Characterization of matrix types of ultramatricial algebras

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ABSTRACT. For any equivalence relation $\equiv$ on positive integers such that $nk \equiv mk$ if and only if $n \equiv m$, there is an abelian group G such that the endomorphism ring of G^n and G^m are isomorphic if and only if $n \equiv m$. However, G^n and G^m are not isomorphic if $n \neq m$.

CONTENTS

1. Introduction

We construct partially ordered abelian groups such that the orbit of a distinguished element under the automorphism group is prescribed; the precise statement is Theorem 1.1 in Subsection 1.1. The prescribed orbit controls the *matrix type* of a ring, i.e., which matrix algebras over the ring are isomorphic, hence we can characterize the matrix types of ultramatricial algebras over any principal ideal domain, see Theorem 1.2 in Subsection 1.2. If the ground ring is $\mathbb{Z}$ then these algebras

are realizable as endomorphism rings of torsion-free abelian groups, which groups therefore have the property stated in the abstract, see Corollary 1.3.

We are indebted to Péter Vámos who draw our attention to this wonderful problem.

1.1. Dimension groups. An *order unit* in a partially ordered abelian group is a positive element u such that for every element x there is a positive integer n such that $nu \geq x$. A *dimension group* $(D, \leq, u)$ is a countable partially ordered abelian group $(D, \leq)$ with order unit u such that every finite subset of D is contained in a subgroup, which is isomorphic to a direct product of finitely many copies of $(\mathbb{Z}, \leq)$ as a partially ordered abelian group.

Our main result, which will be proven from Section 3 on, is:

Theorem 1.1. *Let $H \leq \mathbb{Q}_+^{\times}$ be a subgroup of the multiplicative group of the positive rational numbers. Then there is a dimension group $(D, \leq, u)$ whose group of order-preserving automorphisms is isomorphic to H. Furthermore, under this isomorphism every element of H acts on u by multiplication by itself as a rational number.*

In the special case when H is generated by a set S of prime numbers, one may choose D to be the ring $\mathbb{Z}[S^{-1}]$ and $u = 1$, see [7, Proposition 4.2].

1.2. Ultramatricial algebras. An *ultramatricial algebra* over a field or principal ideal domain F is an F-algebra which is a union of an upward directed countable set of F-subalgebras, which are direct products of finitely many matrix algebras over F.

1.2.1. Matrix types of rings. Let $M_n(R)$ denote the ring of $n \times n$ matrices over the ring R. Obviously, $M_n(R)$ is ultramatricial if R is ultramatricial. The *matrix type* of a ring R is the equivalence on positive integers describing which matrix algebras over R are isomorphic:

$$(1) \qquad \mathrm{mt}(R) := \{(n, m) \mid M_n(R) \cong M_m(R)\}.$$

Clearly, if $(n, m) \in \mathrm{mt}(R)$ then $(mk, nk) \in \mathrm{mt}(R)$ for all positive integers m, n, k. The converse also holds for ultramatricial algebras but probably not for all rings.

The next theorem states that all such equivalences indeed arise as matrix types of ultramatricial algebras:

Theorem 1.2. *Let F be a field or principal ideal domain and $\equiv$ be an equivalence relation on the set of positive integers. Then the following are equivalent:*

(i) *For all positive integers n, m and k*

$$(2) \qquad n \equiv m \iff nk \equiv mk.$$

(ii) *There is a (unique) subgroup H of the multiplicative group $\mathbb{Q}_+^{\times}$ of positive rational numbers such that for all positive integers n and m:*

$$(3) \qquad n \equiv m \iff \frac{n}{m} \in H.$$

(iii) *There exists an ultramatricial F-algebra with matrix type $\equiv$.*

The second statement is clearly a reformulation of the first one, which is useful for explicit construction of equivalences, as pointed out by the referee.

The equivalence of the second and third statements is a simple consequence of our main result, Theorem 1.1, as we will explain in the next section.

1.2.2. Basis types of rings. In a similar vein, the *basis type* of a ring R characterizes which finite rank free modules are isomorphic:

$$(4) \qquad \qquad \mathrm{bt}(R) := \{(n,m)|R^n \cong R^m\}.$$

The analogue of Theorem 1.2 for basis types is Theorem 1 of [3]: every equivalence relation with the property $n \equiv m$ if and only if $n + k \equiv m + k$ is the basis type of a ring of infinite matrices.

Clearly, if $R^n \cong R^m$ then $R^{n+k} \cong R^{m+k}$. All equivalence relation with this property arise as basis type of a ring: see [6].

The basis type is obviously smaller than the matrix type. It seems plausible that this is the only relation between the two types. For example, ultramatricial algebras have trivial basis type i.e., free modules of different finite rank are not isomorphic.

