, no infinite square root chains in G. An infinite square root chain in G must have some nonzero values in a coordinate other than $\mathbb{Z}$, say the $\mathbb{Z}_{2^n}$ coordinate. The values in that coordinate would form a nontrivial infinite square root chain in $\mathbb{Z}_{2^n}$ which does not exist.(One easy way to prove this is by induction). Thus, T answers King's question.

Analogously we can produce transformations with qth roots of all orders but no infinite qth root chains, where q is any positive integer except 1.

1.2. A simple map with no prime factors. Applying Theorem 1 to $G = \mathbb{Z} \oplus \bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ we obtain a simple map with no prime factors as originally constructed by del Junco. See [Jun98] for details.

1.3. A transformation with $C(T) = \mathbb{Q}$ or $\mathbb{Z}^d$. Each application of our theorem produces a transformation with countable but (usually) nontrivial centralizer. When $G = \mathbb{Z}$ we have constructed Ornstein's rank-one mixing transformation that only commutes with its powers [Orn67]. When $G = \mathbb{Z}^d$ we obtain a transformation with centralizer $\mathbb{Z}^d$, and when $G = \mathbb{Q}$, the transformation has the rationals as centralizer. The latter is possible because $\mathbb{Q}/\mathbb{Z}$ is a locally finite group.

1.4. Transformations with a fixed set of roots.

Question 1. Can you construct a transformation with only a specified set of roots?

Akcoglu and Baxter [AB69] published an interesting theorem on this topic in 1969. We offer an alternate proof, using Theorem 1.

Theorem 2 (Akcoglu and Baxter)**.** *Let P be any set of primes. There exists a weak mixing transformation T, so that T has a pth root if and only if all prime factors of p are in P.*

Proof. Consider the set H of rational numbers (in lowest form) whose denominator is 1 or has all its prime factors in P. H is a subgroup of $\mathbb{Q}$ that includes $\mathbb{Z}$. Apply Theorem 1 to obtain a transformation T which has a pth root if and only if $1/p$ is in H, that is, if all prime divisors of p are in P. □

The transformation T we constructed in this proof is actually mixing. Also notice our T will have an infinite number of roots. Define $T \xrightarrow{p} S$ to mean S is a pth root of T. Then T also has infinite root chains. For example, if $p \in P$, the following is a pth root chain:

$$T \xrightarrow{p} T^{1/p} \xrightarrow{p} T^{1/p^2} \xrightarrow{p} T^{1/p^3} \dots$$

One could also use Theorem 1 and a carry group to construct a transformation that satisfies Theorem 2 yet has no infinite root chains.

1.5. A transformation with countably many nonisomorphic square roots. Let G be the root group of $r = \{2, 2, 2, 2, 2, \dots\}$. Apply the main theorem to G to obtain a simple mixing transformation T corresponding to $(1, 0, \dots)$ in G. T has countably many square roots, each corresponding to an element $(0, \dots, 1, 0, \dots)$ in G. Let S and Q be two distinct square roots of T. Assume ϕ is an isomorphism between S and Q. Then $\phi S = Q\phi$ which implies that $\phi S^2 = Q^2\phi$, and thus $\phi T = T\phi$. Since ϕ commutes with T it is in $C(T)$, which is isomorphic to the commutative group G. S and Q are also in this commutative group $C(T)$ so $\phi S = Q\phi$ implies $S = Q$. This contradicts the assumption that S and Q were distinct. Thus any two square roots of T are nonisomorphic.

1.6. Future directions. Here are a few of the many natural questions arising from this work.

Question 2. Is the full rank-one group action, constructed in Theorem 1, a mixing action?

In separate work [Mad] we constructed rank-one mixing $\mathbb{Z}^d$ actions so that all times are simple. It is possible to extend our main theorem to ensure the $\mathbb{Z}^d$-subaction is mixing with all times simple.

Question 3. Given a group G and subgroup H can you construct an action G so that the H-subaction is simple and only commutes with the entire group action?

Our theorem gives an answer for the case $H = \mathbb{Z}$ and a certain class of countable abelian groups G. A more general construction, especially when G is nonabelian, could produce very interesting examples. del Junco outlines several in [Jun98].

Question 4. Which groups have rank-one mixing actions?

Our techniques could lead us to a class of countable abelian groups which have such actions. Can this be extended to any nonabelian groups? To solvable groups?

Acknowledgements. This paper is based on research which is part of the author's University of Toronto Ph.D. Thesis, written under Andres del Junco. Thanks are also due to Mustafa Akcoglu and Jonathan King.

2. Definitions and preliminaries

We use := to indicate definition (or assignment). All our transformations are on $\mathbb{X} := ([0,1), \beta, \mu)$ which is isomorphic to the unit interval with the Borel σ-algebra and Lebesgue measure. Use $|A|$ to denote absolute value when A is a number and cardinality when A is a finite set. Denote the set of integers $\{0, 1, 2, \dots, n\}$ by $[0, n]$. Similarly $[0, n)$ denotes $\{0, 1, 2, \dots, n-1\}$. Any other notations are introduced as needed. Unnumbered definitions are in italics.

2.1. Joinings and simplicity. Let T be a measure preserving transformation on the measure space $\mathbb{X}$. If λ is a $T \times T$ invariant measure on (X^2, β^2) such that for all $A \in \beta$,

$$\lambda(X \times A) = \lambda(A \times X) = \mu(A),$$

then λ is a *self-joining* of T. Every transformation has self-joinings such as product measure, $\mu \times \mu$, or μ lifted onto the diagonal, denoted Δ. More precisely, $\Delta := \mu \circ J^{-1}$ where $J \colon X \to X \times X$ and $J(x) = (x, x)$. If S is any measure preserving transformation that commutes with T then $(I \times S)\Delta$ is also a self-joining. These are called *graph joinings*, because $(I \times S)\Delta$ is μ lifted onto the graph of S.

We say T is *simple* if the only ergodic self-joinings of T are product measure or graph joinings.

These definitions in the literature are properly named 2-fold self-joinings, and 2-simple. As we will not discuss higher-order joinings we have opted for the simpler names.

2.2. Rank-one group actions. We follow the definitions and notations for group actions as in [Jun98], [PR91] and [YJ00]. All our groups are amenable, countable and have the discrete topology. In our main theorem G is also abelian, though definitions in this subsection do not assume the group is commutative. Let $\pounds$ be a homomorphism from the group G into the set of invertible measure preserving transformations on X.

$$\pounds \colon G \to M(X)$$
$$g \to \pounds^g.$$

We call $\pounds$ a measure preserving action of the group G. The range of $\pounds$ is denoted times($\pounds$). An individual transformation $\pounds^g$ is called a *time* of the action. Let $C(\pounds)$ denote the centralizer, that is, all invertible measure-preserving transformations that commute with all of the times of the action.

Originally, towers of rank-one transformations were indexed by intervals in $\mathbb{Z}$. Ferenczi, in [Fer85], credits Thouvenot with the idea that towers could be indexed by other sets. Ferenczi used a special Følner sequence in $\mathbb{Z}$ to define a $\mathbb{Z}$ action and called it funny rank-one. Generalizing this idea to other groups we have rank-one group actions.

Definition 2 (Rank-one group action). $\pounds$ is called rank-one (with respect to a Følner sequence $\{A_n\}$ in G) if:

1. For all $n \in \mathbb{N}$ there is a partition P_n of X, $P_n = \{E^n_g \mid g \in A_n\} \cup \{X \backslash X_n\}$, where $X_n = \cup_{g \in A_n} E^n_g$.
2. $\mu(X \backslash X_n) \xrightarrow{n} 0$.
3. $P_n \xrightarrow{n} \epsilon$.
4. $\pounds_g E^n_h = E^n_{gh}$ when $h \in A_n \cap g^{-1} A_n$.

$P_n \xrightarrow{n} \epsilon$ means that given measurable B in X, for all n there is a set B_n made up of a union of sets in P_n and $\mu(B \triangle B_n) \xrightarrow{n} 0$. For each $g \in A_n$, E_g^n is an interval of length a_n where $a_n|A_n| \xrightarrow{n} 1$.

