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Arens-regularity of algebras arising from tensor norms

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Abstract. We investigate the Arens products on the biduals of certain algebras of operators on nonreflexive Banach spaces. To be precise, we study the α-nuclear operators, where α is a tensor norm. This includes the approximable and nuclear operators, and we use these, together with the 2-nuclear operators, as motivating examples. The structure of the two topological centres of the bidual are studied, and typical results are that for the approximable operators, the two topological centres are always distinct, neither contains the other, and both strictly contain the original algebra. In contrast, on a nonpathological Banach space, the topological centres of the bidual of the nuclear operators coincide. Our methods allow us to also study the algebra of compact operators, even when the compacts are not equal to the approximable operators.

Contents

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1. Introduction

A Banach algebra $\mathcal{A}$ is a Banach space, and so embeds isometrically in its bidual $\mathcal{A}''$. This raises the question of whether there exists a Banach algebra product on $\mathcal{A}''$ extending the product on $\mathcal{A}$. In [Are51], Arens showed that there are two natural products $\Box$ and $\Diamond$ on $\mathcal{A}''$ which extend the product on $\mathcal{A}$. These are termed the *first* and *second Arens products*, respectively, and when $\Box$ and $\Diamond$ coincide, we say that $\mathcal{A}$ is *Arens regular*. Arens regularity is an interesting property in that it passes to closed subalgebras and quotient algebras: it is rare for properties of Banach algebras to display such permanence.

If $\mathcal{A}$ is reflexive as a Banach space, then $\mathcal{A}$ is trivially Arens regular. Consider the space $l^1(\mathbb{Z})$. If we define a product pointwise, then $l^1(\mathbb{Z})$ is Arens regular. Conversely, if we use the convolution product on $l^1(\mathbb{Z})$ coming from the additive group $\mathbb{Z}$, then $l^1(\mathbb{Z})$ is not Arens regular: in fact, the two Arens products agree only on the original algebra (see [Neu04] for an interesting proof of this result, which was first considered in [LL88]). In the positive direction, every C*-algebra is Arens regular. We hence see that the relationship between the algebraic properties of $\mathcal{A}$, the Banach space geometry of $\mathcal{A}$, and the Arens regularity of $\mathcal{A}$ is rather complex.

We shall be interested in algebras of operators. Let E be a Banach space, and let $\mathcal{B}(E)$ be the algebra of all bounded linear operators on E. Then, if $\mathcal{A} \subseteq \mathcal{B}(E)$ is a closed subalgebra containing all the finite-rank operators, and $\mathcal{A}$ is Arens regular, then E must be reflexive (see [You76]). Conversely, the author showed in [Daw04] that if E is *super-reflexive*, then $\mathcal{B}(E)$ is Arens regular. Let $\mathcal{K}(E)$ be the ideal of compact operators on E, and let $\mathcal{A} \subseteq \mathcal{K}(E)$ be a closed algebra. Then $\mathcal{A}$ is Arens regular when E is reflexive. Indeed, if $\mathcal{A}$ is *any* Arens regular Banach algebra, then $\mathcal{A}$ is isometrically isomorphic to a closed subalgebra of $\mathcal{B}(E)$ for some reflexive Banach space E (see [You76] or [Kai81]).

This paper considers the question of what happens when we consider such algebras contained in the compact operators, and when E is not reflexive. To make this question more precise, we shall consider the *topological centres* $\mathfrak{Z}_t^{(1)}(\mathcal{A}'')$ and $\mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, defined as

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Phi\Box\Psi = \Phi\Diamond\Psi \ (\Psi \in \mathcal{A}'')\},$$
$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi\Box\Phi = \Psi\Diamond\Phi \ (\Psi \in \mathcal{A}'')\}.$$

We shall see below that $\mathcal{A}$ is contained in either topological centre, while if one topological centre is equal to the whole of $\mathcal{A}''$, then $\mathcal{A}$ is Arens regular. We say that $\mathcal{A}$ is *left strongly Arens irregular* if $\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \mathcal{A}$; similarly we have *right strongly Arens irregular* and *strongly Arens irregular*. These definitions were introduced in [LÜ96], and studied in detail in [DL04], for example.

We shall study algebras of operators arising from tensor norms. Briefly, let $\mathcal{F}(E)$ be the algebra of finite-rank operators on E, let E' be the dual of E, and let $E'\otimes E$ be the tensor product of E' and E. Then $E'\otimes E$ is naturally identified with $\mathcal{F}(E)$ (see below for further details). Let α be a, in some sense, reasonable norm on $E'\otimes E$, and let $E'\widehat{\otimes}_\alpha E$ be the completion of the resulting normed space. Then

$E'\widehat{\otimes}_\alpha E$ maps into $\mathcal{K}(E)$, and if we equip the image $\mathcal{N}_\alpha(E)$ with the quotient norm, we get a Banach algebra. It is this algebra which we shall study. The advantage of this approach is that we can treat the algebras of approximable operators (the closure of $\mathcal{F}(E)$) and nuclear operators (arising from the "maximal" α allowed) in a unified manner. The key tool is the Gröthendieck Composition Theorem, which shows why calculations involving the approximable operators, or the nuclear operators, often seem very similar, despite the rather different characterisations of these algebras.

Let $\mathcal{A}(E)$, the *approximable operators*, be the closure of $\mathcal{F}(E)$ in $\mathcal{K}(E)$. As a typical result, we show that when E is not reflexive, the topological centres of $\mathcal{A}(E)''$ are always distinct, that neither contains the other, and that both strictly contain $\mathcal{A}(E)$. The same is true of $\mathcal{K}(E)$, even when $\mathcal{A}(E) \neq \mathcal{K}(E)$. The situation becomes more complicated when we consider the nuclear operators, $\mathcal{N}(E)$ (see later for a formal definition), where for well-behaved spaces E, we have that the topological centres of $\mathcal{N}(E)''$ are equal, and strictly contain $\mathcal{N}(E)$. However, for other spaces E, the topological centres of $\mathcal{N}(E)''$ can behave exactly like those of $\mathcal{A}(E)''$ (as it is possible to have $\mathcal{N}(E) = \mathcal{A}(E)$). In fact, we show that for any reasonable tensor norm α, $\mathcal{N}_\alpha(E)$ is not (left or right) strongly Arens irregular (see Theorem 5.14). The same holds for any tensor norm α when E'' is well-behaved, but we leave open the case when both E'' and α are pathological.

In the next section, we shall define the Arens products and study the Arens representations. We shall then quickly survey the definitions and results from the theory of tensor norms. The rest of the paper then studies the topological centres of algebras arising from tensor norms. One a first reading, the many long calculations in the latter sections may be skipped, as we hope the results are easy accessible without the proofs.

We note that many of the results in this area (especially for approximable operators) have been (re-)discovered multiple times, and that some results are "folklore". We hope that one function of this paper is to bring together some of these results in a unified manner.

2. Arens products and representations

Let E be a Banach space, and denote by E' its dual space. We use the dual-pair notation $\langle\cdot,\cdot\rangle$, so we write $\langle\mu, x\rangle = \mu(x)$ for $\mu \in E'$ and $x \in E$. We write $E_{[1]}$ for the closed unit ball of E, and more generally set $E_{[t]} = \{x \in E : \|x\| \leq t\}$ for $t > 0$. For a subspace $F \subseteq E$, we set

$$F^\circ = \{\mu \in E' : \langle\mu, x\rangle = 0 \ (x \in F)\}.$$

Similarly, for a subspace $G \subseteq E'$, we set

$${}^\circ G = \{x \in E : \langle\mu, x\rangle = 0 \ (\mu \in G)\}.$$

Then, for example, we naturally identify $(E/F)'$ with F°. Recall that there is a canonical isometry $\kappa_E : E \to E''$ defined by $\langle\kappa_E(x), \mu\rangle = \langle\mu, x\rangle$ for $\mu \in E'$ and $x \in E$. We say that E is *reflexive* if κ_E is a isomorphism.

Let $\mathcal{A}$ be a Banach algebra. A *Banach left $\mathcal{A}$-module* E is a Banach space together with a bilinear map $\mathcal{A} \times E \to E; (a, x) \mapsto a \cdot x$ such that

$$a \cdot (b \cdot x) = ab \cdot x, \quad \|a \cdot x\| \leq \|a\|\|x\| \qquad (a, b \in \mathcal{A}, x \in E).$$

Then we get a norm-decreasing homomorphism $\pi : \mathcal{A} \to \mathcal{B}(E)$, that is, a *representation*, given by $\pi(a)(x) = a \cdot x$ for $a \in \mathcal{A}$ and $x \in E$. Similarly we get the notion of a *Banach right $\mathcal{A}$-module*. A *Banach $\mathcal{A}$-bimodule* E is a Banach left $\mathcal{A}$-module which is also a Banach right $\mathcal{A}$-module such that $(a \cdot x) \cdot b = a \cdot (x \cdot b)$ for $a, b \in \mathcal{A}$ and $x \in E$.

When E is a Banach left $\mathcal{A}$-module, we define E' to be a Banach right $\mathcal{A}$-module by setting

$$\langle \mu \cdot a, x \rangle = \langle \mu, a \cdot x \rangle \qquad (a \in \mathcal{A}, x \in E, \mu \in E').$$

Similar results hold for right- and bimodules. In particular, $\mathcal{A}$ is a bimodule over itself, and so $\mathcal{A}'$ is a Banach $\mathcal{A}$-bimodule.

We define norm-decreasing bilinear maps $\mathcal{A}'' \times \mathcal{A}' \to \mathcal{A}'$ and $\mathcal{A}' \times \mathcal{A}'' \to \mathcal{A}'$ by

$$\langle \Phi \cdot \mu, a \rangle = \langle \Phi, \mu \cdot a \rangle, \quad \langle \mu \cdot \Phi, a \rangle = \langle \Phi, a \cdot \mu \rangle \qquad (a \in \mathcal{A}, \mu \in \mathcal{A}', \Phi \in \mathcal{A}'').$$

We then define norm-decreasing bilinear maps $\Box, \Diamond : \mathcal{A}'' \times \mathcal{A}'' \to \mathcal{A}''$ by

$$\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi, \Psi \cdot \mu \rangle, \quad \langle \Phi \Diamond \Psi, \mu \rangle = \langle \Psi, \mu \cdot \Phi \rangle \qquad (\mu \in \mathcal{A}', \Phi, \Psi \in \mathcal{A}'').$$

These are then the Arens products: we may check that these products are associative, and that $\kappa_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}''$ becomes a homomorphism when $\mathcal{A}''$ is given either Arens product. Furthermore, we have that

$$a \cdot \Phi = \kappa_{\mathcal{A}}(a) \Box \Phi = \kappa_{\mathcal{A}}(a) \Diamond \Phi \qquad (a \in \mathcal{A}, \Phi \in \mathcal{A}''),$$

and similarly with the orders reversed. This shows that $\kappa_{\mathcal{A}}(\mathcal{A}) \subseteq \mathfrak{Z}_t^{(i)}(\mathcal{A}'')$ for $i = 1, 2$.

A good survey of results about the Arens products is [DH79]; [CY61] is the first systematic study of Arens products. See also [Dal00, Section 2.6] or [Pal94, Section 1.4].

It follows from Goldstein's Theorem that for $\Phi \in \mathcal{A}''$, there exists a net (a_α) in $\mathcal{A}$ such that $\|a_\alpha\| \leq \Phi$ for each α, and $\langle \Phi, \mu \rangle = \lim_\alpha \langle \mu, a_\alpha \rangle$ for each $\mu \in \mathcal{A}'$. Similarly, let $\Psi \in \mathcal{A}''$ define a bounded net (b_β) in $\mathcal{A}$. We may show that

$$\langle \Phi \Box \Psi, \mu \rangle = \lim_\alpha \lim_\beta \langle \mu, a_\alpha b_\beta \rangle, \quad \langle \Phi \Diamond \Psi, \mu \rangle = \lim_\beta \lim_\alpha \langle \mu, a_\alpha b_\beta \rangle \qquad (\mu \in \mathcal{A}'),$$

which illustrates why we get two, in general distinct, products which depend both upon the algebraic structure of $\mathcal{A}$, and the topological structure (so that, for example, there exist commutative algebras $\mathcal{A}$ which are not Arens regular).