Every ultramatricial algebra R over $F = \mathbb{Z}$ is a countable reduced torsion-free ring, and hence is the endomorphism ring of a torsion-free abelian group G by [1, Theorem A]. Then $M_n(R)$ is the endomorphism ring of G^n. Obviously G^n and G^m are not isomorphic if $n \neq m$ since there is no invertible $n \times m$ matrix over R. Hence we have the statement in the abstract as an immediate corollary to the last theorem:

Corollary 1.3. *Let $\equiv$ be an equivalence relation on positive integers with the property that $n \equiv m$ if and only if $nk \equiv mk$ for all positive integers n, m and k. Then there is a torsion-free abelian group G such that the endomorphism ring of G^n and G^m are isomorphic if and only if $n \equiv m$. Moreover, G^n and G^m are isomorphic if and only if $n = m$.*

2. Equivalence of dimension groups and ultramatricial algebras

Dimension groups and ultramatricial algebras over a fixed field or principal ideal domain are essentially the same. In this section, we recall this equivalence, which shows that Theorem 1.2 follows from Theorem 1.1. For more details and proofs see [7, Proposition 4.1] or [5, Chapter 15, Lemma 15.23, Theorems 15.24 and 15.25], which assume that the ground ring is a field, but the arguments also work when it is a principal ideal domain.

First we define the functor K_0 from the category of rings to the category of pre-ordered abelian groups with a distinguished order unit. The isomorphism classes of finitely generated projective left modules over a ring R form a commutative monoid where the binary operation is the direct sum. The quotient group of the monoid is denoted by $K_0(R)$. Declaring the isomorphism classes of projective modules to be nonnegative, $K_0(R)$ becomes a preordered group. The isomorphism class u of R, the free module of rank 1, is an order unit of $K_0(R)$.

If $f\colon R \to S$ is a ring homomorphism then let $K_0(f)$ map the isomorphism class of a projective module P to that of $S \otimes_R P$. So $K_0(f)$ is a homomorphism preserving both the order and the order unit.

If R is an ultramatricial algebra then $K_0(R)$ is a dimension group. Conversely, every dimension group is isomorphic to the K_0 of an ultramatricial algebra. If R and S are ultramatricial algebras then every morphism between $K_0(R)$ and $K_0(S)$ is of the form $K_0(f)$ for some algebra homomorphism $f\colon R \to S$. However, f is not unique in general. Nevertheless, every isomorphism between $K_0(R)$ and $K_0(S)$ comes from an isomorphism between R and S.

Thus, by restriction, K_0 is essentially an equivalence between the category of ultramatricial algebras over a given field and the category of dimension groups with morphisms the group homomorphisms preserving both the order and the order unit.

Now we examine how the matrix type of an ultramatricial algebra can be recovered from its dimension group. The standard Morita equivalence between R and $M_n(R)$ induces an isomorphism between the K_0 groups. However, this isomorphism does not preserve the order unit, in fact $K_0(M_n(R))$ is $(K_0(R), \leq, nu)$ where u is the order unit of $K_0(R)$.

So if R is an ultramatricial algebra, then the dimension groups $K_0(M_n(R))$ and $K_0(M_m(R))$ (and hence the algebras $M_n(R)$ and $M_m(R)$) are isomorphic if and only if there is an order-preserving automorphism of $K_0(R)$ sending nu to mu. Obviously, for any dimension group $(D, \leq, u)$ whether an order-preserving automorphism maps nu to mu, depends only on the factor m/n. Such factors m/n form a subgroup of the multiplicative group $\mathbb{Q}_+^\times$ of the positive rationals.

So the classification of matrix types of ultramatricial algebras is equivalent to the classification of subgroups of $\mathbb{Q}_+^\times$ which arise from dimension groups in the above construction. Theorem 1.1 states that all subgroups arise, and Theorem 1.2 is just the translation of it to the language of matrix types of ultramatricial algebras.

3. Overview of the construction

In the rest of the paper we prove Theorem 1.1. In this section we outline the main ideas of the proof and leave the details for the following sections. The next section fixes notations used frequently in the rest of the paper. Section 5 recalls a construction of abelian groups. The actual proof is contained in the rest of the sections, which are organized so that they can be read independently. At the beginning of every section, we shall refer to its main proposition, which will be the only statement used in other sections. The same is true for subsections.

To prove Theorem 1.1, we fix a subgroup H of the positive rationals and construct a dimension group D for it. We search for D (as an abelian group without any order) in the form $D := \mathbb{Q}u \oplus G$ where H acts on the direct sum componentwise. We let H act on $\mathbb{Q}u$ by multiplication as required to act on the order unit. A key observation (Proposition 7.1) is that if the only automorphisms of G are the elements of H and their negatives, then for any partial order $\leq$ on D, which is preserved by H and makes $(D, \leq, u)$ a dimension group, the order-preserving automorphism group of D is only H. So $(D, \leq, u)$ satisfies the theorem.