2.3. Cutting and stacking rank-one actions. We will describe a cutting and stacking style of construction that produces a rank-one action $\pounds$ of G, given a special kind of Følner sequence in G called almost-tiling. It is not known if all rank-one actions can be obtained in this manner.

Definition 3 (Almost-tiling Følner sequences)**.** An almost-tiling Følner sequence consists of two sequences of sets $(\{A_n\}, \{C_n\})$ in G such that:

1. A_n is a Følner sequence.
2. $A_n C_n = \cup_{c \in C_n} A_n c$ is a disjoint union.
3. $A_n C_n \subset A_{n+1}$.
4. $\prod_{n=1}^{\infty} \frac{|A_{n+1}|}{|A_n C_n|} < \infty$.

This is an inductive construction. At stage n (also called time-n) of the construction, the n-tower consists of $|A_n|$ levels, each a distinct interval in the real line that is left-closed and right-open. Levels are indexed by A_n and all have equal length. An individual level is denoted E_a^n, the ath level ($a \in A_n$) of the n-tower.

The action $\pounds$ is defined partially at time-n by the n-tower. For $g \in A_n {A_n}^{-1}$, $\pounds^g$ translates E_a onto E_b if $g = ba^{-1}$. Thus $\pounds^g$ is defined on the levels of the n-tower indexed by $(g^{-1}A_n) \cap A_n$. Since A_n is a Følner sequence, as $n \to \infty$, $\pounds^g$ becomes defined a.e. To build the $(n+1)$-tower from the n-tower, each level E_a^n is divided into $|C_n|$ equal intervals, each closed on the left and open on the right. We assume C_n is ordered $\{c_1, c_2, c_3, \dots\}$ and label our new intervals from left to right as

$$E_{ac_1}^{n+1}, E_{ac_2}^{n+1}, E_{ac_3}^{n+1} \dots .$$

The $(n+1)$-tower is formed by these intervals, together with some additional intervals, E_b^{n+1} for all b in $A_{n+1} \backslash A_n C_n$. The additional intervals were called spacers in the classical rank-one construction. Because of property three of the almost-tiling Følner sequence, we only introduce a finite amount of measure in the whole construction.

It's not hard to see that the partially defined action on the $(n+1)$-tower is consistent with the partially defined action on the n-tower. By normalizing we can assume the action is defined on $[0,1)$ with probability Lebesgue measure μ.

2.4. Some notations for rank-one actions. The entire space on which the action takes place is called X. The subset of X that is the n-tower is called X_n. The n-tower is composed of left-closed, right-open intervals labeled E_a^n, where a is in A_n. Let $E_s^n := X/\ X_n$. The s stands for spacer.

The partition P_n divides X into levels E_i^n of the n-tower and the complement E_s^n. View P_n as a function from X to $A_n^* := A_n \cup \{s\}$, defined by $P_n(x) = j$ if $x \in E_j^n$ for $j \in A_n^*$. Define $P_k^n : A_n^* \to A_k^*$ by $P_k^n(u) = v$ if $E_u^n \subset E_v^k$. This corresponds to the natural map, based on inclusion, from the n-tower to the k-tower ($n > k$). The value $P_k^n(u)$ is the P_k name of a level u in the n-tower. Thus, $P_k^n \circ P_n = P_k$.

Definition 4 (P-name of x)**.** Let be P be a partition on X, so that $P : X \to A$, and $x \in X$. Then the function $G \to A$ defined by $g \to P(\pounds^g x)$ is the P-name of x.

Definition 5 (n-block)**.** Given a P_k-name of x (for $k < n$) and a fixed $c \in G$, if for all $u \in A_n$ we find that $P_k(\pounds^{uc}x) = P_k^n(u)$ then $\{P_k(\pounds^{uc}x) \mid u \in A_n\}$ is an n-block indexed by $A_n c$.

The set C_n determines how n-blocks are situated in the $(n+1)$-tower.

We often compare names of different points x and y. If the function from G to $A \times A$ is given by $g \to (P(\pounds^g x), P(\pounds^g y))$, then this function is the $P \times P$ name of (x, y). We refer to the symbols in the first coordinate as *upper* and the symbols in the second coordinate as *lower*.

Definition 6 (Overlap of n-blocks)**.** If an n-block occurs at $A_n c$ in the P_k name of x and an n-block occurs at $A_n c'$ in the P_k name of y then $A_n c \cap A_n c'$ indexes an n-block overlap in the $P_k \times P_k$ name of (x, y). The overlap is given by

$$\{(P_k(\pounds^h x), P_k(\pounds^h y)) \mid h \in A_n c \cap A_n c'\}.$$

2.5. Comparing measures on finite sets. If f is a map from a probability space (A, μ, α) to a finite set B, then $\mathrm{Dist}_{a \in A} f(a)$ denotes the probability measure on B given by $\mathrm{Dist}_{a \in A} f(a) := \alpha \circ f^{-1}$.

For $C \subset A$, the *conditional measure* is defined by $\alpha_C(U) := \alpha(U \cap C)/\alpha(C)$. If C is a measurable subset of A define $\mathrm{Dist}_{a \in C} f(a) := \alpha_C \circ f^{-1}$. We also denote this by $\mathrm{Dist}(f(a) \mid a \in C)$. For convenience and readability we also use

$$\mathrm{Dist}_{a \in C}(f(a) | g(a) = k)$$

in place of

$$\mathrm{Dist}_{\{a \in C | g(a) = k\}} f(a).$$

For finite A the normalized counting measure on A is denoted $\mathrm{Unif}\, A$.

Example. Define a measure λ on A_n by $\mathrm{Dist}_{x \in X_n}(P_n(x))$. So if E_p^n is a level of the n-tower then

$$\lambda(p) = \mu_{X_n}(P_n^{-1}(p)) = \frac{\mu(X_n \cap E_p^n)}{\mu(X_n)} = \frac{1}{m}.$$

Thus $\mathrm{Dist}_{x \in X_n}(P_n(x)) = \mathrm{Unif}\, A_n$.

Example. Let $X_i : \Omega \to A$ for $i = 1, 2, 3, \ldots$ be a countable family of independent random variables each with uniform distribution over a finite alphabet A. Then for fixed $w \in \Omega$, $\mu := \mathrm{Dist}(X_i(w) \mid i \in [1, m])$ is a measure on A. Precisely,

$$\mu(a) = \frac{|\{i \in [1, m] \mid X_i(w) = a\}|}{m}.$$

To compare two probability measures, p, q, on a finite set A, use the norm

$$\|p - q\| := \sum_{a \in A} |p(a) - q(a)|.$$

The following five lemmas about finite measures, which we use repeatedly, are stated without proof.

Lemma 3. *If p and q are probability measures on A and $\rho : A \to B$ then*

$$\|p \circ \rho^{-1} - q \circ \rho^{-1}\| \leq \|p - q\|.$$

Lemma 4. *If $C \subset A$ and $p(C) \geq 1 - \epsilon$, then $\|p - p_C\| \leq 2\epsilon$.*

Let $\Pi : A \times B \to A$ be the projection. If p is a probability measure on $A \times B$, then the *marginal measure* $\overline{p} := \Pi(p)$ is the projection of p onto A. So $\overline{p}(a) = \sum_b p(a, b)$. For $a \in A$ define the *fibre measure* p_a as p conditioned on $\Pi^{-1}(a)$. So

$$p_a(b) = \frac{p(a,b)}{\overline{p}(a)}.$$

Lemma 5. *If p and q are probability measures on $A \times B$ and $\overline{p} = \overline{q}$, then*

$$\|p - q\| = \sum_a \overline{p}(a) \, \|p_a - q_a\|.$$

A more useful version of Lemma 5 when $\overline{p}$ is not exactly equal to $\overline{q}$ is:

Lemma 6. *If p and q are probability measures on $A \times B$ then*

$$\|p - q\| \leq \sum_a \overline{p}(a) \|p_a - q_a\| \; + \|\overline{p} - \overline{q}\|.$$

Note that Lemma 5 is a special case of this lemma. When the marginals on A are close and the fibre measures over A are close, Lemma 6 implies the measures will be close.