A bounded net (a_α) in $\mathcal{A}$ is a *bounded approximate identity* when $aa_\alpha \to a$ and $a_\alpha a \to a$ for each $a \in \mathcal{A}$. A functional $\Xi \in \mathcal{A}''$ is a *mixed identity* when

$$\Phi \Box \Xi = \Xi \Diamond \Phi = \Phi \qquad (\Phi \in \mathcal{A}'').$$

A simple calculation shows that this condition is equivalent to

$$\Xi \cdot \mu = \mu \cdot \Xi = \mu \quad (\mu \in \mathcal{A}'), \quad \text{or} \quad a \cdot \Xi = \Xi \cdot a = \kappa_{\mathcal{A}}(a) \quad (a \in \mathcal{A}).$$

Proposition 2.1. *A Banach algebra $\mathcal{A}$ has a bounded approximate identity if and only if $\mathcal{A}''$ has a mixed identity.*

Proof. Give a bounded approximate identity (a_α), any weak*-limit point of this net in $\mathcal{A}''$ will be a mixed identity. The converse follows by applying Goldstein's Theorem and Mazur's Theorem on the weak closure of convex sets. See [Dal00, Proposition 2.9.16] for further details. $\Box$

2.1. Arens representations. We shall now define the *Arens representations* as detailed in, for example, [Pal94, Section 1.4]. We use the language of modules, but the results (once translated) are the same.

Let E and F be Banach spaces, and norm the tensor product $E \otimes F$ by

$$\pi(\tau) = \inf\left\{ \sum_{i=1}^{r} \|x_i\| \|y_i\| : \tau = \sum_{i=1}^{r} x_i \otimes y_i \right\} \qquad (\tau \in E \otimes F).$$

Then $\pi(\cdot)$ is the *projective tensor norm*, and the completion of $(E \otimes F, \pi)$ is $E \widehat{\otimes} F$, the *projective tensor product of E and F*. In this section, we shall use the fact that if $\phi : E \times F \to G$ is a bounded, bilinear map to some Banach space G, then there is a unique bounded linear map $\psi : E \widehat{\otimes} F \to G$ such that $\|\psi\| = \|\phi\|$ and $\psi(x \otimes y) = \phi(x, y)$ for each $x \in E$ and $y \in F$. When $T \in \mathcal{B}(E, F)$ and $S \in \mathcal{B}(G, H)$ for Banach spaces E, F, G and H, there is a unique linear map $T \otimes S : E \widehat{\otimes} G \to F \widehat{\otimes} H$ such that $(T \otimes S)(x \otimes y) = T(x) \otimes S(y)$ for $x \in E$ and $y \in G$.

Let $\mathcal{A}$ be a Banach algebra, and let F be a Banach left $\mathcal{A}$-module. Then F' is a Banach right $\mathcal{A}$-module, and F'' is a Banach left $\mathcal{A}$-module. Thus $F' \widehat{\otimes} F$ and $F'' \widehat{\otimes} F'$ become Banach $\mathcal{A}$-bimodules for the module actions

$$(\mu \otimes x) \cdot a = \mu \cdot a \otimes x, \qquad a \cdot (\mu \otimes x) = \mu \otimes a \cdot x,$$
$$(\Lambda \otimes \mu) \cdot a = \Lambda \otimes \mu \cdot a, \qquad a \cdot (\Lambda \otimes \mu) = a \cdot \Lambda \otimes \mu,$$

for $a \in \mathcal{A}$, $\mu \otimes x \in F' \widehat{\otimes} F$ and $\Lambda \otimes \mu \in F'' \widehat{\otimes} F'$.

Define a bilinear map $\phi_1 : F'' \times F' \to \mathcal{A}'$ by

$$\langle \phi_1(\Lambda, \mu), a \rangle = \langle a \cdot \Lambda, \mu \rangle \qquad (\Lambda \in F'', \mu \in F', a \in \mathcal{A}).$$

Then ϕ_1 extends to a norm-decreasing map $F'' \widehat{\otimes} F' \to \mathcal{A}'$. Similarly define $\phi_2 : F' \widehat{\otimes} F \to \mathcal{A}'$ by

$$\langle \phi_2(\mu \otimes x), a \rangle = \langle \mu, a \cdot x \rangle \qquad (\mu \otimes x \in F' \widehat{\otimes} F, a \in \mathcal{A}).$$

We may check that ϕ_1 and ϕ_2 are $\mathcal{A}$-bimodule homomorphisms.

Then $\phi_1' : \mathcal{A}'' \to \mathcal{B}(F'')$, with the action given by

$$\langle \phi_1'(\Phi)(\Lambda), \mu \rangle = \langle \Phi, \phi_1(\Lambda \otimes \mu) \rangle \qquad (\Phi \in \mathcal{A}'', \Lambda \in F'', \mu \in F').$$

Similarly, $\phi_2' : \mathcal{A}'' \to \mathcal{B}(F')$. We can also verify the following identities:

$$\Phi \cdot \phi_1(\Lambda \otimes \mu) = \phi_1(\phi_1'(\Phi)(\Lambda) \otimes \mu) \qquad (\Phi \in \mathcal{A}'', \Lambda \otimes \mu \in F'' \widehat{\otimes} F'),$$
$$\phi_2(\mu \otimes x) \cdot \Phi = \phi_2(\phi_2'(\Phi)(\mu) \otimes x) \qquad (\Phi \in \mathcal{A}'', \mu \otimes x \in F' \widehat{\otimes} F).$$

Definition 2.2. For a Banach space E, we have the isometric map $\mathcal{B}(E) \to \mathcal{B}(E')$; $T \mapsto T'$, defined by

$$\langle T'(\mu), x \rangle = \langle \mu, T(x) \rangle \qquad (T \in \mathcal{B}(E), x \in E, \mu \in E').$$

For a subset $X \subseteq \mathcal{B}(E)$ write

$$X^a = \{T' : T \in X\} \subseteq \mathcal{B}(E'),$$

so that, in particular, $\mathcal{B}(E)^a$ is a subalgebra of $\mathcal{B}(E')$. We can show that $\mathcal{B}(E)^a = \mathcal{B}(E')$ if and only if E is reflexive. For a Banach algebra $\mathcal{A}$ and $\psi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(E))$, we define $\psi^a \in \mathcal{B}(\mathcal{A}, \mathcal{B}(E'))$ by $\psi^a(b) = \psi(b)'$ for $b \in \mathcal{A}$.

Let $\theta_1 = \phi_1'$ and $\theta_2 = (\phi_2')^a$.

Proposition 2.3. *The maps $\theta_1 : (\mathcal{A}'', \Box) \to \mathcal{B}(F'')$ and $\theta_2 : (\mathcal{A}'', \Diamond) \to \mathcal{B}(F'')$ are norm-decreasing homomorphisms. Thus θ_1 and θ_2 induce a module structure on F'' so that we can, respectively, view F'' as a Banach left $(\mathcal{A}'', \Box)$-module or a Banach left $(\mathcal{A}'', \Diamond)$-module. For $\Phi \in \mathcal{A}''$, we have $\theta_1(\Phi) \circ \kappa_F = \theta_2(\Phi) \circ \kappa_F$.*

Proof. Let $\Phi, \Psi \in \mathcal{A}''$ and $\Lambda \otimes \mu \in F''\widehat{\otimes}F'$. Then we have

$$\begin{aligned}\langle \theta_1(\Phi\Box\Psi)(\Lambda), \mu\rangle &= \langle \Phi, \Psi \cdot \phi_1(\lambda \otimes \mu)\rangle = \langle \Phi, \phi_1(\phi_1'(\Psi)(\Lambda) \otimes \mu)\rangle \\ &= \langle \phi_1'(\Phi)(\phi_1'(\Psi)(\Lambda)), \mu\rangle = \langle \big(\theta_1(\Phi) \circ \theta_2(\Psi)\big)(\Lambda), \mu\rangle.\end{aligned}$$

We will show that ϕ_2' is an anti-homomorphism, so that θ_2 is a homomorphism. For $\Phi, \Psi \in \mathcal{A}''$ and $\mu \otimes x \in F'\widehat{\otimes}F$, we have

$$\begin{aligned}\langle \phi_2'(\Phi\Diamond\Psi)(\mu), x\rangle &= \langle \Psi, \phi_2(\mu \otimes x) \cdot \Phi\rangle = \langle \Psi, \phi_2(\phi_2'(\Phi)(\mu) \otimes x)\rangle \\ &= \langle \phi_2'(\Psi)(\phi_2'(\Phi)(\mu)), x\rangle.\end{aligned}$$

The final claim is a simple calculation. $\Box$

An alternative way to look at these maps is through the use of nets. For $\Phi, \Psi \in \mathcal{A}''$, suppose that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$, with convergence in the weak*-topology on $\mathcal{A}''$. We then have that

$$\theta_1(\Phi\Box\Psi)(\Lambda) = \lim_\alpha \lim_\beta a_\alpha b_\beta \cdot \Lambda \qquad (\Lambda \in F''),$$

where the limit is in the weak*-topology on F''. Similarly, we have

$$\phi_2'(\Phi\Diamond\Psi)(\mu) = \lim_\beta \lim_\alpha \mu \cdot a_\alpha b_\beta \qquad (\mu \in F'),$$

where the limit is in the weak*-topology on F'.

For a general Banach algebra $\mathcal{A}$ and module F, the behaviour of θ_1 applied to $\Diamond$ (or θ_2 applied to $\Box$) has no simple description. However, in a large number of cases, we can say something. Recall that an operator $T : E \to F$ is *weakly-compact* when T maps the unit ball of E into a relatively weakly-compact subset of F. In this case, we write $T \in \mathcal{W}(E, F)$, and write $\mathcal{W}(E, E) = \mathcal{W}(E)$.

Definition 2.4. For $a \in \mathcal{A}$, define $T_a \in \mathcal{B}(F)$ by $T_a(x) = a \cdot x$ for $x \in F$. We say that the action of $\mathcal{A}$ on F is *weakly-compact* if $T_a \in \mathcal{W}(F)$ for every $a \in \mathcal{A}$.

The following definitions appear in [DL04], but we give a more general treatment here; the use of these ideas appears to be "folklore" in that they are certainly known, but there is no definitive source for them (see, for example, [Gro87], [Gro84] or [Pal85], all of which deal with ideals of approximable operators). Let F be a Banach space, and for $T \in \mathcal{B}(F'')$, define $\eta(T) \in \mathcal{B}(F')$ and $\mathcal{Q}(T) \in \mathcal{B}(F'')$ by

$$\eta(T) = \kappa_F' \circ T' \circ \kappa_{F'}, \qquad \mathcal{Q}(T) = \eta(T)' = \kappa_{F'}' \circ T'' \circ \kappa_F'' = \kappa_{F'}' \circ (T \circ \kappa_F)''.$$

Then note that $\eta(T') = T$ for $T \in \mathcal{B}(F')$, so that $\mathcal{B}(F')^a$ is a one-complemented subspace of $\mathcal{B}(F'')$. Define a bilinear operation $\star$ on $\mathcal{B}(F'')$ by $T \star S = \mathcal{Q}(T) \circ S$ for $T, S \in \mathcal{B}(F'')$.