Therefore, all we have to do is to find such a G and $\leq$. Actually, G is already constructed by A. L. S. Corner in [1]. Since we shall use the structure of G to define

the partial order $\leq$, we recall a special case of the construction in Section 5, which is enough for our purposes. See also [4] for the general statement.

Finally, we define a partial order $\leq$ on D making it a dimension group in Subsection 6.2. The basic idea is to explicitly make some subgroups of D order-isomorphic to $\mathbb{Z}^n$ and show that these partial orders on the subgroups are compatible. This will be based on a description of G (Proposition 6.1): we define elements of G, which will be bases of order-subgroups $\mathbb{Z}^n$ of D and state relations between them implying the compatibility of partial orders.

4. Notation

Let x_h denote the element of the group ring $\mathbb{Z}H$ corresponding to $h \in H$. This is to distinguish $x_h a$ the value of x_h at a from ha the element a multiplied by the rational number h.

If the automorphism group of an abelian group is the direct product of a group H and the two element group generated by -1 then we say that the automorphism group is $\pm H$.

5. Abelian groups with prescribed endomorphisms

We revise a special case of A. L. S. Corner's construction of abelian groups with prescribed endomorphism rings. I am grateful to Rüdiger Göbel and his group for teaching me this method.

Let $\hat{M}$ denote the $\mathbb{Z}$-adic completion of the abelian group M. The following result is is a special case of Theorem 1.1 from [2] by taking $A = R$ and $N_k = 0$:

Proposition 5.1. *Let R be a ring with free additive group. Let B be a free R-module of rank at least 2 and $\{w_b : b \in B \setminus \{0\}\}$ a collection of elements of $\hat{\mathbb{Z}}$ algebraically independent over $\mathbb{Z}$. Let the R-module G be*

$$(5) \qquad G := \langle B, Rbw_b : b \in B \setminus \{0\}\rangle_* \subseteq \hat{B}.$$

Then G is a reduced abelian group with endomorphism ring R.

Here $*$ means purification: i.e., we add all the elements x of $\hat{B}$ to

$$E := B \oplus_{b \in B \setminus \{0\}} Rbw_b$$

for which $nx \in E$ for some nonzero integer n. The usual down-to-earth description of G is the following, which we shall use in Subsection 6.1: we select positive integers m_n such that every integer divides $m_1 \ldots m_n$ for n large enough. We choose elements $w_b^{(n)}$ of $\hat{\mathbb{Z}}$ for all nonzero elements b of B and natural numbers n such that $w_b^{(0)} = w_b$ and $w_b^{(n)} - m_n w_b^{(n+1)}$ is an integer. Then G is generated by the submodules B and $Rbw_b^{(n)}$. Obviously, $G_n := B \oplus_{b \in B \setminus \{0\}} Rbw_b^{(n)}$ is a submodule of G and these modules G_n form an increasing chain whose union is the whole G.

Remark 5.2. Note that the automorphism group of G is the group of units of R, which is just $\pm H$ if $R = \mathbb{Z}H$ and H is an orderable group. (It is a famous conjecture that the group of units of $\mathbb{Z}H$ is $\pm H$ for all torsion-free groups H.)

We will be interested in the case when $H \leq \mathbb{Q}_+^\times$ and $R = \mathbb{Z}H$ is a group ring. The free module B will have countable rank. Recall that there are continuum many elements of $\hat{\mathbb{Z}}$ algebraically independent over $\mathbb{Z}$, so the construction works in this case, and we will get a reduced abelian group G with automorphism group $\pm H$.

6. Construction of the dimension group

We now construct our dimension group $D = \mathbb{Q}u \oplus G$ starting from a subgroup $H \leq \mathbb{Q}_+^\times$ of the positive rationals. The main result of the section is Proposition 6.2.

The proposition in Subsection 6.1 defines G and summarizes the (technical) properties of G used in Subsection 6.2 to put the partial order on D.

Let $^n2$ denote the set of sequences of length n whose elements are 0 or 1. If y is a finite sequence of 0 and 1 then let $y0$ denote the sequence obtained from y by adding an additional element 0 to the end. Similarly, we can define $y1$. These sequences will be identified with elements of G.

6.1. The abelian group under the dimension group.
In this subsection we construct an abelian group G satisfying the following properties. Essentially, G will be the underlying abelian group of our dimension group.

Proposition 6.1. *Let $H \leq \mathbb{Q}_+^\times$ be a subgroup of the multiplicative group of the positive rational numbers. Let $R := \mathbb{Z}H$ denote its group ring. Then there exist:*

- *an R-module G, which is also a reduced abelian group,*
- *a finite subset $F_n \subseteq H$ for all positive integer n,*
- *positive integers s_n, t_n, k_n and l_n for $n \geq 0$,*

subject to the following conditions:

(i) $\operatorname{Aut} G = \pm H$.