Conversely, the distance between the measures gives a bound on the distance between the fibre measures.

Lemma 7. *If p and q are probability measures on $A \times B$ then for a fixed $a \in A$,*

$$\|p_a - q_a\| \leq \frac{2}{\overline{p}(a)} \|p - q\|.$$

2.6. Special application of the ergodic theorem. The following is a standard consequence of the mean and pointwise Ergodic Theorems:

Lemma 8. *Suppose (X, β, T, μ) is ergodic, P is a measurable finite partition of X, $\epsilon > 0$, and $\alpha > 0$. Then for μ-almost all x in X, there exists N (depending on x) so that if $E = l + \cup_{m=0}^{M} [mn, mn + L]$ where:*

1. $n \geq N$,
2. $M \in \mathbb{N}$,
3. $\alpha n \leq L \leq n$, *and*
4. $-n \leq l \leq n$,

then

$$\left\| \mathrm{Dist}_{i \in E} \, P(T^i x) - \mathrm{Dist} \; P \right\| < \epsilon.$$

Lemma 8 shows the Ergodic Theorem is valid on sets other than initial segments of the integers. The conclusion of this lemma holds even if E is

$$l + \bigcup_{m=0}^{M} [mn + \delta_m, mn + \delta_m + L],$$

where δ_m satisfies $0 \leq \delta_m \leq (1 - \alpha)n$. The case $\delta_m = 0$, however, when the intervals of length L are regularly spaced, is sufficient for our needs.

3. Main proof

Theorem 9 (Main result)**.** *Let G be a countable abelian group with $\mathbb{Z}^d$ $(d \geq 1)$ as a subgroup so that $G/\mathbb{Z}^d$ is a locally finite group. There exists a rank-one action of G so that the transformation T corresponding to $(1, 0, 0, \ldots, 0)$ in $\mathbb{Z}^d$ is mixing, simple, and only commutes with the other times of the action, that is, $C(T) = G$.*

We prove this theorem in three parts: the construction of the group action in Subsection 3.1, the proof that T is mixing in Subsection 3.2, and the proof that T is simple and $C(T) = G$ in Subsection 3.3.

3.1. Constructing a rank-one action of G.

Let $H := G/\mathbb{Z}^d$. H is a locally finite group. The proof when H is finite or even trivial is an easy modification of the proof given, the case when H is infinite. As H is countable there exists a sequence of finite groups H_n, so that

$$H_1 \subset H_2 \subset H_3 \ldots$$

and

$$H = \bigcup_{n=1}^{\infty} H_n.$$

Our group G has a special structure that we will identify. Consider the cosets of $\mathbb{Z}^d$ in G and for each coset select a unique element in the coset. Use $\psi(g)$ to denote the unique element in the coset $g + \mathbb{Z}^d$. So ψ maps from G into G but is not a homomorphism since $\psi(g_1 + g_2)$ need not be equal to $\psi(g_1) + \psi(g_2)$.

Since $G/\mathbb{Z}^d$ is isomorphic to H, the projection $\Pi_H : G \to H$ is a homomorphism that has $\mathbb{Z}^d$ as its kernel. Define $\phi : H \to G$ so that $\phi \circ \Pi_H = \psi$. Then we can define an operation, also denoted $+$, on $\mathbb{Z}^d \times H$ as follows:

$$(\mathbf{u}, h_1) + (\mathbf{v}, h_2) := (\mathbf{u} + \mathbf{v} + \phi(h_1) + \phi(h_2) - \phi(h_1 + h_2), h_1 + h_2).$$

We think of this as addition in each coordinate with a carry from the H coordinate into the $\mathbb{Z}^d$ coordinate. The *carry*, $\phi(h_1) + \phi(h_2) - \phi(h_1 + h_2)$, is an element of $\mathbb{Z}^d$. Why? If g_1 and g_2 are two elements of G so that $\Pi_H(g_1) = h_1$ and $\Pi_H(g_2) = h_2$ then

$$\begin{aligned} \phi(h_1) + \phi(h_2) - \phi(h_1 + h_2) &= \phi(\Pi_H(g_1)) + \phi(\Pi_H(g_2)) - \phi(\Pi_H(g_1) + \Pi_H(g_2)) \\ &= \psi(g_1) + \psi(g_2) - \phi(\Pi_H(g_1 + g_2)) \\ &= \psi(g_1) + \psi(g_2) - \psi(g_1 + g_2). \end{aligned}$$

Since $\psi(g_1) \in g_1 + \mathbb{Z}^d$ and $\psi(g_2) \in g_2 + \mathbb{Z}^d$, then $\psi(g_1) + \psi(g_2) \in (g_1 + g_2) + \mathbb{Z}^d$. Yet, $\psi(g_1 + g_2)$ is also in $(g_1 + g_2) + \mathbb{Z}^d$, so $\psi(g_1) + \psi(g_2) - \psi(g_1 + g_2)$ is an element of $\mathbb{Z}^d$.

Claim 10. *With this operation, $\mathbb{Z}^d \times H$ is isomorphic to G.*

Proof. Define an isomorphism $\Phi : G \to \mathbb{Z}^d \times H$ as $\Phi(g) := (g - \psi(g), \Pi_H(g))$. Then

$$\begin{aligned}\Phi(g_1) + \Phi(g_2) &= (g_1 - \psi(g_1), \Pi_H(g_1)) + (g_2 - \psi(g_2), \Pi_H(g_2)) \\ &= (g_1 - \psi(g_1) + g_2 - \psi(g_2) + \phi(\Pi_H(g_1)) + \phi(\Pi_H(g_2)) \\ &\quad - \phi(\Pi_H(g_1) + \Pi_H(g_2)), \Pi_H(g_1) + \Pi_H(g_2)) \\ &= (g_1 - \psi(g_1) + g_2 - \psi(g_2) + \psi(g_1) + \psi(g_2) \\ &\quad - \phi(\Pi_H(g_1 + g_2)), \Pi_H(g_1 + g_2)) \\ &= (g_1 + g_2 - \psi(g_1 + g_2), \Pi_H(g_1 + g_2)) \\ &= \Phi(g_1 + g_2).\end{aligned}$$

Thus Φ is a homomorphism. Each $g \in G$ can be uniquely written as $(g - \psi(g)) + \psi(g)$. Since $g - \psi(g) \in \mathbb{Z}^d$ and $\psi(g)$ corresponds to $\Pi_H(g) \in H$, it is clear that Φ is injective and surjective. □

For the reminder of our proof we consider our group G to be presented as $\mathbb{Z}^d \times H$ with the special operation $+$, where we add in each coordinate and carry from the H coordinate into the $\mathbb{Z}^d$ coordinate. An element in G is uniquely represented by $(\mathbf{v}, g)$, where $\mathbf{v}$ is in $\mathbb{Z}^d$ and g is in H; sometimes denoted $\mathbf{v} + g$.

For notational convenience:

1. $\mathbf{u}$ denotes $(u_1, u_2, \dots, u_d)$ a vector in $\mathbb{Z}^d$.
2. $\|\mathbf{u}\|$ is $\max(|u_1|, |u_2|, \dots, |u_d|)$.
3. $a\,\mathbf{v}$ is the usual scalar multiplication.
4. $\mathbf{e}_i$ represents the ith standard basis vector in $\mathbb{Z}^d$.
5. For an integer m, $\mathbf{m}$ denotes $(m, m, m, \dots, m)$.

Thus, $m\mathbf{e}_1$ denotes the $\mathbb{Z}^d$ vector $(m, 0, 0, \dots, 0)$. For $\mathbf{u} = (u_1, u_2, \dots u_d)$ and $\mathbf{v} = (v_1, v_2, \dots v_d)$, if $u_i \le v_i$ for all i, define $[\mathbf{u}, \mathbf{v}]$ to be the set

$$[u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_d, v_d].$$

Define the projection from G onto $\mathbb{Z}^d$ by $\Pi_{\mathbb{Z}^d}(\mathbf{v}, g) := \mathbf{v}$. Although Π_H is a homomorphism, $\Pi_{\mathbb{Z}^d}$ is not.