Proposition 2.5. *The operation $\star$ is a Banach algebra product on $\mathcal{B}(F'')$. When the action of $\mathcal{A}$ on F is weakly-compact, the map $\theta_1 : (\mathcal{A}'', \Diamond) \to (\mathcal{B}(F''), \star)$ is a homomorphism.*

Proof. We see immediately that $\star$ satisfies $\|T \star S\| \leq \|T\|\|S\|$, and that if suffices to show that $(T \star S) \star R = T \star (S \star R)$ for each $R, S, T \in \mathcal{B}(F'')$. We have

$$\eta(S) \circ \eta(T) = \kappa_F' \circ S' \circ \kappa_{F'} \circ \eta(T) = \kappa_F' \circ S' \circ \eta(T)'' \circ \kappa_{F'} = \eta(\mathcal{Q}(T) \circ S),$$

and thus

$$\begin{aligned}(T \star S) \star R &= \mathcal{Q}(T \star S) \circ R = \mathcal{Q}(\mathcal{Q}(T) \circ S) \circ R = \eta(\mathcal{Q}(T) \circ S)' \circ R \\ &= \eta(T)' \circ \eta(S)' \circ R = \mathcal{Q}(T) \circ \mathcal{Q}(S) \circ R = T \star (S \star R).\end{aligned}$$

For $a \in \mathcal{A}$ and $\Lambda \in F''$, we can verify that $a \cdot \Lambda = T_a''(\Lambda)$. As $T_a \in \mathcal{W}(F)$, by Theorem 3.8, we have $T_a''(\Lambda) \in \kappa_F(F)$. Thus let $\kappa_F(y) = a \cdot \Lambda$, so that for $\Phi \in \mathcal{A}''$ and $\Lambda \otimes \mu \in F'' \widehat{\otimes} F'$, we have

$$\begin{aligned}\langle \phi_1(\Lambda \otimes \mu) \cdot \Phi, a \rangle &= \langle \Phi, \phi_1(a \cdot \Lambda \otimes \mu) \rangle = \langle \theta_1(\Phi)(a \cdot \Lambda), \mu \rangle = \langle \theta_1(\Phi)' \kappa_{F'}(\mu), a \cdot \Lambda \rangle \\ &= \langle \theta_1(\Phi)' \kappa_{F'}(\mu), \kappa_F(y) \rangle = \langle \eta(\theta_1(\Phi))(\mu), y \rangle \\ &= \langle \kappa_F(y), \eta(\theta_1(\Phi))(\mu) \rangle = \langle a \cdot \Lambda, \eta(\theta_1(\Phi))(\mu) \rangle \\ &= \langle \phi_1\big(\Lambda \otimes \eta(\theta_1(\Phi))(\mu)\big), a \rangle.\end{aligned}$$

Thus for $\Phi, \Psi \in \mathcal{A}''$ and $\Lambda \otimes \mu \in F'' \widehat{\otimes} F'$, we have

$$\begin{aligned}\langle \theta_1(\Phi \Diamond \Psi)(\Lambda), \mu \rangle &= \langle \Psi, \phi_1(\Lambda \otimes \mu) \cdot \Phi \rangle = \langle \Psi, \phi_1\big(\Lambda \otimes \eta(\theta_1(\Phi))(\mu)\big) \rangle \\ &= \langle \theta_1(\Psi)(\Lambda), \eta(\theta_1(\Phi))(\mu) \rangle = \langle \mathcal{Q}(\theta_1(\Phi)) \theta_1(\Psi)(\Lambda), \mu \rangle,\end{aligned}$$

so that $\theta_1(\Phi \Diamond \Psi) = \theta_1(\Phi) \star \theta_1(\Psi)$. □

Suppose that F is reflexive, so that the action of $\mathcal{A}$ on F is certainly weakly-compact. Then $\star = \circ$ on $\mathcal{B}(F)$, so that θ_1 is a homomorphism $\mathcal{A}'' \to \mathcal{B}(F)$ for either Arens product. In particular, if θ_1 is injective, then $\mathcal{A}$ must be Arens regular.

3. Tensor norms

We shall now sketch the definitions and results about tensor norms which we shall need. We refer the reader to [Rya02] or [DF93] for more details on the topics in these sections. We follow the notation of [Rya02], which occasionally clashes with that of [DF93]. There is a short, self-contained account in [DU77, Chapter VIII] of many of the more important ideas in this section. The reader who knows about the projective and injective tensor norms, and about integral operators, is welcome to skim this section, and to think purely about, say, the projective tensor norm, instead of general tensor norms α.

We have already defined the projective tensor norm. For Banach spaces E and F, we define the *injective tensor norm* ε on $E \otimes F$ by

$$\varepsilon(u, E \otimes F) = \sup\left\{ \left| \sum_{i=1}^n \langle \mu, x_i \rangle \langle \lambda, y_i \rangle \right| : \mu \in E', \lambda \in F', \|\mu\| = \|\lambda\| = 1 \right\},$$

where $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$. We often denote the completion of $E \otimes F$ with respect to ε by $E \check{\otimes} F$. If we identify $E' \otimes F$ with $\mathcal{F}(E, F)$, then it is a simple check that $E' \check{\otimes} F$ is identified with $\mathcal{A}(E, F)$. Furthermore, the norm ε on $E \otimes F$ agrees with the norm induced by the natural embedding on $E \otimes F$ into $\mathcal{F}(E', F)$.

Let FIN be the class of finite-dimensional normed vector spaces. For a normed vector space E, let $\mathrm{FIN}(E)$ be the collection of all finite-dimensional subspaces of E, together with the norm induced by that of E.

Definition 3.1. Let E and F be normed vectors spaces and α be a norm on $E\otimes F$. Then α is a *reasonable crossnorm* if $\varepsilon(u) \leq \alpha(u) \leq \pi(u)$ for each $u \in E \otimes F$.

A *uniform crossnorm* is an assignment to each pair, (E, F), of Banach spaces, of a reasonable crossnorm α on $E \otimes F$ such that we have the following. Let D, E, F and G be Banach spaces, and let $T \in \mathcal{B}(D, E)$, $S \in \mathcal{B}(G, F)$. Then we form the bilinear map

$$T \otimes S : D \times G \to E \otimes F; (x, y) \mapsto T(x) \otimes S(y) \quad (x \in D, y \in G),$$

which extends to $D \otimes G$ by the tensorial property. Then we insist that $\|T \otimes S\| \leq \|T\|\|S\|$ with respect to the norm α on $D \otimes G$ and on $E \otimes F$.

For $u \in E \otimes F$, we often write $\alpha(u, E \otimes F)$, instead of just $\alpha(u)$, to avoid confusion. Let D be a closed subspace of E, let G be a closed subspace of F, and let $u \in D \otimes G$. By considering the inclusion maps $D \to E$ and $G \to F$, we identify u with its image in $E \otimes F$, and hence for a uniform crossnorm α, we see that

$$\alpha(u, E \otimes F) \leq \alpha(u, D \otimes G) \qquad (u \in D \otimes G).$$

Definition 3.2. Let α be a uniform crossnorm. Then α is *finitely generated* if, for each pair of Banach spaces E and F, and each $u \in E \otimes F$, we have

$$\alpha(u, E \otimes F) = \inf\{\alpha(u, M \otimes N) : M \in \mathrm{FIN}(E), N \in \mathrm{FIN}(F), u \in M \otimes N\}.$$

We call a finitely generated uniform crossnorm a *tensor norm*. We denote the completion of the normed space $(E \otimes F, \alpha)$ by $E\widehat{\otimes}_\alpha F$.

Definition 3.3. For Banach spaces E and F, and $u \in E \otimes F$, let $u^t \in F \otimes E$ be defined by $u^t = \sum_{i=1}^n y_i \otimes x_i$ when $u = \sum_{i=1}^n x_i \otimes y_i$. We call u^t the *transpose* of u and often refer to the map $u \mapsto u^t$ as the *swap map*. For a tensor norm α, define α^t by $\alpha^t(u, E \otimes F) = \alpha(u^t, F \otimes E)$, so that α^t is a tensor norm.

Both the injective and projective tensor norms are tensor norms. They are also *symmetric*, in that the swap map leaves them invariant, but this is not true for general tensor norms. The injective tensor norm is *injective* in that $\varepsilon(u, E \otimes F) = \varepsilon(u, G \otimes H)$ for each $u \in E \otimes F$, E a subspace of G, and F a subspace of H. Similarly, the projective tensor norm is *projective* in that, for $u \in G/E \otimes H/F$,

$$\varepsilon(u, G/E \otimes H/F) = \inf\{\varepsilon(v, G \otimes H) : (Q_E \otimes Q_F)(v) = u\},$$

where $Q_E : G \to G/E$ and $Q_F : H \to H/F$ are the quotient maps. The projective tensor norm is not, in general, injective (and vis versa).

3.1. Duals of tensor products and operator ideals. Let $E, F \in \mathrm{FIN}$ so that $E \otimes F$ is finite-dimensional, and thus all norms on $E \otimes F$ are equivalent. There is an isomorphism of vector spaces $\mathcal{B}(E, F') \to (E \otimes F)'$ defined by

$$\langle T, u\rangle = \sum_{i=1}^n \langle T(x_i), y_i\rangle \qquad \left(T \in \mathcal{B}(E, F'), u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F\right).$$

As $F \in \mathrm{FIN}$, $\mathcal{B}(E, F') = \mathcal{F}(E, F') = E' \otimes F'$, so that $(E \otimes F)' = E' \otimes F'$. Explicitly, the duality is

$$\langle u, v\rangle = \sum_{i=1}^n \sum_{j=1}^m \langle \mu_i, x_j\rangle\langle \lambda_i, y_j\rangle,$$

for $u = \sum_{i=1}^n \mu_i \otimes \lambda_i \in E' \otimes F'$ and $v = \sum_{j=1}^m x_j \otimes y_j \in E \otimes F$. As $E' \otimes F' = \mathcal{F}(E, F')$ and $E \otimes F = \mathcal{F}(E', F)$, let $T_u \in \mathcal{F}(E, F')$ and $T_v \in \mathcal{F}(E', F)$ be the operators represented by u and v, respectively. Then we have

$$T_u \circ T_v' = \sum_{j=1}^m y_j \otimes T_u(x_j) \in \mathcal{F}(F'), \qquad \langle u, v \rangle = \sum_{j=1}^m \langle T_u(x_j), y_j \rangle = \operatorname{tr}(T_u \circ T_v'),$$

the trace of $T_u \circ T_v'$. The duality using the trace is often referred to as *trace duality*. Note that $\operatorname{tr}(T_u \circ T_v') = \operatorname{tr}(T_v' \circ T_u)$, a property which is useful in calculations.

Definition 3.4. Let α be a tensor norm. Then the dual tensor norm to α is α', and is given by setting

$$(E \widehat{\otimes}_\alpha F)' = E' \widehat{\otimes}_{\alpha'} F'$$

for $E, F \in \mathrm{FIN}$, and extending α' to all Banach spaces by finite-generation. Define $\check{\alpha}$ to be the tensor norm $(\alpha')^t$, called the *adjoint* of α.

Of course, we can show that α' is a tensor norm. We then have that $\alpha'' = \alpha$, $\varepsilon' = \pi$ and $\pi' = \varepsilon$. So for $E, F \in \mathrm{FIN}$, we have $(E \widehat{\otimes} F)' = \mathcal{B}(E, F') = E' \check{\otimes} F'$ and that $\mathcal{B}(E, F)' = (E' \check{\otimes} F)' = E \widehat{\otimes} F'$.

The picture is more complicated for infinite-dimensional Banach spaces, due to our insisting that tensor norms are finitely-generated (which is necessary to ensure that, for example, $\alpha'' = \alpha$). For a tensor norm α define α^s by the embedding $E \widehat{\otimes}_{\alpha^s} F \to (E' \widehat{\otimes}_\alpha F')'$ for *any* Banach spaces E and F. Thus $\alpha^s = \alpha'$ on FIN, but not, in general, on infinite-dimensional spaces.

Definition 3.5. Let α be a tensor norm such that $(\alpha')^s = \alpha'' = \alpha$ on $E \otimes F$ whenever at least one of E and F are in FIN. Then α is said to be *accessible*.

Suppose further that we always have $(\alpha')^s = \alpha$. Then α is *totally accessible*.

We can show that ε is totally accessible, that π is accessible, and that α is accessible if and only if α' is accessible. Indeed, most common tensor norms are accessible; certainly any defined in [Rya02] are. However, as shown in [DF93, Section 31.6], there do exist tensor norms which are not accessible.

As shown in [Rya02, Chapter 2], for the projective tensor product, we have that $(E \widehat{\otimes} F)' = \mathcal{B}(E, F')$ with the duality as defined above (this follows easily by the definition of the projective tensor norm) for any Banach spaces E and F. As the swap map $E \widehat{\otimes} F \to F \widehat{\otimes} E$ is an isometry, we can naturally identify $(E \widehat{\otimes} F)'$ with $\mathcal{B}(F, E')$ as well as with $\mathcal{B}(E, F')$.