(ii) *Structure of G:*

  (a) *G is a union of an increasing sequence of free submodules G_n.*

  (b) *G_n has base $^n2 \cup \left\{ c_i^{(n)} : i = 1, \ldots, n \right\}$.*

(iii) *Relations describing the inclusion $G_n \subseteq G_{n+1}$:*

  (a) *$y = y0 + s_n^2 \cdot y1$ for all $y \in {}^n2$.*

  (b) *There are integers $n_{h,y}^{(i)}$ for all $i \leq n$, $h \in H$ and $y \in {}^n2$ such that*

$$(6) \qquad c_i^{(n)} - s_n t_n c_i^{(n+1)} = \sum_{\substack{h \in F_i \\ y \in {}^n2}} n_{h,y}^{(i)} x_h \cdot y0, \quad 0 \leq n_{h,y}^{(i)} < s_n t_n.$$

(iv) *Properties of s_n, t_n, k_n and l_n:*

  (a)

$$k_0 = 1, \qquad\qquad k_{n+1} = s_n k_n,$$
$$l_0 = 1, \qquad\qquad l_{n+1} = t_n l_n.$$

  (b) *Every positive integer divides k_n and l_n for n large enough.*

  (c) *$k_n(s_n^2 - 1) \geq l_n s_n t_n \sum_{i=1}^n \sum_{h \in F_i} h$.*

Proof. The construction of the items is easy. One has to take care to define them in the correct order.

Let B be a countable-rank free module over the group ring $R := \mathbb{Z}H$ i.e., $B = \mathbb{Z}H \otimes A$ where A is a free abelian group of countably infinite rank. Using this group we define G by Equation (5) as in Proposition 5.1. (The $\mathbb{Z}$-adic integers w_b can be arbitrary.) The proposition tells us that G is a reduced abelian group and $\operatorname{Aut} G = \pm H$ so (i) is satisfied.

We identify a base of A with the finite sequences of 0 and 1 not ending with 0 (so with the sequences ending with 1 and the empty sequence). In this base, the

sequences of length at most n is a basis of a free subgroup A_n of A and the free R-module $B_n = R \otimes A_n$.

Let us enumerate the elements of $B \setminus \{0\}$ into a sequence $b_1, b_2, \ldots$ such that $b_n \in B_n$. Every element of B can be written uniquely as $\sum_{h \in H} x_h b_h$ where $b_h \in A$ and only finitely many of the b_h are nonzero. We define the *support* of an element of B as the finite set

$$
(7) \qquad \left[\sum_{h \in H} x_h b_h \right] := \{ h \in H \mid b_h \neq 0 \} \quad (b_h \in A).
$$

Let $F_i := [b_i]$ be the support of b_i.

Now we are ready to define our positive integers s_n, t_n, k_n and l_n. Let us impose the following additional condition on them:

$$
(*) \qquad t_n \text{ is divisible by } n, \text{ and } t_n \text{ divides } s_n.
$$

Now the integers can be defined recursively such that (iv)(a), (iv)(c) and (∗) hold. These automatically imply the truth of (iv)(b).

Now we identify the sequences of 0 and 1 with elements of A. We have already done this for the sequences not ending with zero: they form a basis of A. As dictated by (iii)(a), we set

$$
(8) \qquad y0 := y - s_n^2 \cdot y1 \quad y \in {}^n 2.
$$

(This is in fact a recursive definition on the length of y since y may also end with zero.) Thus the sequences of finite length are identified with elements of A such that (iii)(a) holds and the sequences of length n form a basis of A_n as can be easily seen by induction on n.

We turn to the definition of the G_n. Let us choose $\mathbb{Z}$-adic integers $w_i^{(n)}$ for $1 \leq i \leq n$ such that

$$
(9) \qquad w_i^{(0)} := w_{b_i}, \quad w_i^{(n)} - s_n t_n w_i^{(n+1)} \in \mathbb{Z}, \quad w_i^{(n)} \in \hat{\mathbb{Z}}.
$$

We let G_n be the free submodule

$$
(10) \qquad G_n := B_n \oplus \bigoplus_{i=1}^{n} R b_i w_i^{(n)}.
$$

It follows from the definition of G (Equation (5)) that the groups G_n form an increasing sequence of submodules whose union is G.

The only missing entities are the elements $c_i^{(n)}$. We could set $c_i^{(n)} = b_i w_i^{(n)}$ to satisfy (ii)(b) but this may not be appropriate for (iii)(b). Therefore we shall set

$$
(11) \qquad c_i^{(n)} := b_i w_i^{(n)} + b_i^{(n)} \quad b_i^{(n)} \in B_n, i \leq n
$$

for some $b_i^{(n)}$, which will also satisfy (ii)(b). For i fixed, we are going to define the $b_i^{(n)}$ recursively for $n \geq i$ subject to:

(A) $b_i^{(n)} \in B_n$.
(B) $[b_i^{(n)}] \subseteq [b_i] = F_i$.