H_{n+1} is composed of $k_n := |H_{n+1}/H_n|$ cosets of H_n. Choose elements of H_{n+1}, one from each coset, to form $\Gamma_n = \{g_1, g_2, \dots, g_{k_n}\}$. Then

$$H_{n+1} = (g_1 + H_n) \cup (g_2 + H_n) \cup \cdots \cup (g_{k_n} + H_n).$$

To construct a rank-one action of G we specify an almost-tiling Følner sequence $(\{A_n\}, \{C_n\})$. Our Følner sequence will be defined by $A_n := [0, h_n)^d \times H_n$. More precisely define:

1. Spacer length $s_n := n h_{n-1}$.
2. Window length $w_n := h_n + s_n + 2\theta_{n+1}$.
3. New block length $h_{n+1} := N_n w_n$.
4. The maximum norm of the carry:

$$\theta_n := \sup_{g, g' \in H_n} \|\Pi_{\mathbb{Z}^d}((0, g) + (0, g'))\|.$$

This is the $\mathbb{Z}^d$ norm, as the carry is in $\mathbb{Z}^d$.

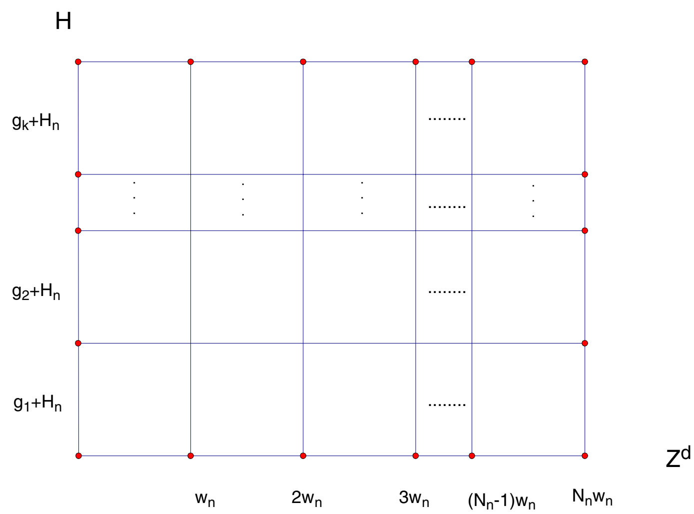

FIGURE 1. Windows in A_{n+1}.

The value of N_n will be specified later. The $(n+1)$-tower A_{n+1} has *windows*

$$[w_n\mathbf{i}, w_n(\mathbf{i}+\mathbf{1})] \times (g + H_n),$$

where $\mathbf{i} \in [1, N_n]^d$ and $g \in \Gamma_n$. A window is identified by its *order* $(\mathbf{i}, g)$. Figure 1 illustrates the case d=1, but is sufficient to envision the general case.

To almost-tile A_{n+1} we place a copy of A_n centrally in each window, with a "random perturbation". More precisely, a spacer function η_n is used to place the copies of A_n in windows of A_{n+1}, that is,

$$\eta_n : [1, N_n]^d \times \Gamma_n \longrightarrow [0, s_n]^d \times H_n,$$

and is defined to be nearly random as detailed later. Now the set of translators C_n is given by

$$C_n := \left\{(w_n\mathbf{i}, g) + (\theta_{n+1}\mathbf{1}, 0) + \eta_n(\mathbf{i}, g) \;\middle|\; \mathbf{i} \in [1, N_n]^d,\ g \in \Gamma_n\right\}.$$

For $c \in C_n$, we describe exactly how cA_n appears in its window. The n-tower, as seen in Figure 2, is composed of many *rows* of the form $[0, h_n)^d \times r$, for $r \in H_n$. The translation of A_n by c has two components. The $\mathbb{Z}^d$ component called the *shift*,

$$\Pi_{\mathbb{Z}^d}(c) = w_n\mathbf{i} + \theta_{n+1}\mathbf{1} + \Pi_{\mathbb{Z}^d}(\eta_n(\mathbf{i}, g)),$$

and the H component called the *shuffle*,

$$\Pi_H(c) = g + \Pi_H(\eta_n(\mathbf{i}, g)).$$

What is the effect of the shift on A_n? It shifts the entire n-block to the center of the window $(\mathbf{i}, 0)$ and a further small translation due to the spacer sequence. If we now apply the shuffle what effect does it have? The shuffle "rotates" the rows of A_n as it places them in the window $(\mathbf{i}, g)$. It also introduces a small $\mathbb{Z}^d$ translation for each row, less than θ_{n+1} in any of the d directions. So the definition of w_n ensures the shuffled n-block is inside the window. Figure 3 illustrates the case $d = 1$.

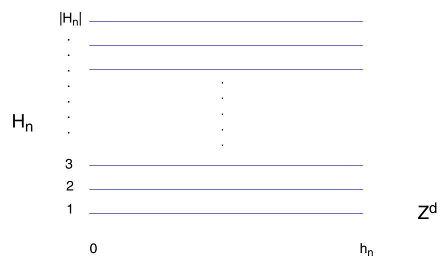

FIGURE 2. A_n is composed of rows.

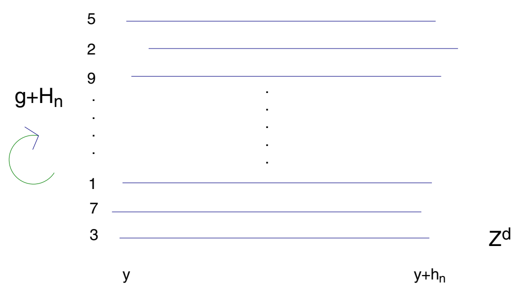

FIGURE 3. *Shuffling* an n-block into a window of A_{n+1}. Here $y = \Pi_{\mathbb{Z}^d}(c)$.

When $d = 2$, cA_n could look like a loosely shuffled deck of cards. For any c and c' in C_n,the n-blocks cA_n and $c'A_n$ are inside their respective windows and thus disjoint.

Let m be a positive integer and let $(\mathbf{i}, g)$ and $(\mathbf{i}', g')$ be two starting windows in A_{n+1}.

Definition 7 (Admissibility)**.** The triple $(m, (\mathbf{i}, g), (\mathbf{i}', g'))$ is called *admissible* if:

1. $\mathbf{i}$, $\mathbf{i}'$, $\mathbf{i} + m\mathbf{e}_1$, and $\mathbf{i}' + m\mathbf{e}_1$ are in $[1, N_n]^d$.
2. $(\mathbf{i}, g)$ is not equal to $(\mathbf{i}', g')$.
3. $m \geq \dfrac{N_n}{n^2}$.

Spacer functions η_n must satisfy a uniformity condition: for all admissible triples $(m, (\mathbf{i}, g), (\mathbf{i}', g'))$, starting at windows $(\mathbf{i}, g)$ and $(\mathbf{i}', g')$ in A_{n+1}, and looking in the m consecutive windows in the positive $\mathbb{Z}$ direction, there is a jointly uniform distribution of spacers. Let $\epsilon_n = \left((s_n)^d |H_n|\right)^{-2}$ which clearly $\xrightarrow{n} 0$. Then we require

$$\left\| \text{Dist}_{k \in [0,m]} \left(\eta_n(\mathbf{i} + k\mathbf{e}_1, g), \eta_n(\mathbf{i}' + k\mathbf{e}_1, g')\right) - \text{Unif}\left([0, s_n]^d \times H_n\right)^2 \right\| \leq \epsilon_n. \tag{1}$$

Why do we want $m \geq N_n/n^2$? The ratio of m to the number of windows in a block required to see uniformity must decrease to zero as n goes to infinity. Yet, m

must increase to infinity to make uniformity possible. Choosing $m > N_n/n^2$ is one way to accomplish this.