Let α be some tensor norm. As $\alpha \leq \pi$ for each pair of Banach spaces E and F, the formal identity map $I_\alpha : E \widehat{\otimes} F \to E \widehat{\otimes}_\alpha F$ is norm decreasing. For $\mu \in (E \widehat{\otimes}_\alpha F)'$, we then have

$$T := I_\alpha'(\mu) \in (E \widehat{\otimes} F)' = \mathcal{B}(E, F').$$

A check shows that

$$\left\langle \mu \sum_{i=1}^n x_i \otimes y_i \right\rangle = \sum_{i=1}^n \langle T(x_i), y_i \rangle \qquad \left(\sum_{i=1}^n x_i \otimes y_i \in E \otimes F \right),$$

so that we can identify $(E \widehat{\otimes}_\alpha F)'$ with a subspace of $\mathcal{B}(E, F')$, denoted by $\mathcal{B}_{\alpha'}(E, F')$, the *$\alpha'$-integral operators*, and give it the norm $\|\cdot\|_{\alpha'}$ induced by the identification of $\mathcal{B}_{\alpha'}(E, F')$ with $(E \widehat{\otimes}_\alpha F)'$. This notation is chosen because we have

$$\mathcal{B}_{\alpha'}(E, F') = (E \widehat{\otimes}_\alpha F)' = E' \widehat{\otimes}_{\alpha'} F' \qquad (E, F \in \mathrm{FIN}).$$

Again, the duality can be explicitly defined by using the trace, at least when dual spaces are being used. For let $u = \sum_{i=1}^n \mu_i \otimes y_i \in E' \widehat{\otimes}_\alpha F$ and $T \in \mathcal{B}_{\alpha'}(E', F')$, and let $S \in \mathcal{F}(E, F)$ be the operator induced by u. Then we have

$$\begin{aligned} \langle T, u \rangle &= \sum_{i=1}^n \langle T(\mu_i), y_i \rangle = \operatorname{tr}\Big(\sum_{i=1}^n \kappa_F(y_i) \otimes T(\mu_i) \Big) = \operatorname{tr}(T \circ S') \\ &= \sum_{i=1}^n \langle T' \kappa_F(y_i), \mu_i \rangle = \operatorname{tr}\Big(\sum_{i=1}^n T' \kappa_F(y_i) \otimes \mu_i \Big) = \operatorname{tr}(S' \circ T). \end{aligned}$$

The ε-integral operators are just the bounded operators. We call the π-integral operators just the *integral operators* and denote them by $\mathcal{I}(E, F') = \mathcal{B}_\pi(E, F')$.

Proposition 3.6. *Let E and F be Banach spaces, let $T \in \mathcal{B}(E, F)$, and let α be a tensor norm. The following are equivalent:*

1. *T is an α-integral operator.*
2. *$\kappa_F T : E \to F''$ is an α-integral operator.*
3. *$T'' : E'' \to F''$ is an α-integral operator.*
4. *$T' : F' \to E'$ is an α^t-integral operator.*

Furthermore $\|T\|_\alpha = \|\kappa_F T\|_\alpha = \|T''\|_\alpha = \|T'\|_{\alpha^t}$.

Let D and G be Banach spaces, let $S \in \mathcal{B}(D, E)$ and $R \in \mathcal{B}(F, G)$. Then $RTS \in \mathcal{B}_\alpha(D, G)$ and $\|RTS\|_\alpha \leq \|R\| \|T\|_\alpha \|S\|$.

Proof. See [Rya02, Section 8.1]. □

Hence we have the following isometric inclusions

$$\mathcal{B}_\alpha(E, F) \subseteq (E \widehat{\otimes}_{\alpha'} F')' = \mathcal{B}_\alpha(E, F'') \subseteq \mathcal{B}_\alpha(E'', F''),$$

noting that $T'' \kappa_E = \kappa_F T$ for any $T \in \mathcal{B}(E, F)$. The final part of the above proposition shows that the α-integral operators are an *operator ideal* in the sense of Pietsch (see [Pie80]). In particular, $\mathcal{B}_\alpha(E)$ is, algebraically, an ideal in $\mathcal{B}(E)$.

Definition 3.7. An *operator ideal* $\mathfrak{U}$ is an assignment, to each pair of Banach spaces E and F, a subspace $\mathfrak{U}(E, F) \subseteq \mathcal{B}(E, F)$ such that:

1. There is a norm u on $\mathfrak{U}(E, F)$ such that $(\mathfrak{U}(E, F), u)$ is a Banach space.
2. $\mathcal{F}(E, F) \subseteq \mathfrak{U}(E, F)$, and for $\mu \in E'$ and $x \in F$, for the one-dimensional operator $\mu \otimes x \in \mathcal{F}(E, F)$, we have that $u(\mu \otimes x) = \|\mu\| \|x\|$.
3. For Banach spaces D and G, $T \in \mathfrak{U}(E, F)$, $S \in \mathcal{B}(D, E)$ and $R \in \mathcal{B}(F, G)$, $RTS \in \mathfrak{U}(D, G)$, and $u(RTS) \leq \|R\| u(T) \|S\|$.

If $\mathfrak{U}(E, F)$ is always a closed subspace of $\mathcal{B}(E, F)$, then we say that $\mathfrak{U}$ is a *closed operator ideal*.

Note that some sources give a more general definition for the term "operator ideal". For each tensor norm α, we see that $\mathcal{B}_\alpha$ is an operator ideal for the norm $\|\cdot\|_\alpha$; it is rarely closed. The assignment $\mathcal{A}(E, F)$ is a closed operator ideal, and by condition (2) we see that it is the smallest closed operator ideal. An operator $T : E \to F$ is *compact* if T maps the unit ball of E into a relatively norm-compact subset of F. In this case we write $T \in \mathcal{K}(E, F)$, and write $\mathcal{K}(E, E) = \mathcal{K}(E)$. Then $\mathcal{K}$ is a closed operator ideal. Similarly, the collection of weakly-compact operators, $\mathcal{W}(E, F)$, is also a closed operator ideal.

Theorem 3.8. *Let E and F be Banach spaces, and $T \in \mathcal{B}(E,F)$. Then $T \in \mathcal{K}(E,F)$ if and only if $T' \in \mathcal{K}(F',E')$. Moreover, the following are equivalent:*
(1) $T \in \mathcal{W}(E,F)$.
(2) $T' \in \mathcal{W}(F',E')$.
(3) $T''(E'') \subseteq \kappa_F(F)$.

Proof. These are standard results, and are Schauder's Theorem and Gantmacher's Theorem, respectively. □

Theorem 3.9. *Let E and F be Banach spaces, and $T \in \mathcal{B}(E,F)$. Then $T \in \mathcal{W}(E,F)$ if and only if there exists a reflexive Banach space G, $R \in \mathcal{B}(E,G)$ and $S \in \mathcal{B}(G,F)$ with $T = S \circ R$. Furthermore, we can choose G, S and R so that R has dense range and the same norm and kernel as T, and such that S is norm-decreasing and injective.*

Proof. This is [DFJP74]; see also the more accessible sketch proof in [Pal94, Section 1.7.8]. □

3.2. Nuclear and integral operators; the approximation property.

Definition 3.10. Let E and F be Banach spaces, and let α be a tensor norm. Then there is a natural, norm-decreasing map $J_\alpha : E'\widehat{\otimes}_\alpha F \to \mathcal{B}(E,F)$ given by

$$J_\alpha(\mu \otimes y)(x) = \langle \mu, x\rangle y \qquad (x \in E, y \in F, \mu \in E'),$$

and linearity and continuity. The image of J_α, equipped with the quotient norm, is the set of *α-nuclear operators*, denoted $\mathcal{N}_\alpha(E,F)$, with norm $\|\cdot\|_{\mathcal{N}_\alpha}$. The *nuclear operators*, $\mathcal{N}(E,F)$, are the π-nuclear operators.

We can check that the α-nuclear operators form an operator ideal. For $E, F \in$ FIN, we have $E'\widehat{\otimes}_\alpha F = (E\widehat{\otimes}_{\alpha'} F')' = \mathcal{B}_\alpha(E,F)$, so that the α-integral and α-nuclear operators coincide for finite-dimensional spaces.

Proposition 3.11. *Let E and F be Banach spaces. Then the map $J_\alpha : E'\widehat{\otimes}_\alpha F \to \mathcal{B}(E,F)$ maps into $\mathcal{B}_\alpha(E,F)$, and the arising inclusion $\mathcal{N}_\alpha(E,F) \to \mathcal{B}_\alpha(E,F)$ is norm-decreasing; that is, $\|T\|_{\mathcal{N}_\alpha} \geq \|T\|_\alpha$ for each $T \in \mathcal{N}_\alpha(E,F)$.*

Proof. This follows from the finite-generation of α-integral operators. There is a short discussion in [Rya02, Section 8.1]. □

To say more on the relationship between $\mathcal{N}_\alpha$ and $\mathcal{B}_\alpha$ we need to study ideas which have their roots in the initial study of tensor norms in [Grot53] and [Sch50].

Definition 3.12. Let E be a Banach space. Then E has the *approximation property* if the map $J_\pi : E'\widehat{\otimes} E \to \mathcal{N}(E)$ is injective.

There are numerous equivalent definitions of the approximation property: see [Rya02, Proposition 4.6] or [DU77, Section 3, Chapter VIII] for example. We can show that for $1 \leq p \leq \infty$ and any measure μ, $L^p(\mu)$ and $L^p(\mu)'$ have the approximation property (indeed, they have the metric approximation property as defined below). Furthermore, $C(X)$ and $C(X)'$ have the (metric) approximation property for each compact space X. There are spaces without the approximation property (the first was constructed in [Enf73]). In fact, for $p \neq 2$, l^p contains subspaces without the approximation property (see [Sza78]), and $\mathcal{B}(l^2)$ does not have the approximation property (see [Sza81]).

Proposition 3.13. *Let E and F be Banach spaces, and let $T \in \mathcal{B}(E,F)$. Suppose that E' has the approximation property and $T' \in \mathcal{N}(F',E')$. Then $T \in \mathcal{N}(E,F)$.*

Proof. This is [Rya02, Proposition 4.10]. □

We wish to give a more concrete description of $\mathcal{I}(E,F)$.

Theorem 3.14. *Let E and F be Banach spaces, and let $T \in \mathcal{B}(E,F)$. Then the following are equivalent:*

(1) $T \in \mathcal{I}(E,F)$.
(2) $T' \in \mathcal{I}(F',E')$.
(3) *There exists a finite measure space (Ω,Σ,ν) and operators $S : E \to L^\infty(\nu)$ and $R : L^1(\nu) \to F''$ such that if $I : L^\infty(\nu) \to L^1(\nu)$ is the formal identity map, then $RIS = \kappa_F T$:*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "S"] & F \arrow[r, "\kappa_F"] & F'' \\ L^\infty(\nu) \arrow[rr, "I"] & & L^1(\nu). \arrow[u, "R"'] \end{tikzcd}$$

Furthermore, $\|T\|_\pi = \|T'\|_\pi = \inf \nu(\Omega)\|S\|\|R\|$ where the infimum is taken over all factorisations as above.

Proof. See [Rya02, Theorem 3.10]. □

Corollary 3.15. *Let E be a Banach space, and let $T \in \mathcal{I}(E)$. Then T is weakly-compact and* completely continuous (*that is, T takes weakly-convergent sequences to norm-convergent sequences*). *Thus the composition of two integral operators is compact, and so $\mathcal{I}(E) \neq \mathcal{B}(E)$ when E is infinite-dimensional.*

Proof. This follows directly from the factorisation given in the above theorem. For further details, see [Rya02, Proposition 3.20]. □

Note that for an infinite-dimensional Banach space, we have

$$(E'\widehat{\otimes}E)' = \mathcal{B}(E'), \qquad (E'\check{\otimes}E)' = \mathcal{I}(E'),$$

so we immediately see that π and ε are not equivalent norms on $E'\otimes E$. A construction by Pisier, [Pis83] (or [Pis86] for a more readable account) gives a separable Banach space P such that $P\check{\otimes}P = P\widehat{\otimes}P$. The space P does not have the approximation property, but it does satisfy $\mathcal{N}(P) = \mathcal{A}(P)$. In particular, the integral norm on $\mathcal{I}(P')$ is equivalent to the operator norm, as $\mathcal{N}(P)'$ is isometrically a subspace of $\mathcal{B}(P')$, namely $(\ker J_\pi)^\circ = \{T \in \mathcal{B}(P') : \langle T,u\rangle = 0 \ (J_\pi(u) = 0)\}$.

Following the theme of factorising maps, we have the following.