(C) For suitable integers $n_{h,y}^{(i)}$:

$$(12) \qquad b_i\left(w_i^{(n)} - s_n t_n w_i^{(n+1)}\right) + b_i^{(n)} = s_n t_n b_i^{(n+1)} + \sum_{h \in F, y \in {}^n 2} n_{h,y}^{(i)} x_h \cdot y0,$$

$$0 \le n_{h,y}^{(i)} < s_n t_n.$$

Note that the last equation is just a reformulation of (iii)(b) in terms of the $b_i^{(n)}$.

Now we carry out the recursive definition. We can start with $b_i^{(i)} := 0$. Observe that (C) determines how to define $b_i^{(n+1)}$: the left-hand side is an element of B_{n+1}, a free abelian group with basis $x_h y$ for $h \in H$ and y a sequence of 0 and 1 of length $n+1$. We divide the coefficient of every $x_h y$ by $s_n t_n$. The quotient gives the coefficient of $x_h y$ in $b_i^{(n+1)}$ and the remainder is a coefficient of the big sum on the right. Since the support of the left-hand side is contained in F_i by induction, the same is true for $b_i^{(n+1)}$ and the sum on the right-hand side. The left-hand side is actually contained in B_n not only B_{n+1}. This means that the coefficients of sequences of length $n+1$ ending with 1 are divisible by s_n^2 by (iii)(a) and hence by $s_n t_n$ since t_n divides s_n. So in the sum on the right-hand side, the coefficient of sequences ending with 1 is zero. Thus we have defined $b_i^{(n+1)}$ according to the requirements. □

6.2. The partial order. In this subsection we define a partial order on D which will make it a dimension group.

Proposition 6.2. *Let $H \le \mathbb{Q}_+^\times$ be a subgroup of the multiplicative group of the positive rational numbers acting on $\mathbb{Q}u$ by multiplication. Suppose G is a group satisfying the conditions of Proposition 6.1. Let H act on $D := \mathbb{Q}u \oplus G$ componentwise. Then there is a partial order $\le$ on D such that $(D, \le, u)$ is a dimension group on which H acts by order-preserving automorphisms.*

The dimension group D in the proposition satisfies all requirements of Theorem 1.1. We shall see in the next section that the group of order-preserving automorphisms of D is exactly H. The other requirements are obviously satisfied.

Proof. We define the partial order on a larger group, the divisible hull $\mathbb{Q}D$ of D. For all natural number n and finite subset F of H we define a subgroup of D:

$$(13) \qquad D_{n,F} := \mathbb{Z}\frac{u}{k_n} \oplus \bigoplus_{\substack{h \in F \\ y \in {}^n 2}} \mathbb{Z}x_h y \oplus \bigoplus_{\substack{i=1 \\ h \in F}}^{n} \mathbb{Z}x_h c_i^{(n)}.$$

We define the partial order on $\mathbb{Q}D_{n,F}$ as the product order

$$(14) \qquad (\mathbb{Q}D_{n,F}, \le) := (\mathbb{Q}v_{n,F}, \le) \times \prod_{\substack{h \in F \\ y \in {}^n 2}} (\mathbb{Q}x_h y, \le) \times \prod_{\substack{i=1 \\ h \in F}}^{n} (\mathbb{Q}x_h c_i^{(n)}, \le)$$

where

$$(15) \qquad v_{n,F} := \frac{u}{k_n} - \sum_{h \in F} h^{-1} x_h \left(k_n \sum_{y \in {}^n 2} y + l_n \sum_{i=1}^{n} c_i^{(n)}\right).$$

Note that u is an order unit of $\mathbb{Q}D_{n,F}$.

The subgroups $D_{n,F}$ form a directed system whose union is the whole D. It follows that the subgroups $\mathbb{Q}D_{n,F}$ form a directed system whose union is $\mathbb{Q}D$.

We are in a position now to reduce the proof to two lemmas stated below. The first one states that the inclusions between the $\mathbb{Q}D_{n,F}$ are order-embeddings. It follows that $\mathbb{Q}D$ has a unique partial order which extends the partial order of all the $\mathbb{Q}D_{n,F}$. Under this partial order u is clearly an order-unit. The partial order is preserved by H since any $h \in H$ maps $\mathbb{Q}D_{n,F}$ bijectively onto $\mathbb{Q}D_{n,hF}$ and this bijection is an order-isomorphism of the two subgroups. (Note that $x_h v_{n,F} = h v_{n,hF}$.) The second lemma claims that a cofinal set of the $D_{n,F}$ is order-isomorphic to a direct product of finitely many copies of $(\mathbb{Z}, \leq)$, and thus $(D, \leq, u)$ have all the properties claimed.

All in all, the proposition is proved modulo the following two lemmas. $\qquad\square$

Lemma 6.3. *The inclusions between the subgroups $\mathbb{Q}D_{n,F}$ are order-embeddings.*

Lemma 6.4. *A cofinal set of the groups $D_{n,F}$ is order-isomorphic to a finite direct power of $(\mathbb{Z}, \leq)$.*

We consider first the inclusions.