The condition (1) on η_n may appear very restrictive. How can we be sure such functions exist? The length of the spacer sequence, N_n, is not yet fixed. As the length of a sequence of random letters from a finite alphabet grows, it is exponentially more likely that its letters are uniformly distributed. From this fact we can construct η_n.

Our spacer sequence η_n will be a realization of a sequence of N_n independent random variables each uniformly distributed over the alphabet $A = ([0, s_n]^d \times H_n)^2$. We show that for large enough N_n there is a high probability that (1) is satisfied.

Lemma 11 (Exponential convergence of random sequences to uniformity)**.** *Given a probability space* (Ω, P)*, let* $X_1, X_2, X_3, \ldots$ *be independent random variables each with uniform distribution over a finite alphabet* A*. Specifically,* $X_i : \Omega \longrightarrow A$ *and* $P\{X_i = a\} = \frac{1}{|A|}$ *for all* a *in* A*. Then for all* $\epsilon > 0$*, there exists* M *and a constant* c *so that if* $m \geq M$ *then*

$$P\{\|\mathrm{Dist}(X_i \mid i \in [0, m]) - \mathrm{Unif}\ A\| \geq \epsilon\} < e^{-cm}. \tag{2}$$

Proof. This is a consequence of the Central Limit Theorem. See [Orn67] for an elementary proof or [Rud79] for a proof using Stirling's Formula. □

Each admissible triple $(m, (\mathbf{i}, g), (\mathbf{i}', g'))$ imposes a constraint on the spacer sequence. Apply Lemma 11 with $A = ([0, s_n]^d \times H_n)^2$ and $\epsilon = \epsilon_n$ to determine c and M. If g is not equal to g', then clearly the m windows to the right of these starting windows are distinct. So by Lemma 11, if $m > M$ then the probability that (1) is not satisfied is less than e^{-cm}.

If g equals g', then it may be that the m windows to the right of the starting points do overlap. If so, divide $[0, m]$ into two sets A and B, where $|A|$ and $|B|$ are both greater than $m/3$, and so that the sets $\{\mathbf{i} + k\mathbf{e}_1 | k \in A\}$ and $\{\mathbf{i}' + k\mathbf{e}_1 | k \in A\}$ are disjoint. Similarly for B, $\{\mathbf{i} + k\mathbf{e}_1 | k \in B\}$ and $\{\mathbf{i}' + k\mathbf{e}_1 | k \in B\}$ are disjoint. Then the probability that

$$\left\| \mathrm{Dist}_{k \in A}\ (\eta_n(\mathbf{i} + k\mathbf{e}_1, g), \eta_n(\mathbf{i}' + k\mathbf{e}_1, g')) - \mathrm{Unif}\ \left([0, s_n]^d \times H_n\right)^2 \right\| \leq \epsilon_n. \tag{3}$$

is less than $e^{-cm/3}$, and likewise for B. So the probability that (1) is not true for an admissible triple $(m, (\mathbf{i}, g), (\mathbf{i}', g'))$ is less than the probability that (3) is not true for $k \in A$ or $k \in B$, which is less than $2e^{-cm/3}$.

Since $m > \dfrac{N_n}{n^2}$ we can conclude that the probability of (1) not being true is less than $2e^{\frac{-cN_n}{3n^2}}$, provided $\dfrac{N_n}{3n^2} > M$. Remembering $k = |H_{n+1}/H_n|$, there are less than $N_n((N_n)^d k)^2$ admissible triples so the probability that (1) is true for all of them is less than

$$1 - 2(N_n)^{2d+1} k^2 e^{-cN_n/3n^2}. \tag{4}$$

So, take N_n sufficiently large that $\dfrac{N_n}{3n^2} > M$ and (4) is greater than zero. This means a realization of η_n will exist to satisfy (1). Additionally we require that N_n be large enough to ensure that $\theta_{n+3} < h_{n+1}$. This means that θ_{n+1} will be much less than s_n, a fact that will be important in later scanning arguments.

The sequences $\{A_n\}$ and $\{C_n\}$ thus defined constitute an almost-tiling Følner sequence. By also requiring that N_n grow exponentially with n, we guarantee that the resulting measure space is finite. Thus, we have $\pounds$ a rank-one action of G.

3.2. *T* is mixing. To show T is mixing we must show $\mu(T^i A \cap B) \longrightarrow \mu(A)\mu(B)$ for all measurable sets A and B. Since A and B can be approximated by levels of the k-tower, for large k, it suffices to show that for each k, P_k is mixed by T. More precisely, we want to show

$$\left\|\mathrm{Dist}_x\left(P_k(T^i x), P_k(x)\right) - \mathrm{Dist}_{(x,y)}\left(P_k(x), P_k(y)\right)\right\| \tag{5}$$

tends to 0 as $i \to \infty$.

Each level of the k tower is a union of levels of the $(n-1)$-tower (when $n-1 > k$). We can link i to n by requiring

$$w_n - (s_n + w_{n-1} + \theta_{n+1}) \le i \le w_{n+1} - (s_{n+1} + w_n + \theta_{n+2}). \tag{6}$$

So (5) is bounded by

$$\left\|\mathrm{Dist}_x\left(P_{n-1}(T^i x), P_{n-1}(x)\right) - \mathrm{Dist}_{(x,y)}\left(P_{n-1}(x), P_{n-1}(y)\right)\right\|. \tag{7}$$

Why is this true? The measures in (5) are probability measures on the set $A_k^* \times A_k^*$. Because of Lemma 3 with the map $P_k^{n-1} \times P_k^{n-1}$, (5) is less than or equal to (7). Thus to prove T is mixing it suffices to show (7) tends to zero as n and thus i go to infinity. Why link i to n? When i is small compared to h_{n+1}, we can investigate the first measure in (7) by examining an $(n+1)$-block and its overlap with the $(n+1)$-block shifted by T^i. When i is much larger than w_n, but still smaller than w_{n+1}, we examine the overlap of $(n+2)$-blocks.

The key idea is that when we shift the $(n+1)$-tower by an offset i in the $\mathbb{Z}$ coordinate, we see nearly uniform $P_{n-1} \times P_{n-1}$ symbols on the overlap.

Definition 8 (Row overlaps)**.** The $(n+1)$-block is composed of $|H_{n+1}|$ rows, each of the form $[0, h_n]^d \times g$ for fixed $g \in H_{n+1}$. An overlap of rows is indexed by $[\mathbf{0}, \mathbf{m}]$ and two points $\mathbf{a}$ and $\mathbf{a}'$. This is a map

$$[\mathbf{0}, \mathbf{m}] \to ([\mathbf{0}, \mathbf{h_n}], g) \times ([\mathbf{0}, \mathbf{h_n}], g') \tag{8}$$

which is defined by $\mathbf{u} \to ((\mathbf{a} + \mathbf{u}, g), (\mathbf{a}' + \mathbf{u}, g'))$. Thus the overlap is a pairing of $[\mathbf{a}, \mathbf{a} + \mathbf{m}] \times g$ in the upper block with $[\mathbf{a}', \mathbf{a}' + \mathbf{m}] \times g'$ in the lower block.

Lemma 12 (Row overlap measures)**.** *Consider an overlap of two rows in A_{n+1}, as above, where $|m_i| > \frac{N_n}{n^2} w_n$ for all i. If $g + H_n$ equals $g' + H_n$ and $\|\mathbf{a} - \mathbf{a}'\| > w_n - (s_n + w_{n-1} + \theta_{n+1})$, or if $g + H_n$ is not equal to $g' + H_n$ then*

$$\left\|\mathrm{Dist}_{\mathbf{u} \in [\mathbf{0}, \mathbf{m}]}\left(P_{n-1}^{n+1}(\mathbf{a} + \mathbf{u}, g), P_{n-1}^{n+1}(\mathbf{a}' + \mathbf{u}, g')\right) - \mathrm{Unif}\left(A_{n-1} \times A_{n-1}\right)\right\| \tag{9}$$

tends to 0 *as* $n \to \infty$.