Definition 3.16. Let E be a Banach space such that for each $T \in \mathcal{B}(L^1([0,1]),E)$, there exists $S \in \mathcal{B}(L^1([0,1]),l^1)$ and $R \in \mathcal{B}(l^1,E)$ with $R\circ S = T$. Then E has the *Radon–Nikodým property.*

There are many equivalent formulations of the Radon–Nikodým property, see, for example, [DU77, Chapter VII, Section 6]. In particular, we have the following. Recall that a Banach space F is *separable* if F contains a dense, countable subset.

Theorem 3.17. *Let E be a Banach space. Then the following are equivalent:*

(1) *E' has the Radon–Nikodým property.*
(2) *Every separable subspace of E has a separable dual.*

In particular, $l^\infty(I)$ does not have the Radon–Nikodým property for any infinite set I. However, all separable dual spaces do have the Radon–Nikodým property.

Let E be a reflexive Banach space. Then E has the Radon–Nikodým property.

Proof. See [DU77, Chapter VII, Section 6]. □

To us, the Radon–Nikodým property is important because of the following.

Theorem 3.18. *Let E be a Banach space such that E' has the Radon–Nikodým property. Then, for each Banach space F, $\mathcal{N}(F, E') = \mathcal{I}(F, E')$ with the same norm.*

Proof. See [Rya02, Section 5.3]. □

Corollary 3.19. *Let E and F be Banach spaces, with E' or F' having the Radon–Nikodým property. Then $(F\check{\otimes}E)' = \mathcal{I}(F, E') = \mathcal{N}(F, E')$. If E' or F' have the approximation property, then $(F\check{\otimes}E)' = F'\widehat{\otimes}E'$.*

In particular, if E is a Banach space with E' or E'' having the Radon–Nikodým property, then $\mathcal{A}(E)' = (E'\check{\otimes}E)' = \mathcal{N}(E')$.

Proof. For all Banach spaces E and F, we have $(F\check{\otimes}E)' = \mathcal{I}(F, E')$, so that $(F\check{\otimes}E)' = \mathcal{N}(F, E')$ when E' has the Radon–Nikodým property. As $F\check{\otimes}E$ and $E\check{\otimes}F$ are isometrically isomorphic, we also have the result when F' has the Radon–Nikodým property. □

We hence see that, if E' has the Radon–Nikodým property and the approximation property, then $\mathcal{A}(E)' = E''\widehat{\otimes}E'$. Thus $\mathcal{A}(E)'' = (E''\widehat{\otimes}E')' = \mathcal{B}(E'')$. If $T \in \mathcal{A}(E)$, and $\Phi \otimes \mu \in E''\widehat{\otimes}E'$, we have

$$\langle \Phi \otimes \mu, T\rangle = \langle \Phi, T'(\mu)\rangle = \langle T''(\Phi), \mu\rangle = \langle \kappa_{\mathcal{A}(E)}(T), \Phi \otimes \mu\rangle.$$

We hence see that $\kappa_{\mathcal{A}(E)}(T) = T''$ for each $T \in \mathcal{A}(E)$. In particular, if E is reflexive and has the approximation property (so that E' has the Radon–Nikodým property and the approximation property) then $\mathcal{A}(E)'' = \mathcal{B}(E)$ and $\kappa_{\mathcal{A}(E)}$ is just the inclusion map $\mathcal{A}(E) \to \mathcal{B}(E)$. We shall shortly study these ideas in far greater detail.

Definition 3.20. Let E be a Banach space. Then E has the *bounded approximation property* if, for some $M > 0$, for each compact set $K \subseteq E$ and each $\varepsilon > 0$, there is $T \in \mathcal{F}(E)$ with $\|T\| \leq M$ and $\|T(x) - x\| < \varepsilon$ for each $x \in K$. If we can take $M = 1$, then E has the *metric approximation property.*

There are Banach spaces with the approximation property, but without the bounded approximation property (see [FJ73]). The following shows that the bounded approximation property is really a statement about nuclear and integral operators.

Theorem 3.21. *Let E be a Banach space. Then the following are equivalent:*

(1) *E has the bounded approximation property with bound M.*

(2) *For each Banach space F, the map*

$$E\widehat{\otimes}F \xrightarrow{\kappa_E\otimes\kappa_F} E''\widehat{\otimes}F'' \xrightarrow{J_\pi} \mathcal{N}(E',F'') \longrightarrow \mathcal{I}(E',F'') = \mathcal{A}(E,F')'$$

is bounded below by M^{-1}.

(3) *The map*

$$E\widehat{\otimes}E' \xrightarrow{\kappa_E\otimes\mathrm{Id}_{E'}} E''\widehat{\otimes}F' \xrightarrow{J_\pi} \mathcal{N}(E') \longrightarrow \mathcal{I}(E') = \mathcal{A}(E)'$$

is bounded below by M^{-1}.

Proof. This follows from the proof of [Rya02, Theorem 4.14]. □

Corollary 3.22. *Let E be a Banach space such that E' has the bounded approximation property. Then E has the bounded approximation property with a smaller* (*or equal*) *bound.*

Proof. Compare with [Rya02, Corollary 4.15], or see [DF93, Section 16.3]. □

Proposition 3.23. *Let E be a Banach space such that E' has the bounded approximation property with bound M. Then, for every Banach space F, the map*

$$E'\widehat{\otimes}F \xrightarrow{\mathrm{Id}_{E'}\otimes\kappa_F} E'\widehat{\otimes}F'' \xrightarrow{J_\pi} \mathcal{N}(E,F'') \longrightarrow \mathcal{I}(E,F'') = (E\check{\otimes}F')'.$$

is bounded below by M^{-1}.

Proof. This follows as above. □

Corollary 3.24. *Let E and F be Banach spaces such that at least one of E' or F has the bounded approximation property. Then $\mathcal{N}(E,F) = E'\widehat{\otimes}F$ is a closed subspace of $\mathcal{I}(E,F)$.*

Proof. If E' has the bounded approximation property, then $E'\widehat{\otimes}F = \mathcal{N}(E,F)$, and, by the above proposition, the map $E'\widehat{\otimes}F \to \mathcal{I}(E',F'')$ is bounded below. We can then show that this map takes values in $\mathcal{I}(E',F)$ and that $\mathcal{I}(E',F)$ is a closed subspace of $\mathcal{I}(E',F'')$. The argument in the case when F has the bounded approximation property is similar. □

Proposition 3.25. *Let E be a Banach space which has the approximation property, does not have the bounded approximation property, and be such that E' is separable. Then there exists $T \in \mathcal{B}(E)\setminus\mathcal{N}(E)$ with $T' \in \mathcal{N}(E')$.*

Proof. This is [FJ73, Proposition 3]. □

The metric approximation property also has links to accessibility of tensor norms.

Proposition 3.26. *Let α be a tensor norm. Then α is accessible if and only if $(\alpha')^s = \alpha$ on $E\otimes F$ whenever at least one of E and F has the metric approximation property.*

If E or F has only the bounded approximation property, then $(\alpha')^s$ and α are merely equivalent on $E\otimes F$ for an accessible tensor norm α.

Proof. See [Rya02, Section 7.1]. The statement about the bounded approximation property is an obvious generalisation. □

This allows us to extend Corollary 3.24. First note that this corollary actually states that the π-nuclear operators form a closed subspace of the π-integral operators, at least under some conditions. The property of π which allows this is the fact that π is accessible.

Proposition 3.27. *Let α be an accessible tensor norm. Then $\mathcal{N}_\alpha(E,F)$ is a subspace of $\mathcal{B}_\alpha(E,F)$ whenever E' or F has the metric approximation property.*

Proof. It is enough to show that for $T \in \mathcal{F}(E,F)$ we have $\alpha(T, E'\otimes F) = \|T\|_\alpha$. We have $\|T\|_\alpha = \|T''\|_\alpha$, where $T'' \in \mathcal{B}_\alpha(E'',F'') = (E''\widehat{\otimes}_{\alpha'} F')'$. Thus the embedding $E' \otimes F = \mathcal{F}(E,F) \to \mathcal{B}_\alpha(E,F)$ induces the same norm on $E'\otimes F$ as does the embedding $E'\otimes F \to (E''\widehat{\otimes}_{\alpha'} F')'$. This, however, is precisely the definition of the norm $(\alpha')^s$. We are hence done, as we know that $(\alpha')^s = \alpha$ on $E'\otimes F$, given that α is accessible and E' or F has the metric approximation property. $\square$

Proposition 3.28. *Let α be a totally accessible tensor norm. Then $\mathcal{N}_\alpha(E,F)$ is a subspace of $\mathcal{B}_\alpha(E,F)$ for any Banach spaces E and F.*

Proof. This is exactly the same as the above proof. $\square$

Proposition 3.29. *Let α be an accessible tensor norm. Then $\mathcal{N}_\alpha(E,F)$ is a closed* (*but not necessarily isometric*) *subspace of $\mathcal{B}_\alpha(E,F)$ whenever E' or F has the bounded approximation property.*

Proof. This follows by using Proposition 3.26. $\square$

We now give another application of these sorts of argument.

Theorem 3.30. *Let E be a reflexive Banach space, or let $E = F'$ for some Banach space F such that F' is separable. If E has the approximation property, then E has the metric approximation property.*

Proof. See [Rya02, Corollary 5.51] for the details of the following sketch. Suppose that $E = F'$, so that E has the Radon–Nikodým property by Theorem 3.17. By Theorem 3.21, and using the fact that E has the approximation property, we wish to prove that the map $\mathcal{N}(F') \to \mathcal{I}(F'')$ is an isometry onto its range. However, we know that $\mathcal{N}(F') = \mathcal{I}(F')$ and that the natural map $\mathcal{I}(F') \to \mathcal{I}(F'')$ is an isometry, so we are done. The argument when E is reflexive is similar. $\square$

Finally, we collect some miscellaneous results.

Theorem 3.31. *Let E, F and G be Banach spaces. Then we have:*

(1) *If $T \in \mathcal{I}(E,F)$ and $S \in \mathcal{W}(F,G)$, then $ST \in \mathcal{N}(E,G)$.*
(2) *If $S \in \mathcal{W}(E,F)$ and $T \in \mathcal{I}(F,G)$, then $\kappa_G TS \in \mathcal{N}(E,G'')$. Furthermore, if E' has the approximation property, then $TS \in \mathcal{N}(E,G)$.*

Proof. For (1), from Theorem 3.9, we see that as S is weakly-compact, we can find a reflexive Banach space D and $S_1 \in \mathcal{B}(F,D)$, $S_2 \in \mathcal{B}(D,G)$ so that $S = S_2S_1$. Then $S_1T \in \mathcal{I}(E,D)$, and as D is reflexive, $\mathcal{I}(E,D) = \mathcal{N}(E,D)$. Thus S_1T is nuclear, so $S_2S_1T = ST$ is also nuclear.

For (2), again factor S through a reflexive space D as $S = S_2S_1$. Then $S_2'T' \in \mathcal{I}(G',D') = \mathcal{N}(G',D')$, as D' is reflexive, so that $S'T' \in \mathcal{N}(G',E')$. Then $\kappa_G TS = T''S''\kappa_G$ is also nuclear. When E' has the approximation property, by Proposition 3.13, we see that as $S'T'$ is nuclear, so is TS. $\square$

Theorem 3.32 (Gröthendieck Composition Theorem). *Let α be a tensor norm, E, F and G be Banach spaces, $T \in \mathcal{B}_{\alpha'}(E,F)$ and $S \in \mathcal{B}_{\alpha^t}(F,G)$. If α is accessible or F has the metric approximation property, then $ST \in \mathcal{I}(E,G) = \mathcal{B}_\pi(E,G)$ with $\|ST\|_\pi \leq \|S\|_{\alpha^t}\|T\|_{\alpha'}$. If F has the bounded approximation property with bound M, then $ST \in \mathcal{I}(E,G) = \mathcal{B}_\pi(E,G)$ with $\|ST\|_\pi \leq M\|S\|_{\alpha^t}\|T\|_{\alpha'}$.*

Proof. See [Rya02, Theorem 8.5] while considering Proposition 3.26. The comment about the bounded approximation property is again an obvious generalisation. □

3.3. 2-nuclear operators. We shall now introduce the Chevet–Saphar tensor norms (see [Rya02, Section 6.2]) which lead to the p-nuclear operators in the same way in which the projective tensor norm leads to the nuclear operators.