Proof of Lemma 6.3. We claim it is enough to prove that $\mathbb{Q}D_{n,F}$ is an order-subgroup of $\mathbb{Q}D_{n+1,F'}$ if F' contains F and FF_i for $i \leq n$. Because it will follow by induction on $m - n$ that for every n and F and $m > n$ there is a finite subset S of H such that $\mathbb{Q}D_{n,F}$ is an order-subgroup of $\mathbb{Q}D_{m,F'}$ if F' contains S. Hence, if $\mathbb{Q}D_{n,F}$ is a subgroup of $\mathbb{Q}D_{k,C}$ then both are order-subgroups of $\mathbb{Q}D_{m,F'}$ for suitable m and F', hence $\mathbb{Q}D_{n,F}$ must be an order-subgroup of $\mathbb{Q}D_{k,C}$.

Now we prove the claim that $\mathbb{Q}D_{n,F}$ is an order-subgroup of $\mathbb{Q}D_{n+1,F'}$ if F' contains F and FF_i for all $i \leq n$. For this, it is good to have the following general example of an order-embedding of $(\mathbb{Q}, \leq)^m$ into $(\mathbb{Q}, \leq)^{m+k}$ given by a matrix:

$$
(16) \qquad \begin{pmatrix} > 0 & 0 & \dots & 0 & \geq 0 & \dots & \geq 0 \\ 0 & > 0 & 0 & \vdots & \geq 0 & \dots & \geq 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & > 0 & \geq 0 & \dots & \geq 0 \end{pmatrix}.
$$

On the first m coordinates this is an order-isomorphism: every coordinate is multiplied by a positive number. On the last k coordinates the map is an arbitrary order-preserving map. By permuting the coordinates, we may complicate the map.

All in all, we see that a homomorphism $(\mathbb{Q}, \leq)^m \to (\mathbb{Q}, \leq)^M$ is an order-embedding if the canonical basis elements of the domain have only nonnegative coordinates in the codomain and every basis element has a positive coordinate, which coordinate is zero for the other basis elements.

We show that the inclusion of $\mathbb{Q}D_{n,F}$ into $\mathbb{Q}D_{n+1,F'}$ is an order-embedding of the above type using the direct product decomposition (14). To this end, we express the generators of $\mathbb{Q}D_{n,F}$ as linear combination of the generators of $\mathbb{Q}D_{n+1,F'}$ (which

in particular shows that $\mathbb{Q}D_{n,F}$ is really a subgroup of $\mathbb{Q}D_{n+1,F'}$):

$$(17) \qquad x_h y = x_h \cdot y0 + s_n^2 x_h \cdot y1 \qquad\qquad\qquad y \in {}^n 2, h \in F$$

$$(18) \qquad x_h c_i^{(n)} = s_n t_n x_h c_i^{(n+1)} + \sum_{\substack{t \in F_i \\ y \in {}^n 2}} n_{t,y}^{(i)} x_{ht} \cdot y0 \qquad\qquad i \leq n, h \in F$$

$$(19) \quad v_{n,F} = s_n v_{n+1,F'} + \sum_{h \in F'} s_n l_{n+1} h^{-1} x_h c_{n+1}^{(n+1)} + \sum_{\substack{h \in F' \setminus F \\ i \leq n}} s_n l_{n+1} h^{-1} x_h c_i^{(n+1)}$$

$$+ \sum_{\substack{h \in F \\ y \in {}^n 2}} k_n (s_n^2 - 1) h^{-1} x_h \cdot y0 + \sum_{\substack{h \in F' \setminus F \\ y \in {}^{n+1} 2}} k_n s_n^2 h^{-1} x_h y$$

$$- \sum_{\substack{h \in F, i \leq n \\ t \in F_i, y \in {}^n 2}} l_n n_{t,y}^{(i)} h^{-1} x_{ht} \cdot y0.$$

These are easy consequences of the formulas under (iii) of Proposition 6.1 and the definition (15) of $v_{n,F}$. All the coordinates of the above generators are obviously nonnegative except for the coefficient of $x_h y0$ of $v_{n,F}$ for $y \in {}^n 2$ and $h \in F'$. So let us consider the coefficient of $h^{-1} x_h y0$ in Equation (19): from the second row comes $k_n(s_n^2 - 1)$ or $k_n s_n^2$ depending on whether h is contained in F. From the last row $-l_n n_{t,y}^{(i)} t$ comes for all $t \in F_i$ and $i \leq n$ for which ht^{-1} lies in F. All in all, the coefficient is at least

$$(20) \qquad k_n(s_n^2 - 1) - \sum_{t \in F_i, i \leq n} l_n n_{t,y}^{(i)} t \geq k_n(s_n^2 - 1) - \sum_{t \in F_i, i \leq n} l_n s_n t_n t \geq 0$$

by (iv)(c) from Proposition 6.1.