An overlap of rows that satisfies the conclusion of Lemma 12 is called *good.* Using this lemma we can complete the proof that T is mixing.

Proof that *T* is mixing. When i is in the range

$$w_n - (s_n + w_{n-1} + \theta_{n+1}) \le i \le w_n + s_{n+1} + \theta_{n+2},$$

i is small compared to h_{n+1}, so the overlap between A_{n+1} and $(i\mathbf{e_1} + A_{n+1})$ is large. Let B equal to $A_{n+1} \cap (i\mathbf{e_1} + A_{n+1})$ and let X_B represent the levels of the $(n+1)$-tower indexed by B. Statement (7) is less than

$$\begin{aligned}\left\|\mathrm{Dist}_x(P_{n-1}(T^i x), P_{n-1}(x)) - \mathrm{Dist}_{x\in X_B}(P_{n-1}(T^i x), P_{n-1}(x))\right\| \\ + \left\|\mathrm{Dist}_{x\in X_B}(P_{n-1}(T^i x), P_{n-1}(x)) - \mathrm{Unif}(A_{n-1}\times A_{n-1})\right\| \\ + \left\|\mathrm{Unif}(A_{n-1}\times A_{n-1}) - \mathrm{Dist}_{(x,y)}(P_{n-1}(x), P_{n-1}(y))\right\|.\end{aligned}$$

Lemma 4 shows the first summand is less than $2(1-\mu(X_B))$ which $\xrightarrow{n} 0$. Why? Notice that X_B is a large fraction of X_{n+1}. More precisely

$$\frac{\mu(X_B)}{\mu(X_{n+1})} \geq \frac{h_{n+1} - w_n - s_{n+1} - \theta_{n+2}}{h_{n+1}}.$$

This fraction $\xrightarrow{n} 1$ so we see that $1 - \mu(X_B) \xrightarrow{n} 0$. Lemma 4 also shows the third summand is less than $2(1-\mu(X_{n-1}))$ which $\xrightarrow{n} 0$. The second summand is equivalent to

$$\left\|\mathrm{Dist}_{\mathbf{u}\in B}(P^{n+1}_{n-1}(\mathbf{u} + i\mathbf{e}_1), P^{n+1}_{n-1}(\mathbf{u})) - \mathrm{Unif}(A_{n-1}\times A_{n-1})\right\|, \tag{10}$$

and this distribution over the overlap indexed by B is a weighted average of distributions over overlaps of rows in A_{n+1}. They are all good overlaps so Lemma 12 shows that (10) $\xrightarrow{n} 0$.

When i is in the range

$$w_n + s_{n+1} + \theta_{n+2} \leq i \leq w_{n+1} - (s_{n+1} + \theta_{n+2} + w_n), \tag{11}$$

we consider the overlap of $(n+2)$-towers. Let B be $A_{n+2} \cap (i\mathbf{e_1} + A_{n+2})$, and let X_B represent levels of the $(n+2)$-tower indexed by B. Statement (7) is less than

$$\begin{aligned}\left\|\mathrm{Dist}_x(P_{n-2}(T^i x), P_{n-2}(x)) - \mathrm{Dist}_{x\in X_B}(P_{n-1}(T^i x), P_{n-1}(x))\right\| \\ + \left\|\mathrm{Dist}_{x\in X_B}(P_{n-1}(T^i x), P_{n-1}(x)) - \mathrm{Unif}(A_{n-1}\times A_{n-1})\right\| \\ + \left\|\mathrm{Unif}(A_{n-1}\times A_{n-1}) - \mathrm{Dist}_{(x,y)}(P_{n-1}(x), P_{n-1}(y))\right\|.\end{aligned}$$

Again, Lemma 4 shows the first and third summands $\xrightarrow{n} 0$. The second summand is equal to

$$\left\|\mathrm{Dist}_{\mathbf{u}\in B}(P^{n+1}_{n-1}(\mathbf{u} + i\mathbf{e}_1), P^{n+1}_{n-1}(\mathbf{u})) - \mathrm{Unif}(A_{n-1}\times A_{n-1})\right\|. \tag{12}$$

Most of the overlap indexed by B is composed of overlaps of $(n+1)$-blocks. These $(n+1)$-blocks are not in their standard positions. They have been shifted and shuffled into their respective windows of the $(n+2)$-tower. This means their overlaps are not indexed by a simple set of the form $(i' + A_{n+1}) \cap A_{n+1}$, rather, they are the union of overlaps of rows of the $(n+1)$-block. Because of the range for i, most of the overlap indexed by B is composed of good overlaps of rows of $(n+1)$-blocks. They are good because the range for i guarantees that the row overlaps start in different windows or are too short to be significant. Thus, the portion of B not in good overlaps $\xrightarrow{n} 0$. The distribution of the P_{n-1} symbols in the overlap indexed by B will be a weighted average of distributions over good row overlaps. So (12) is a weighted average of distributions like (9) which $\xrightarrow{n} 0$ by Lemma 12. This concludes the proof that T is mixing. □

We turn to the proof of Lemma 12. Rather than distributions on overlaps of rows like the map from $[\mathbf{0},\mathbf{m}]$ to $([\mathbf{0},\mathbf{h_n}],g)\times([\mathbf{0},\mathbf{h_n}],g')$ defined by

$$\mathbf{u}\to((\mathbf{a}+\mathbf{u},g),(\mathbf{a}'+\mathbf{u},g))\,, \tag{13}$$

we consider sets formed by fixing a position in the window and looking at that position, in both the upper and lower block, in some number of consecutive windows in the positive $\mathbb{Z}$ direction.

Definition 9 (Span)**.** A span is a map from $[0,M]$ to $([\mathbf{0},\mathbf{h_n}],g)\times([\mathbf{0},\mathbf{h_n}],g')$ of the form

$$i\to((\mathbf{b}+iw_n\mathbf{e}_1,g),(\mathbf{b}'+iw_n\mathbf{e}_1,g')) \tag{14}$$

where:

1. $\mathbf{b}$ and $\mathbf{b}'$ are in $[\mathbf{0},\mathbf{h_{n+1}}]$.
2. g and g' are in H_{n+1}.
3. $(\mathbf{b}+Mw_n\mathbf{e}_1)$ and $(\mathbf{b}'+Mw_n\mathbf{e}_1)$ are in $[\mathbf{0},\mathbf{h_{n+1}}]$.
4. $M>N_n/n^2$.

Let $R(b,m)$ be b mod m. Call a vector $\mathbf{b}$ *good* if each coordinate b_i satisfies

$$s_n+\theta_{n+1}\le\ R(b_i,w_n)\ \le w_n-(s_n+\theta_{n+1}).$$

If both $\mathbf{b}$ and $\mathbf{b}'$ are good and in windows of different order in A_{n+1}, we call (14) a *good span*. The constraint on b_i means that if $\mathbf{b}$ is a good vector and $g\in H_{n+1}$ then $(\mathbf{b},g)$ is contained in C_nA_n. That's because $\mathbf{b}$ is far enough away from the boundary of the $\mathbb{Z}^d$ component of the window that $(\mathbf{b},g)$ is always included in the n-block in the window. So the $P_n\times P_n$ symbols in good spans are never spacers.

Overlaps like (13) are unions of spans of nearly equal size. So (9) is an almost equally weighted average of distributions over spans D,

$$\left\|\mathrm{Dist}_{d\in D}\left(P^{n+1}_{n-1}\times P^{n+1}_{n-1}\right)(d)-\mathrm{Unif}\left(A_{n-1}\times A_{n-1}\right)\right\|. \tag{15}$$

Most of the spans considered are good but those that are not have weight at most $2d(s_n+\theta_{n+1}+w_{n-1})/w_n$. Since their weight $\xrightarrow{n}0$, to establish Lemma 12 it suffices to prove the following:

Lemma 13. *If D is a good span then* (15) $\xrightarrow{n}0$.