Following the notation of [AS93], let E be a Banach space, let $1 \leq p \leq \infty$, let (x_i) be a sequence in E, and define

$$N_p(x_i) = \begin{cases} \Big(\sum_i \|x_i\|^p \Big)^{1/p} & : 1 \leq p < \infty, \\ \sup_i \|x_i\| & : p = \infty. \end{cases}$$

Similarly, define

$$\varepsilon_p(x_i) = \begin{cases} \sup\Big\{ \Big(\sum_i |\langle \mu, x_i \rangle|^p \Big)^{1/p} : \mu \in E', \|\mu\| \leq 1 \Big\} & : 1 \leq p < \infty, \\ \sup_i \|x_i\| & : p = \infty. \end{cases}$$

We may check that $\varepsilon_p(x_i)$ agrees with the norm of the operator $T : l^q \to E$ defined by $T(a) = \sum_{n=1}^\infty a_n x_n$ for $a = (a_n) \in l^q$, where $p^{-1} + q^{-1} = 1$. This follows as by $l^p - l^q$ duality,

$$\|T\| = \sup\left\{ \left| \sum_{n=1}^\infty a_n \langle \mu, x_n \rangle \right| : \mu \in E', \|\mu\| \leq 1, a \in l^q, \|a\| \leq 1 \right\} = \varepsilon_p(x_i).$$

The *Chevet–Saphar tensor norms* are defined, for Banach spaces E and F, $1 \leq p \leq \infty$, and $u \in E \otimes F$, as

$$d_p(u) = \inf\left\{ \varepsilon_q(x_i) N_p(y_i) : u = \sum_{i=1}^n x_i \otimes y_i \right\},$$
$$g_p(u) = \inf\left\{ N_p(x_i) \varepsilon_q(y_i) : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

Clearly we have that $d_p^t = g_p$. That these are tensor norms follows from a simple calculation; see [Rya02, Proposition 6.6].

We shall be mainly interested in the tensor norms d_2 and g_2, as these have much nicer properties than for other values of p. We say that an operator $T : E \to F$ is *p-summing* if there exists some constant $C > 0$ such that for all finite sequences $(x_i)_{i=1}^n$ in E, we have that $N_p(T(x_i)) \leq C\varepsilon_p(x_i)$, that is,

$$\sum_{i=1}^n \|T(x_i)\|^p \leq C^p \sup_{\mu \in E', \|\mu\| \leq 1} \sum_{i=1}^n |\langle \mu, x_i \rangle|^p.$$

The least such $C > 0$ is the p-summing norm of T, $\pi_p(T)$. We denote the class of p-summing operators from E to F as $\mathcal{P}_p(E,F)$, and $(\mathcal{P}_p(E,F), \pi_p)$ is a Banach space (see [Rya02, Section 6.3]). Indeed, we have that $(E\widehat{\otimes}_{d_p}F)' = \mathcal{P}_q(E, F')$, so that the d_p'-integral operators are precisely the p-summing operators.

We recall that we have $(E\widehat{\otimes}_\alpha F)' = \mathcal{B}_{\alpha'}(E, F')$ and hence that $(F\widehat{\otimes}_{\alpha^t} E)' = \mathcal{B}_{(\alpha^t)'}(F, E')$. By definition, $E\widehat{\otimes}_\alpha F$ and $F\widehat{\otimes}_{\alpha^t} E$ are isometrically isomorphic, so that $\mathcal{B}_{\alpha'}(E, F')$ and $\mathcal{B}_{(\alpha^t)'}(F, E')$ are also isometrically isomorphic. We may check that this isomorphism, $\phi : \mathcal{B}_{\alpha'}(E, F') \to \mathcal{B}_{(\alpha^t)'}(F, E')$, satisfies

$$\phi(T) = T' \circ \kappa_F, \quad \phi^{-1}(S) = S' \circ \kappa_E \qquad \big(T \in \mathcal{B}_{\alpha'}(E, F'), S \in \mathcal{B}_{(\alpha')^t}(E, F')\big).$$

We hence see that $T \in \mathcal{B}_{g_p'}(E, F') = (E\widehat{\otimes}_{g_p} F)'$ if and only if $T'\kappa_F \in \mathcal{P}_q(F, E')$, which we write, slightly inaccurately, as $(E\widehat{\otimes}_{g_p} F)' = \mathcal{P}_q(F, E')$.

Theorem 3.33 (Pietsch Domination Theorem)**.** *Let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$. The following are equivalent:*

(1) *T is 2-summing.*
(2) *There exists a compact Hausdorff space K, a regular Borel probability measure ν on K, and operators $U : E \to C(K)$ and $V : L_2(K, \nu) \to F$ such that the following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "V"] & F \\ C(K) \arrow[r, "J_2"] & L^2(K,\nu). \arrow[u, "U"'] \end{tikzcd}$$

Here $J_2 : C(K) \to L^2(K, \nu)$ is the inclusion map.

Furthermore, in this case, the infimum of $\|U\|\|V\|$ taken over all such factorizations is $\pi_2(T)$, and this infimum is obtained. We may replace $C(K)$ by $L^\infty(\nu)$ if we so wish.

Proof. See [Rya02, Theorem 6.19] and [Rya02, Proposition 6.23]. $\square$

There is an analogous statement for p-summing operators, but here we have to replace mapping from $L^p(\nu)$ to F by mapping from $L^p(\nu)$ to $l^\infty(I)$ for some index set I such that F embeds isometrically into $l^\infty(I)$ (see [Rya02, Theorem 6.19]). It is for this reason that we shall concentrate on 2-summing operators.

As a corollary, we immediately see that a p-summing operator is both weakly-compact and completely-continuous, and hence that the identity operator on a infinite-dimensional Banach space is never p-summing. For further details on p-summing operators and their many uses in Banach space theory, see [DJT95].

The *p-nuclear* operators are then precisely the g_p-nuclear operators, namely the operators in the range of the quotient map

$$E'\widehat{\otimes}_{g_p} F \to \mathcal{A}(E, F).$$

We denote these by $\mathcal{N}_p(E, F)$. If we replace g_p by d_p, then we obtain the *right p-nuclear operators*, which we shall (in a nonstandard way) denote by $\mathcal{N}_p^r(E, F)$. We hence see that $\mathcal{N}_p(E, F)'$ is isometrically a subspace of $\mathcal{P}_q(F, E')$, while $\mathcal{N}_p^r(E, F)'$ is isometrically a subspace of $\mathcal{P}_q(E, F')$. Furthermore, we see that when E has the approximation property, $N_p(E)$ is equal to $E'\widehat{\otimes}_{g_p} E$, and so $N_p(E)' = \mathcal{P}_q(E, E'')$, while $N_p^r(E)' = \mathcal{P}_q(E')$, which is perhaps more natural.

As $d_1 = g_1 = \pi$, we see that the (right) 1-nuclear operators are just the nuclear operators. However, for other values of p, the p-nuclear operators (as we might expect) have properties which differ from those of the nuclear (or approximable) operators. For example, by [AS93, Corollary 3.2], when $1 < p \leq \infty$, the space $E\widehat{\otimes}_{g_p}F$ is reflexive when E and F are. We hence see that $\mathcal{N}_p(E)$ is reflexive when E is, and $1 < p \leq \infty$. In particular, $\mathcal{N}_p(E)$ is trivially Arens-regular in this case. It is not known if $\mathcal{N}(E)$ or $\mathcal{A}(E)$ can be reflexive when E is infinite-dimensional, but this is known to be impossible if E has the approximation property (see [Rya02, Theorem 4.21]).

Proposition 3.34. *The tensor norms d_p and g_p are accessible for every p. Furthermore, d_2 and g_2 are totally accessible, and $d_2' = g_2$ so that $g_2' = d_2$.*

Proof. See [Rya02, Proposition 7.21] and [Rya02, Corollary 7.16]. □

We shall also be interested in g_p (and d_p)-integral operators, that is, the dual of $E\widehat{\otimes}_{g_p'}F$. There are precisely the *p-integral* operators, defined as $\mathcal{I}_p(E, F') = (E\widehat{\otimes}_{g_p'}F)' = \mathcal{B}_{g_p}(E, F')$ (we get the d_p-integral operators by using the fact that $d_p = g_p^t$). As for integral operators and p-summing operators, we have a factorization scheme. Indeed, we simply replace the space $L^1(\nu)$ occurring in Theorem 3.14 by $L^p(\nu)$. For further details, see [Rya02, Theorem 7.22]. As any closed subspace of a Hilbert space is 1-complemented, we see that the 2-integral operators are exactly the 2-summing operators. This is not true for other values of p, unless the Banach space in question has special properties.

Proposition 3.35. *The composition of any two 2-summing operators is nuclear.*

Proof. This is, for example, [Rya02, Corollary 8.6], but we sketch the proof here. Let $T \in \mathcal{P}_2(E, F) = \mathcal{B}_{g_2}(E, F)$ and $S \in \mathcal{P}_2(F, G) = \mathcal{B}_{g_2}(F, G)$, so that by the Gröthendieck Composition Theorem, as $d_2^t = g_2 = d_2'$, we see that $S \circ T$ is an integral operator. Furthermore, we have a factorization:

$$\begin{tikzcd}
E \arrow[rr, "T"] \arrow[d, "U"] & & F \arrow[rr, "S"] \arrow[dr, "W"] & & G \\
C(K) \arrow[r, "J_2^K"] & L^2(K, \nu) \arrow[ur, "V"] & & C(L) \arrow[r, "J_2^L"] & L^2(L, \upsilon). \arrow[u, "X"]
\end{tikzcd}$$

As J_2^L is a 2-summing operator, we see that $J_2^L \circ W \circ T$ is integral. Then X, as it maps from a Hilbert space, is weakly-compact, so $S \circ T = X \circ J_2^L \circ W \circ T$ is nuclear, by Theorem 3.31. □

We finish by investigating the effect of the Radon–Nikodým property on 2-nuclear operators, which was studied in [AS93].

Proposition 3.36. *Let E and F be Banach spaces such that E' has the Radon–Nikodým property. Then $\mathcal{N}_2(E, F) = \mathcal{P}_2(E, F) = \mathcal{B}_{g_2}(E, F)$ and $\mathcal{N}_2^r(F, E') = \mathcal{B}_{d_2}(F, E')$.*

Proof. This follows from [AS93, Proposition 1.1], as in the notation of that paper, for $1 \leq p < \infty$, $\mathcal{N}_p(E, F) = \mathcal{SI}_p(E, F)$, and from Theorem 3.33, we see immediately that $\mathcal{N}_2(E, F) = \mathcal{P}_2(E, F)$.

As d_2 is totally accessible, by Proposition 3.28, $\mathcal{N}_2^r(F,E')$ is a closed subspace of $\mathcal{B}_{d_2}(F,E')$. Let $T \in \mathcal{B}_{d_2}(F,E')$, so that $T'\kappa_E \in \mathcal{P}_2(E,F') = \mathcal{N}_2(E,F')$, and hence $\kappa_E' T'' \in \mathcal{N}_2^r(F'',E')$. Thus $T = \kappa_E'\kappa_{E'}T = \kappa_E' T''\kappa_F \in \mathcal{N}_2^r(F,E')$, as required. □

4. Arens products on operator ideals

We now make a first study of the Arens products on algebras of nuclear operators, and more generally on operator ideals.

Let $\mathfrak{U}$ be an operator ideal. The Arens regularity of $\mathfrak{U}(E)$ is closely related to the topology of E, a fact first shown (in less generality) in [You76, Theorem 3]. See also [Dal00, Section 2.6].

Theorem 4.1. *Let $\mathfrak{U}$ be an operator ideal, and let E be a Banach space such that $\mathfrak{U}(E)$ is Arens regular. Then E is reflexive.*

Proof. This follows from the proof of [Dal00, Theorem 2.6.23]. □

The converse is not true in full generality, for there exist reflexive Banach spaces E such that $\mathcal{B}(E)$ is not Arens regular (see [You76, Corollary 1]). However, for $\mathcal{A}(E)$ and $\mathcal{K}(E)$, we do have a converse, again first shown in [You76]. This will be proved below, in Theorem 5.39.