Now we check that each of the above generators of $\mathbb{Q}D_{n,F}$ has a positive coordinate in $\mathbb{Q}D_{n+1,F'}$ which coordinate is zero for the other generators. This coordinate is $x_h y1$ for $x_h y$ where $h \in F$ and $y \in {}^n 2$; it is $x_h c_i^{(n+1)}$ for $x_h c_i^{(n)}$ where $h \in F$ and $i \leq n$; finally, it is $v_{n+1,F'}$ for $v_{n,F}$.

Thus we have proved that the inclusions between the $\mathbb{Q}D_{n,F}$ are order-embeddings. □

Now we return to our second lemma, namely that a cofinal subset of the $D_{n,F}$ are order-isomorphic to a finite power of $\mathbb{Z}$.

Proof of Lemma 6.4. If n is a natural number and F is a finite subset of H such that for all $h \in H$ the rational numbers $k_n h^{-1}$ and $l_n h^{-1}$ are actually integers then the coefficients of the $x_h y$ in (15) are integers and hence

$$(21) \qquad (D_{n,F}, \leq) := (\mathbb{Z}v_{n,F}, \leq) \times \prod_{\substack{h \in F \\ y \in {}^n 2}} (\mathbb{Z}x_h y, \leq) \times \prod_{\substack{i=1 \\ h \in F}}^{n} (\mathbb{Z}x_h c_i^{(n)}, \leq).$$

We show that such $D_{n,F}$ form a cofinal system i.e., every $D_{n,F}$ is contained in a $D_{m,F'}$ which has the above property. This is easy once we know that $D_{n,F}$ is contained in $D_{m,F'}$ if $m \geq n$ and F' contains F and FF_i for $i \leq n$. This last statement follows form the fact that $x_h c_i^{(n)}$ is contained in $D_{m,F}$ if $m \geq n$ and h and hF_i are contained in F. This fact can be proved by induction on $m - n$: the

case $m = n$ is obvious because $h \in F$. If $m > n$ then $x_h c_i^{(n+1)}$ is contained in $D_{m,F}$ by induction and $x_h(c_i^{(n)} - s_n t_n c_i^{(n+1)})$ is also contained in $D_{m,F}$ by Equation (6) since hF_i is contained in F. $\square$

7. Automorphisms of dimension groups

In this section we prove that H is the full group of order-preserving automorphisms of D constructed in Proposition 6.2, which finishes the proof of our main theorem. This is a special case of the following proposition, which we are going to prove in this section. Note that $D/\mathbb{Q}u = G$ has the required automorphism group.

Proposition 7.1. *Let $(D, \leq, u)$ be a dimension group of rank at least 3 on which a group H acts by order-preserving automorphisms (the order unit need not be preserved). Let us suppose that the maximal divisible subgroup of D is $\mathbb{Q}u$. Furthermore, let us assume that*

$$\text{(22)} \qquad\qquad \text{Aut}(D/\mathbb{Q}u) = \pm H = \mathbb{Z}/2\mathbb{Z}(-1) \times H,$$

i.e., the automorphisms of $D/\mathbb{Q}u$ are those induced by H and their negatives. Then $\text{Aut}(D, \leq) = H$. In other words, all the order-preserving automorphisms of D are those coming from H.

We base our proof on the comparison of multiples of u with elements of D. This can be described by some rational numbers:

Definition 7.2. Let $(D, \leq, u)$ be a dimension group. Then for every element d of D we denote by $r(d)$ the least rational number q such that $qu \geq d$. Similarly, let $l(d)$ denote the greatest rational number q with the property $qu \leq d$. In other words, for all rational numbers q:

$$\text{(23)} \qquad\qquad qu \geq d \iff q \geq r(d),$$

$$\text{(24)} \qquad\qquad qu \leq d \iff q \leq l(d).$$

We collect the main (and mostly obvious) properties of the functions r and l in the following lemma:

Lemma 7.3. *Let $(D, \leq, u)$ be a dimension group and d and element of it. Then the following hold:*

(a) *The numbers $r(d)$ and $l(d)$ exist.*
(b) *We have $l(d) = r(d) = 0$ if and only if $d = 0$.*
(c) *$r(-d) = -l(d)$ and $l(-d) = -r(d)$.*
(d) *For all rational numbers s,*

$$\text{(25)} \qquad\qquad r(d + su) = r(d) + s,$$

$$\text{(26)} \qquad\qquad l(d + su) = l(d) + s.$$

(e) *If Φ is an order-preserving automorphism of D and $\Phi(u) = qu$ then*

$$\text{(27)} \qquad\qquad l(\Phi(d)) = ql(d),$$

$$\text{(28)} \qquad\qquad r(\Phi(d)) = qr(d).$$

(f) *If D has rank at least 3, the function $d \mapsto l(d) + r(d)$ from D to the additive group of rational numbers is not additive. (It is additive if the rank of D is at most 2.)*