Proof. Let D be as in (14), and let the starting points $(\mathbf{b},g)$ and $(\mathbf{b}',g')$ be in windows $(\mathbf{I},g_I)$ and $(\mathbf{I}',g_{I'})$ respectively. Here g_I and $g_{I'}$ are in Γ_n so that $g+H_n=g_I+H_n$ and $g'+H_n=g_{I'}+H_n$. Let ν be the measure

$$\mathrm{Dist}_{d\in D}\left(P^{n+1}_{n-1}\times P^{n+1}_{n-1}\right)(d).$$

The marginal measure $\overline{\nu}$ observes the P_{n-1} symbols at the fixed position in the windows of the upper $(n+1)$-block. Because the position is fixed, the value of the spacer function in that window determines the symbol seen. We look at this position for M consecutive windows along the positive $\mathbb{Z}$ axis. Over that range the spacers are nearly uniformly distributed,

$$\left\|\mathrm{Dist}_{k\in[0,M]}\ \eta_n(\mathbf{I}+k\mathbf{e}_1,g_I)-\mathrm{Unif}([0,s_n]^d\times H_n)\right\|\le\epsilon_n,$$

so we have the effect of "scanning" a region of the n-block nearly uniformly. Why? The measure $\overline{\nu}$ is equal to

$$\mathrm{Dist}_{k\in[0,M]}\,P^n_{n-1}\left((\mathbf{b}+kw_n\mathbf{e}_1,g)-kw_n\mathbf{I}-g_I-\eta_n(\mathbf{I}+i\mathbf{e}_1,g_I)\right).$$

Since the H component of the spacers are uniform over H_n we scan a portion of each row of A_n. Because the spacers are jointly uniform over $[0, s_n]^d$ for each row of A_n we scan a portion that is $[0, s_n]^d$ in size. The region scanned is like the shuffled block seen in Figure 3, but we scan only a portion of each row that is $[0, s_n]^d$ in size. Also because we have arranged for the maximum carry θ_{n+1} to be small compared to s_n, the scanned region of A_n contains $n^d|H_n/H_{n-1}|$ complete copies of the $(n-1)$-block, and the portion scanned that is not in those blocks $\xrightarrow{n} 0$. Thus we observe nearly uniform P_{n-1} symbols at our fixed window position, that is, $\|\overline{\nu} - \text{Unif } A_{n-1}\| \xrightarrow{n} 0$.

Now examine the fibres of ν. The fibre measure ν_a looks only at the windows in the upper $(n+1)$-block where we see a fixed symbol a,

$$\nu_a := \text{Dist}_{i\in[0,M]} \left(P^{n+1}_{n-1}(\mathbf{b}' + iw_n\mathbf{e}_1, g') \mid \; P^{n+1}_{n-1}(\mathbf{b} + iw_n\mathbf{e}_1, g) = a\right).$$

The fixed symbol a only occurs for a finite number of spacer values $\gamma_1, \gamma_2, \ldots, \gamma_t$. Because the spacers are jointly uniform,

$$\left\|\text{Dist}_{k\in[0,M]} \left(\eta_n(\mathbf{I} + kw_n\mathbf{e}_1, g_I), \eta_n(\mathbf{I}' + kw_n\mathbf{e}_1, g')\right) - \text{Unif}([0, s_n]^d \times H_n)^2\right\| \leq \epsilon_n,$$

for a fixed spacer value in the window of the upper block, Lemma 7 shows the distribution of spacers in the windows of the lower block are also nearly uniform,

$$\left\| \text{Dist}_{k\in[0,M]} \left(\eta_n(\mathbf{I}' + kw_n\mathbf{e}_1, g') \,|\, \eta_n(\mathbf{I} + kw_n\mathbf{e}_1, g) = \gamma_j \right) - \text{ Unif }\left([0, s_n]^d \times H_n\right) \right\| \leq \epsilon_n s_n^d |H_n|.$$

Since $\epsilon_n = \left(s_n^d|H_n|\right)^2$ then $\epsilon_n s_n^d|H_n| = \sqrt{\epsilon_n}$ which also $\xrightarrow{n} 0$. Thus, we get nearly uniform scanning of a region of the n-block by observing the windows of the lower block at our fixed position. By the same nearly uniform scanning argument, we see nearly uniform P_{n-1} symbols. This occurs for each γ_j; so taking a weighted average, $\|\nu_a - \text{Unif } A_{n-1}\| \xrightarrow{n} 0$.

Since both $\overline{\nu}$ and ν_a are nearly Unif A_{n-1}, by Lemma 6

$$\|\nu - \text{Unif}(A_{n-1} \times A_{n-1})\| \xrightarrow{n} 0. \qquad \square$$

Thus Lemma 13 is true which concludes the proof of Lemma 12.

3.3. *T* is simple and *C(T) = G*. Let λ be an ergodic self-joining of T. We show that λ is either the product measure $\mu \times \mu$, or an off diagonal $(I \times \pounds^g)\Delta$. Since $(I \times S)\Delta$ is an ergodic joining for all $S \in C(T)$, this implies $C(T)$ is merely times($\pounds$) (all the transformations in the action of G).

Take $\epsilon > 0$, $\alpha = 1/4$ and a partition Q of $X \times X$; apply Lemma 8. Then for λ-almost all (x, x') there exists N (depending on (x, x')) so that if E is a set of the form $E = l + \cup_{i=0}^{M}[iw_n, iw_n + L]$ where:

1. $n \geq N$,
2. $M \in \mathbb{N}$,
3. $\alpha w_n \leq L \leq w_n$, and
4. $-w_n \leq l \leq w_n$,

then

$$\| \text{Dist}_{i\in E}\, Q(T^i x, T^i x') - \text{Dist}_{(y,y')}\, Q(y, y')\| < \epsilon.$$

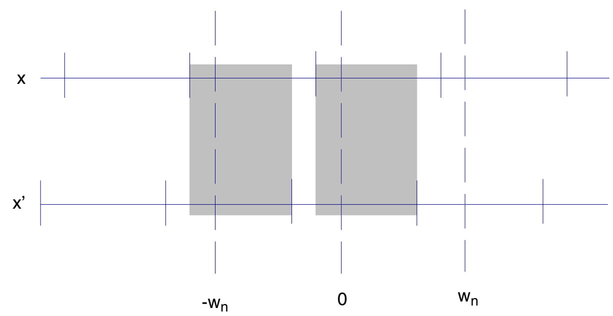

FIGURE 4. Overlap of $\mathbb{Z}$-n-windows in the names of x and x'.

A point (x, x') that satisfies the above for all rational $\epsilon > 0$, all partitions $P_k \times P_k$ and $\alpha = 1/4$, we call *generic*. Since this is only a countable number of applications of the lemma, λ-almost all points are generic. Fix one such point $(x, x') \in X \times X$.

Definition 10 (Time-n order of x)**.** If $x \in X_{n+1}$ then it is contained in a level of the tower E_u^n for $u \in A_{n+1}$. Let $\Theta_n(x)$ denote the time-n order of x, defined by

$$\Theta_n(x) := (\mathbf{i}, g),$$

where $(\mathbf{i}, g)$ is the window of A_{n+1} that contains u.

Definition 11 ($\mathbb{Z}$-n-window)**.** For a point $x \in X_{n+1}$ the $\mathbb{Z}$-n-window is $[a, b]$, a maximal interval in $\mathbb{Z}$, so that $\Theta_n(T^i x)$ is constant for $i \in [a, b]$.

Definition 12 (Centrally Located)**.** Say x is centrally located at time-n if

$$N_n/n^2 < i_1 < N_n - N_n/n^2,$$

where $\Theta_n(x) = (\mathbf{i}, g)$. Define eventually centrally located to mean there exists an N depending on x so that for all times $n \geq N$, x is centrally located at time n.

By the Borel-Cantelli lemma, since the amount of measure we are outlawing is finite, μ-almost all x are eventually centrally located. We want x and x' to be away from the edge of the n-tower in the $\mathbb{Z}$ coordinate.