We now combine the Arens representations (recall the definitions from Section 2.1) with our knowledge of tensor norms. For most of the rest of this paper, we shall study the algebras $\mathcal{N}_\alpha(E)$ for various E and α. We shall now show how to use the maps ϕ_1 and θ_1 defined in Section 2.1 to get an interesting picture of $\mathcal{N}_\alpha(E)''$, at least for "well-behaved" α and E.

We start by defining the map ϕ_1 in a slightly more subtle manner. By the tensorial property, ϕ_1 is a map $F''\otimes F' \to \mathcal{A}'$. We can use this to define a seminorm on $F''\otimes F'$ by

$$\|u\|_0 = \|\phi_1(u)\| = \sup\{|\langle \phi_1(u), a\rangle| : a \in \mathcal{A}_{[1]}\} \qquad (u \in F''\otimes F').$$

Definition 4.2. Let $\mathcal{A}$ be a Banach algebra and F be a Banach left $\mathcal{A}$-module. Suppose that, for each $u = \sum_{i=1}^n \Lambda_i \otimes \mu_i \in F''\otimes F'$, we have

$$\begin{aligned}\sup\left\{\left|\sum_{i=1}^n \langle a\cdot\Lambda_i, \mu_i\rangle\right| : a \in \mathcal{A}_{[1]}\right\} &\geq \sup\left\{\left\|\sum_{i=1}^n \langle \Lambda_i, \lambda\rangle \mu_i\right\| : \lambda \in F'_{[1]}\right\} \\ &= \varepsilon(u, F''\otimes F').\end{aligned}$$

Then we say that $(\mathcal{A},F)$ is *tensorial*.

The reason we make this definition is the following. Let $(\mathcal{A},F)$ be tensorial. Then $\|\cdot\|_0$ is a norm on $F''\otimes F'$, and clearly $\varepsilon(u,F''\otimes F') \leq \|u\|_0 \leq \pi(u,F''\otimes F')$ for each $u \in F''\otimes F'$. Thus $\|\cdot\|_0$ is a reasonable crossnorm on $F''\otimes F'$.

Now, we might wonder if $\|\cdot\|_0$ is a tensor norm. Of course, we have not defined $\|\cdot\|_0$ on all spaces; this is a minor issue, as there are ways to extend to all pairs of Banach spaces, just by using the mapping property. However, even then we might not get a tensor norm, as we need $\|\cdot\|_0$ to be finitely generated. However, for the Banach algebras which we will study, we can say more.

Proposition 4.3. *Let E be a Banach space, let α be a tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then $(\mathcal{A},E)$ is tensorial. When α' is totally accessible, or α is accessible* *and E' has the metric approximation property, $\|\cdot\|_0$ is actually the nuclear norm $\|\cdot\|_{\mathcal{N}_{\alpha'}}$. When α is accessible and E' has the bounded approximation property, $\|\cdot\|_0$ is equivalent to $\|\cdot\|_{\mathcal{N}_{\alpha'}}$.*

Proof. As $\mathcal{A} = \mathcal{N}_\alpha(E)$ is a quotient of $E'\widehat{\otimes}_\alpha E$, we see that $\mathcal{A}'$ is, isometrically, a subspace of $\mathcal{B}_{\alpha'}(E')$, namely

$$\mathcal{A}' = (\ker J_\alpha)^\circ = \{T \in \mathcal{B}_{\alpha'}(E') : \langle T, v\rangle = 0 \ (v \in E'\widehat{\otimes}_\alpha E, J_\alpha(v) = 0)\}.$$

By Proposition 3.11, for $u \in E'' \otimes E'$, we have

$$\|u\|_0 = \|u\|_{\alpha'} \leq \|u\|_{\mathcal{N}_{\alpha'}} \leq \alpha'(u, E'' \otimes E'),$$

where we identify u with the operator in $\mathcal{F}(E')$ it induces. Let $u = \sum_{i=1}^n \Lambda_i \otimes \mu_i$, and let $x \in E$ and $\mu \in E'$. Then we have

$$\left|\sum_{i=1}^n \langle \Lambda_i, \mu\rangle\langle \mu_i, x\rangle\right| = |\langle u, \mu \otimes x\rangle| \leq \|u\|_0 \|\mu \otimes x\|_{\mathcal{N}_\alpha} = \|u\|_0 \|\mu\| \|x\|,$$

so that $\|u\|_0 \geq \varepsilon(u, E' \otimes E)$, and thus we see that $(\mathcal{A}, E)$ is tensorial.

When α' is totally accessible or α is accessible and E' has the metric approximation property, by Propositions 3.27 and 3.28, we immediately have

$$\|u\|_0 = \|u\|_{\alpha'} = \|u\|_{\mathcal{N}_{\alpha'}} \qquad (u \in E'' \otimes E'),$$

so that $\|\cdot\|_0 = \|\cdot\|_{\mathcal{N}_{\alpha'}}$. Similarly, Proposition 3.29 completes the proof. $\square$

Let α be a tensor norm and E be a Banach such that α' is totally accessible, or E' has the metric approximation property (we can generalise this to the bounded approximation property in a simple way). Then $\|\cdot\|_0 = \|\cdot\|_{\mathcal{N}_{\alpha'}}$, so that, by continuity, ϕ_1 extends to a map $E''\widehat{\otimes}_{\alpha'}E' \to \mathcal{N}_\alpha(E)'$, and we see that ϕ_1 agrees with the map $J_{\alpha'}$, so that ϕ_1 is a quotient operator. Thus, in particular, $\theta_1 : \mathcal{N}_\alpha(E)'' \to \mathcal{N}_{\alpha'}(E')' = (\ker J_{\alpha'})^\circ \subseteq \mathcal{B}_\alpha(E'')$ is an isometry.

When α is a general tensor norm and E is a general Banach space, we only have that $\|\cdot\|_0 \leq \|\cdot\|_{\mathcal{N}_{\alpha'}}$. However, we can still extend ϕ_1 by continuity to a map $\phi_1 : E''\widehat{\otimes}_{\alpha'}E' \to \mathcal{N}_\alpha(E)'$, but now ϕ_1 is only norm-decreasing. We can check that ϕ_1 still agrees with the map $J_{\alpha'}$; that is, for $u \in E''\widehat{\otimes}_{\alpha'}E'$, we have that $\phi_1(u)$ and $J_{\alpha'}(u)$ are the same operator in $\mathcal{B}_{\alpha'}(E')$, but the natural norms associated with these operators are different. Thus we also still have $\theta_1 : \mathcal{N}_\alpha(E)'' \to (\ker J_{\alpha'})^\circ \subseteq \mathcal{B}_\alpha(E'')$, but again, θ_1 is no longer an isometry, merely norm-decreasing.

Example 4.4. Let E be a Banach space such that E' has the bounded approximation property. Then $\phi_1 : E''\widehat{\otimes}E' \to \mathcal{A}(E)'$ is an isomorphism onto its range (if E' has the metric approximation property, then ϕ_1 is even an isometry). Thus $\theta_1 : \mathcal{A}(E)'' \to (\ker J_\pi)^\circ = \{0\}^\circ = \mathcal{B}(E'')$ is surjective. As $\mathcal{A}(E)$ clearly has weakly-compact action on E, we see that $\theta_1 : (\mathcal{A}(E)'', \Box) \to \mathcal{B}(E'')$ and $\theta_1 : (\mathcal{A}(E)'', \Diamond) \to (\mathcal{B}(E''), \mathcal{Q})$ are homomorphisms. In particular, let $\Xi \in \mathcal{A}(E)''$ be such that $\theta_1(\Xi) = \mathrm{Id}_{E''}$. Then we have

$$\theta_1(\Phi\Box\Xi) = \theta_1(\Phi), \qquad \theta_1(\Xi\Diamond\Phi) = \mathcal{Q}(\mathrm{Id}_{E''}) \circ \theta_1(\Phi) = \theta_1(\Phi) \qquad (\Phi \in \mathcal{A}(E)'').$$

In fact, using Proposition 5.3, we can show that Ξ is a mixed identity for $\mathcal{A}(E)''$.

5. Topological centres of biduals of operator ideals

We shall continue the study of topological centres of biduals of operators ideals which, in the case of the approximable operators, was started in [DL04]. This work will also allow us to say when some operator ideals are Arens regular. We note that some of the following work is similar to work done in [Gro87], where Grosser studies *multipliers* of algebras of approximable operators. As Grosser points out in this paper, many of these ideas and results have entered folklore (for example, the maps η and $\mathcal{Q}$). Grosser does not study topological centres, but presumably he could have drawn the conclusions which are found in [DL04], for example. We will instead develop the theory for general tensor norms, and study more general Banach spaces than those studied in [Gro87] or [DL04].

Let E be a reflexive Banach space with the metric approximation property (this is not much of a restriction, by Theorem 3.30). We shall see later, for example in Corollary 5.27 (compare with Example 4.4 above), that $\mathcal{A}(E)'' = \mathcal{B}(E)$ both as a Banach space and algebraically, so that $\mathcal{A}(E)$ is Arens regular, and $\mathcal{A}(E)''$ has a mixed identity, so that $\mathcal{A}(E)$ has a bounded approximate identity (see Proposition 2.1). Actually, we can take a more direct (and less circular) route. In [GW93], the question of when $\mathcal{A}(E)$ has a bounded approximate identity is investigated. It is worth noting that a lot of parallel development has occurred in this area; [GW93] is the best summary of available results, but many results were first proved elsewhere, and we urge the interested reader to consult this paper for further details.

Theorem 5.1. *Let E be a Banach space. Then the following are equivalent:*

(1) *E' has the bounded approximation property.*
(2) *$\mathcal{A}(E)$ has a bounded approximate identity.*
(3) *$\mathcal{A}(E')$ has a bounded left approximate identity.*
(4) *$\mathcal{A}(E)''$ has a mixed identity.*

Proof. The first three equivalences follow from [GW93, Theorem 3.3]. The equivalence of (4) and (2) follow by standard results (see Proposition 2.1). Alternatively, these results follow from Example 4.4 and standard properties of nuclear and integral operators. □

We will now turn our attention to ideals of α-nuclear operators for tensor norms α. Eventually we will come a full circle and use the above theorem. Our basic tool will be the Gröthendieck Composition theorem (Theorem 3.32), which will allow us, under many circumstances, to study integral operators (which are the dual of approximable operators, which hints as to why the above theorem will become useful).

Definition 5.2. Let E be a Banach space and α be a tensor norm. We say that (E,α) is a *Gröthendieck pair* if α is accessible or E has the bounded approximation property. In this case, $K(E,\alpha)$ is the constant arising from the Gröthendieck Composition theorem, so that $K(E,\alpha) = 1$ when α is accessible, and otherwise E has the bounded approximation property with bounded $K(E,\alpha)$.

Let E be a Banach space and α be a tensor norm. As in Proposition 4.3, $\mathcal{N}_\alpha(E)'$ is a subspace of $\mathcal{B}_{\alpha'}(E')$, and we can view $\phi_1 : E''\widehat{\otimes}_{\alpha'}E' \to \mathcal{N}_\alpha(E)'$ as a norm-decreasing map, which agrees, algebraically, with $J_{\alpha'}$.

Proposition 5.3. *Let E be a Banach space and α be a tensor norm. Let $\mathcal{A} = \mathcal{N}_\alpha(E)$, the α-nuclear operators on E, so that $\mathcal{A}'$ is a subspace of $\mathcal{B}_{\alpha'}(E')$. Then we have*

$$S \cdot R = R' \circ S \qquad R \cdot S = S \circ R' \qquad (R \in \mathcal{A}, S \in \mathcal{A}'),$$
$$\Phi \cdot S = \eta(\theta_1(\Phi) \circ S') \qquad S \cdot \Phi = \eta(\theta_1(\Phi)) \circ S \qquad (S \in \mathcal{A}', \Phi \in \mathcal{A}'').$$

Furthermore, we have that $\theta_1 : \mathcal{A}'' \to \mathcal{B}_\alpha(E'')$ is a norm-decreasing map.