Proof. To prove the existence of $l(d)$ and $r(d)$, we may restrict ourselves to an order-subgroup $(\mathbb{Z}, \leq)^k$ containing d and u. Such a subgroup exists by the definition of dimension group. Clearly, $u = (n_1, \ldots, n_k)$ remains an order unit in the subgroup i.e., its coordinates n_i are positive. For every element $(m_1, \ldots, m_k)$ of $(\mathbb{Z}, \leq)^k$ the functions r and l are clearly well-defined and have the values

$$(29) \qquad\qquad r(m_1, \ldots, m_k) = \max_{1 \leq i \leq k} \frac{m_i}{n_i},$$

$$(30) \qquad\qquad l(m_1, \ldots, m_k) = \min_{1 \leq i \leq k} \frac{m_i}{n_i}.$$

These formulas also show that $r(d) = l(d) = 0$ if and only if $d = 0$. If D has rank at least 3 then there is an order-subgroup $(\mathbb{Z}, \leq)^k$ of D containing u with $k \geq 3$. We can deduce from the above formulas that $r + l$ is not additive even when restricted to such a subgroup. For example, for the elements e_i whose ith coordinate is 1 and all the other coordinates 0, we have $r(e_1) = r(e_2) = r(e_1 + e_2) = 1$ and $l(e_1) = l(e_2) = l(e_1 + e_2) = 0$, and so $r(e_1 + e_2) + l(e_1 + e_2) \neq r(e_1) + l(e_1) + r(e_2) + l(e_2)$.

The remaining items (c), (d) and (e) are obvious. $\qquad\square$

Now we start proving the proposition. First we split the order-preserving automorphism group of D.

Lemma 7.4. *With the hypothesis of Proposition* 7.1, *let*

$$\Gamma \colon \operatorname{Aut}(D, \leq) \to \operatorname{Aut}(D/\mathbb{Q}u)$$

be the canonical map, i.e., $\Gamma(f)$ is the automorphism induced by f on the quotient. Then there is a semidirect product

$$(31) \qquad\qquad \operatorname{Aut}(D, \leq) = \Gamma^{-1}(\mathbb{Z}/2\mathbb{Z}) \rtimes H,$$

where $\mathbb{Z}/2\mathbb{Z}$ is generated by -1.

Proof. Note that $\mathbb{Q}u$ is invariant under automorphisms since it is the largest divisible subgroup so Γ is well-defined. Note that the composition

$$(32) \qquad H \longrightarrow \operatorname{Aut}(D, \leq) \xrightarrow{\Gamma} \operatorname{Aut}(D/\mathbb{Q}u) = \mathbb{Z}/2\mathbb{Z} \times H \longrightarrow H$$

is the identity, which implies the claimed decomposition as a semidirect product. Here the first arrow is the inclusion of H given by the H-action on D and the last arrow is projection onto the second coordinate. $\qquad\square$

Now we show that the first term of the semidirect product is trivial, which finishes the proof.

To this end, we choose an order-preserving automorphism $\Phi \in \Gamma^{-1}(\mathbb{Z}/2\mathbb{Z})$ and show that it is the identity. By the choice of Φ, there is a number $\varepsilon = \pm 1$ such that the image of $\Phi - \varepsilon$ is contained in $\mathbb{Q}u$. Moreover, since $\mathbb{Q}u$ is invariant there is a positive rational number q such that $\Phi(u) = qu$.

Our first task is to show that $q = 1$. Therefore we select a nonzero element d in the kernel of $\Phi - \varepsilon$. Since $\Phi - \varepsilon$ maps to a 1-rank group $\mathbb{Q}u$ and the rank of D is greater than 1, such an element d exists. Now we use Lemma 7.3 (e). If $\varepsilon = 1$, we obtain $r(d) = qr(d)$ and $l(d) = ql(d)$ and thus $q = 1$ since $r(d)$ and $l(d)$ are not both zero. If $\varepsilon = -1$ then $r(d) = -ql(d)$ and $l(d) = -qr(d)$. Again, since at least one of $r(d)$ and $l(d)$ is not zero and q is positive, q must be 1.

So far we know that $\Phi(u) = u$. Let d be an arbitrary element of D. Then there is a rational number s depending on d such that $\Phi(d) = \varepsilon d + su$. Our next task is to determine s.

We apply Lemma 7.3 again, but this time item (d) of it. If $\varepsilon = 1$ then $r(d) = r(d) + s$ and $l(d) = l(d) + s$. We conclude that $s = 0$ for all d. In other words, Φ is the identity. If $\varepsilon = -1$ then we have $r(d) = s - l(d)$ and $l(d) = s - r(d)$. Thus $s = r(d) + l(d)$ for all d, which means that $\Phi(d) = d - (r(d) + l(d))u$. So $r + l$ is an additive function contradicting Lemma 7.3 (f).

Hence we have proved that $\Phi = 1$ and this finishes the proof.

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