Fix a pair (x, x') that is generic and both x and x' are eventually centrally located. Examining the structure of the $P_k \times P_k$ names of (x, x') will reveal the nature of λ. The placement of $\mathbb{Z}$-n-windows in the name of x' relative to those in the name of x, will be of particular importance. When x and x' are both centrally located, $\mathbb{Z}$-n-windows for x and x' fill the interval $[-N_n w_n/n^2, N_n w_n/n^2]$.

Definition 13 (Overlap of $\mathbb{Z}$-n-windows)**.** An overlap of $\mathbb{Z}$-n-windows is a maximal interval $[a, b]$ such that $\Theta_n(T^i x)$ and $\Theta_n(T^i x')$ are constant for i in $[a, b]$.

Definition 14 (Offset time)**.** Call n an offset time if there is an overlap of length at least $w_n/4$ in $[-w_n, w_n]$ so that the overlapping $\mathbb{Z}$-n-windows (in x and x') have different orders.

Remarkably, the occurrence of offset times completely determines the structure of λ.

Claim 14 (Finitely many offset times)**.** *If there are only a finite number of offset times then* $\lambda = (I \times \pounds^g)\Delta$*, for some* $g \in G$*.*

Proof. Let N be sufficiently large so that for all $n \geq N$, n is not an offset time, and x and x' are centrally located. We know $[-w_N, w_N]$ is filled with $\mathbb{Z}$-n-windows for both x and x', so it must contain at least one overlap of size $> w_n/4$. Since N is not an offset time this overlap is from windows whose order must be the same. If x and x' do not have the same order at time-N their orders differ by no more than $(1, 0, \ldots, 0) \in \mathbb{Z}^d$. The $\mathbb{Z}$-windows of x and x' at time-$(N+1)$ must overlap almost completely, certainly $> w_{N+1}/4$ in length. Since $N+1$ is not an offset time $\Theta_{N+1}(x) = \Theta_{N+1}(x')$. We repeat, inductively, to see that for all $n \geq N+1$, $\Theta_n(x) = \Theta_n(x')$, that is, x and x' have the same order. If x is not equal to $\pounds^g x'$ for some $g \in A_{N+1} - A_{N+1}$ then at some time $n > N$, x and x' would be in different windows. But that's not possible. This implies $x = \pounds^g x'$ for some $g \in G$. Thus $\lambda = (I \times \pounds^g)\Delta$. □

Claim 15 (Infinitely many offset times)**.** *If there is an infinite sequence of offset times* $\{n_j\}_{j=1}^{\infty}$*, then* $\lambda = \mu \times \mu$*.*

Proof. We can assume that for n_j where $j \geq 1$, x and x' are centrally located. Consider the name of (x, x') with respect to partitions $P_k \times P_k$. In the interval $[-w_{n_j}, w_{n_j}]$ there will be an overlap of $\mathbb{Z} - n_j$-windows of length greater than $w_{n_j}/4$. Since n_j is an offset time, this is an overlap of windows of different orders and because x and x' are centrally located this overlap pattern is repeated at least $N_{n_j}/{n_j}^2$ times to the right. Let L_j be the width of the overlap. Thus we can define a set E_{n_j} of the form $l + \cup_{i=0}^{M}[iw_{n_j}, iw_{n_j} + L_j]$ where:

1. $M = N_{n_j}/{n_j}^2$.
2. $w_{n_j}/4 \leq L_j \leq w_{n_j}$.
3. $-w_{n_j} \leq l \leq w_{n_j}$.

Because (x, x') was chosen to be generic for λ over sets like E_{n_j}, for all partitions $P_k \times P_k$,

$$\lim_{j\to\infty} \operatorname{Dist}_{i \in E_{n_j}}(P_k(T^i x), P_k(T^i x')) = \operatorname{Dist}_\lambda P_k \times P_k. \tag{16}$$

If we also knew that for all k,

$$\lim_{j\to\infty} \operatorname{Dist}_{i \in E_{n_j}}(P_k(T^i x), P_k(T^i x')) = \operatorname{Dist}_{\mu\times\mu} P_k \times P_k, \tag{17}$$

then, since the partitions $P_k \times P_k$ generate the σ-algebra, from (16) and (17) we could conclude that $\lambda = \mu \times \mu$.

To establish (17) we will show

$$\left\|\operatorname{Dist}_{i \in E_{n_j}}(P_k(T^i x), P_k(T^i x')) - \operatorname{Dist}_{\mu\times\mu}(P_k \times P_k)\right\| \xrightarrow{n_j} 0.$$

Proceeding similarly to the proof that T is mixing, it suffices to show

$$\left\|\operatorname{Dist}_{i \in E_{n_j}}(P_{n_j-1}(T^i x), P_{n_j-1}(T^i x')) - \operatorname{Dist}_{\mu\times\mu}(P_{n_j-1} \times P_{n_j-1})\right\| \xrightarrow{n_j} 0.$$

This is because P_{n_j-1} refines P_k when n_j is large enough. Of course

$$\left\|\operatorname{Dist}_{\mu\times\mu}(P_{n_j-1} \times P_{n_j-1}) - \operatorname{Unif}(A_{n_j-1} \times A_{n_j-1})\right\|$$

is less than $2\mu(X^C_{n_j-1})$ which $\xrightarrow{n_j} 0$. So it suffices to show that

$$\left\|\text{Dist}_{i\in E_{n_j}}(P_{n_j-1}(T^i x), P_{n_j-1}(T^i x')) - \text{Unif}(A_{n_j-1}\times A_{n_j-1})\right\| \xrightarrow{n_j} 0. \tag{18}$$

But the measure

$$\text{Dist}_{i\in E_{n_j}}(P_{n_j-1}(T^i x), P_{n_j-1}(T^i x'))$$

is an average of distributions over a fixed position in the windows, a span. A small proportion of the spans comprising E_{n_j} are not good, at most $4(s_{n_j}+\theta_{n_j+1})/w_{n_j}$, which goes to 0. Since n_j is an offset time the rest are good. By Claim 13, statement (18) tends to 0. □

References

[AB69] M. Akcoglu and J. Baxter, *Roots of ergodic transformations*, J. Math. Mech., **19** (1969/70), 991–1003, MR 0414819 (54 #2911), Zbl 0197.33501.

[Fer85] S. Ferenczi, *Systemés de rang un gaughe*, Ann. Inst. H. Poincaré Probab. Statist., **21** (1985), 177–186, MR 0798894 (87a:28020), Zbl 0575.28013.

[Jun98] A. del Junco, *A simple map with no prime factors*, Israel J. Math, **104** (1998), 301–320, MR 1622315 (99d:28037), Zbl 0915.28011.

[Kin00] J.L. King, *The generic transformation has roots of all orders.* Dedicated to the memory of Anzelm Iwanik. Colloq. Math. **84/85** (2000), part 2, 521–547, MR 1784212 (2001h:37005), Zbl 0972.37001.

[Mad] B. Madore *A rank-one $\mathbb{Z}^2$ action where all times are mixing*, in preparation.

[Orn67] D.S. Ornstein, *On the root problem in ergodic theory*, Proc. Sixth Berkeley Symp. Math. Stat. Prob., Vol. II: Probability Theory, University of California Press, Berkeley, 1972, 347–356, MR 0399415 (53 #3259), Zbl 0262.28009.

[PR91] K. Park and E.A. Robinson Jr., *The joinings within a class of $\mathbb{Z}^2$ actions*, J. D'Analyse Math., **57** (1991), 1–36, MR 1191741 (94f:28024), Zbl 0774.28010.

[Rud79] D. Rudolph, *An example of a measure preserving map with minimal self joinings and applications*, J. D'Analyse Math., **35** (1979), 97–122, MR 0555301 (81e:28011), Zbl 0446.28018.

[YJ00] R. Yassawi and A. del Junco, *Multiple mixing of rank one actions*, Canad. J. Math, **52**, (2000), 332–347, MR 1755781 (2001g:28031), Zbl 0957.28008.

DEPARTMENT OF MATHEMATICS, SUNY POTSDAM, POTSDAM, NY 13676-2294, USA
madorebf@potsdam.edu

This paper is available via http://nyjm.albany.edu:8000/j/2004/10-11.html.