When (E', α) is a Gröthendieck pair, for $S \in \mathcal{A}'$ and $\Phi \in \mathcal{A}''$, we also have

$$\Phi \cdot S \in \mathcal{I}(E'), \qquad \|\Phi \cdot S\|_\pi \leq K(E', \alpha)\|S\|\|\Phi\|.$$

Similarly, when (E'', α) is a Gröthendieck pair, we have

$$S \cdot \Phi \in \mathcal{I}(E'), \qquad \|S \cdot \Phi\|_\pi \leq K(E'', \alpha)\|S\|\|\Phi\|.$$

Proof. The first part is a simple calculation. For $\Phi \in \mathcal{A}'', S \in \mathcal{A}'$ and $R = \mu \otimes x \in \mathcal{A}$, we have

$$\begin{aligned}
\langle \Phi \cdot S, R \rangle &= \langle \Phi, \phi_1(R' \circ S) \rangle = \langle \Phi, \phi_1(S'(\kappa_E(x)) \otimes \mu) \rangle \\
&= \langle \theta_1(\Phi)(S'(\kappa_E(x))), \mu \rangle = \langle \eta(\theta_1(\Phi) \circ S'), R \rangle, \\
\langle S \cdot \Phi, R \rangle &= \langle \Phi, \phi_1(S \circ R') \rangle = \langle \Phi, \phi_1(\kappa_E(x) \otimes S(\mu) \rangle \\
&= \langle \theta_1(\Phi)(\kappa_E(x)), S(\mu) \rangle = \langle \eta(\theta_1(\Phi)) \circ S, R \rangle.
\end{aligned}$$

Thus we get the second part by linearity and continuity. That $\theta_1 : \mathcal{A}'' \to \mathcal{B}_\alpha(E'')$ is norm-decreasing follows by the discussion after Proposition 4.3.

For $\Phi \in \mathcal{A}''$, we have that $\theta_1(\Phi)' \in \mathcal{B}_{\alpha^t}(E''')$ and so

$$\eta(\theta_1(\Phi)) = \kappa_E' \circ \theta_1(\Phi)' \circ \kappa_{E'} \in \mathcal{B}_{\alpha^t}(E')$$

with $\|\eta(\theta_1(\Phi))\|_{\alpha^t} \leq \|\theta_1(\Phi)\|_\alpha$. Then the Gröthendieck Composition theorem says that, when (E', α) is a Gröthendieck pair, for $S \in \mathcal{A}'$ and $\Phi \in \mathcal{A}''$, we have $S \cdot \Phi = \eta(\theta_1(\Phi)) \circ S \in \mathcal{I}(E')$, and

$$\|S \cdot \Phi\|_\pi \leq K(E', \alpha)\|S\|_{\alpha'}\|\eta(\theta_1(\Phi))\|_{\alpha^t} \leq K(E', \alpha)\|S\|_{\alpha'}\|\Phi\|.$$

Similarly, when (E'', α) is a Gröthendieck pair, $\theta_1(\Phi) \in \mathcal{B}_\alpha(E'')$ and $S' \in \mathcal{B}_{\breve{\alpha}}(E'')$, so that $\theta_1(\Phi) \circ S' \in \mathcal{I}(E'')$ and $\|\theta_1(\Phi) \circ S'\|_\pi \leq K(E'', \alpha)\|\Phi\|\|S\|_{\alpha'}$. Hence

$$\Phi \cdot S = \kappa_E' \circ S'' \circ \theta_1(\Phi)' \circ \kappa_{E'} \in \mathcal{I}(E'),$$

and $\|\Phi.S\|_\pi \leq K(E'', \alpha)\|S\|_{\alpha'}\|\Phi\|$. □

For a tensor norm α, we turn $E'''\widehat{\otimes}_\alpha E''$ into a Banach algebra in the obvious way, by extending the multiplication from $\mathcal{F}(E'')$. Thus, for $u, v \in E'''\widehat{\otimes}_\alpha E''$, we have

$$u \circ v = (\mathrm{Id}_{E'''} \otimes J_\alpha(u))(v).$$

In particular, $J_\alpha : E'''\widehat{\otimes} E'' \to \mathcal{N}_\alpha(E'')$ becomes a homomorphism. We can also define $\star$ as a Banach algebra multiplication on $E'''\widehat{\otimes}_\alpha E''$ by setting

$$u \star v = (\mathrm{Id}_{E'''} \otimes \mathcal{Q}(J_\alpha(u)))(v) \qquad (u, v \in E'''\widehat{\otimes}_\alpha E'').$$

Theorem 5.4. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. There exist norm-decreasing homomorphisms*

$$\psi_1 : (E'''\widehat{\otimes}_\alpha E'', \circ) \to (\mathcal{A}'', \Box), \qquad \psi_2 : (E'''\widehat{\otimes}_\alpha E'', \star) \to (\mathcal{A}'', \Diamond),$$

such that $\theta_1 \circ \psi_1 = J_\alpha$ and $\theta_1 \circ \psi_2 = \mathcal{Q} \circ J_\alpha$. For $i = 1, 2$ and $T \in \mathcal{A}$, if $u \in E' \widehat{\otimes}_\alpha E$ is such that $T = J_\alpha(u)$, then we have $\psi_i(u'') = \psi_i((\kappa_{E'} \otimes \kappa_E)(u)) = \kappa_{\mathcal{A}}(T)$.

Proof. For $T \in \mathcal{F}(E'')$ and $S \in \mathcal{A}' \subseteq \mathcal{B}_{\alpha'}(E')$, we have that $\eta(T \circ S'), \eta(S' \circ T) \in \mathcal{F}(E')$. Thus we can define

$$\langle \psi_1(T), S \rangle = \operatorname{tr}(\eta(T \circ S')), \qquad \langle \psi_2(T), S \rangle = \operatorname{tr}(\eta(S' \circ T)) \qquad (T \in \mathcal{F}(E''), S \in \mathcal{A}').$$

We then have, recalling that $\mathcal{F}(E'') = E''' \otimes E''$, and that $\eta(S' \circ T) = \eta(T) \circ S$,

$$\begin{aligned}
|\langle \psi_1(T), S \rangle| &= |\operatorname{tr}(\kappa_E' \circ S'' \circ T' \circ \kappa_{E'})| = |\operatorname{tr}(\kappa_{E'} \circ \kappa_E' \circ S'' \circ T')| \\
&= |\langle \kappa_{E'} \circ \kappa_E' \circ S'', T \rangle| \leq \alpha(T, E''' \otimes E'') \|\kappa_{E'} \circ \kappa_E' \circ S''\|_{\alpha'} \\
&\leq \alpha(T, E''' \otimes E'') \|S''\|_{\alpha'} = \alpha(T, E''' \otimes E'') \|S\|_{\alpha'}, \\
|\langle \psi_2(T), S \rangle| &= |\operatorname{tr}(\eta(T) \circ S)| = |\operatorname{tr}(\kappa_E' \circ T' \circ \kappa_{E'} \circ S)| \\
&= |\operatorname{tr}(\kappa_{E'} \circ S \circ \kappa_E' \circ T')| \leq \alpha(T, E''' \otimes E'') \|\kappa_{E'} \circ S \circ \kappa_E'\|_{\alpha'} \\
&\leq \alpha(T, E''' \otimes E'') \|S\|_{\alpha'}.
\end{aligned}$$

Consequently, for $i = 1, 2$, $\|\psi_i(T)\| \leq \alpha(T)$, so that ψ_i extends by continuity to a norm-decreasing map $E''' \widehat{\otimes}_\alpha E'' \to \mathcal{A}''$.

For $\Lambda \in E'', \mu \in E'$ and $T \in \mathcal{N}_\alpha(E'')$, we have

$$\begin{aligned}
\langle \theta_1(\psi_1(T))(\Lambda), \mu \rangle &= \langle \psi_1(T), \phi_1(\Lambda \otimes \mu) \rangle = \operatorname{tr}(\eta(T \circ (\kappa_{E'}(\mu) \otimes \Lambda))) = \langle T(\Lambda), \mu \rangle, \\
\langle \theta_1(\psi_2(T))(\Lambda), \mu \rangle &= \langle \psi_2(T), \phi_1(\Lambda \otimes \mu) \rangle = \operatorname{tr}(\eta(T) \circ \phi_1(\Lambda \otimes \mu)) = \langle \Lambda, \eta(T)(\mu) \rangle.
\end{aligned}$$

Thus we see that $\theta_1 \circ \psi_1 = J_\alpha$ and $\theta_1 \circ \psi_2 = \mathcal{Q} \circ J_\alpha$.

For $T = \mu \otimes x \in \mathcal{A}$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}
\langle \psi_1(T''), S \rangle &= \operatorname{tr}(\eta(T'' \circ S')) = \operatorname{tr}(S \circ T') = \langle S, T \rangle = \langle \kappa_{\mathcal{A}}(T), S \rangle, \\
\langle \psi_2(T''), S \rangle &= \operatorname{tr}(\eta(T'') \circ S) = \operatorname{tr}(T' \circ S) = \operatorname{tr}(S \circ T') = \langle \kappa_{\mathcal{A}}(T), S \rangle.
\end{aligned}$$

By linearity, for $i = 1, 2$, we have $\psi_i(T'') = \kappa_{\mathcal{A}}(T)$ for $T \in E' \otimes E$. Thus, for $T = J_\alpha(u) \in \mathcal{N}_\alpha(E)$, suppose that (u_n) is a sequence in $E' \otimes E$ with $\alpha(u_n - u) \to 0$. For $i = 1, 2$, we have

$$\psi_i(u'') = \lim_{n\to\infty} \psi_i(u_n'') = \lim_{n\to\infty} \kappa_{\mathcal{A}}(u_n) = \kappa_{\mathcal{A}}(T),$$

as required.

We defer a calculation to Lemma 5.6 to follow. We claim that, for $T_1, T_2 \in \mathcal{B}(E'')$ and $S \in \mathcal{B}(E')$, we have $\eta(T_1 \circ T_2 \circ S') = \eta(T_1 \circ \mathcal{Q}(T_2 \circ S'))$. Then, for $T_1, T_2 \in \mathcal{F}(E'')$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}
\langle \psi_1(T_1) \Box \psi_1(T_2), S \rangle &= \langle \psi_1(T_1), \eta(\theta_1(\psi_1(T_2)) \circ S') \rangle = \langle \psi_1(T_1), \eta(T_2 \circ S') \rangle \\
&= \operatorname{tr}(\eta(T_1 \circ \mathcal{Q}(T_2 \circ S')) = \operatorname{tr}(\eta(T_1 \circ T_2 \circ S')) \\
&= \langle \psi_1(T_1 \circ T_2), S \rangle.
\end{aligned}$$

We see that $\psi_1 : (E''' \widehat{\otimes}_\alpha E'', \circ) \to (\mathcal{A}'', \Box)$ is a homomorphism.

Similarly, for $T_1, T_2 \in \mathcal{F}(E'')$ and $S \in \mathcal{A}'$, we have

$$\eta(\mathcal{Q}(T_1) \circ T_2) = \eta(\eta(T_1)' \circ T_2) = \eta(T_2) \circ \eta(T_1),$$

so that

$$\begin{aligned}\langle \psi_2(T_1)\Diamond\psi_2(T_2), S\rangle &= \langle \psi_2(T_2), \eta(\theta_1(\psi_2(T_1)))\circ S\rangle = \langle \psi_2(T_2), \eta(T_1)\circ S\rangle \\ &= \mathrm{tr}(\eta(T_2)\circ\eta(T_1)\circ S) = \mathrm{tr}(\eta(\mathcal{Q}(T_1)\circ T_2)\circ S) \\ &= \mathrm{tr}(\eta(T_1\star T_2)\circ S) = \langle \psi_2(T_1\star T_2), S\rangle.\end{aligned}$$

We see that $\psi_2 : (E'''\widehat{\otimes}E'', \star) \to (\mathcal{A}'', \Diamond)$ is a homomorphism. □

It would have been more natural to define the above maps from $\mathcal{N}_\alpha(E'')$. However, in general we cannot do this, as the next example shows.

Example 5.5. Let E be a Banach space with the approximation property such that E' does not have the approximation property. For example, let $E = l^2\widehat{\otimes}l^2$, so that $E' = \mathcal{B}(l^2)$ does not have the approximation property by [Sza81], but E does by [Rya02, Section 4.3]. Then let $\mathcal{A} = \mathcal{N}(E) = E'\widehat{\otimes}E$, so that $\mathcal{A}' = \mathcal{B}(E')$. Thus, if we had defined $\psi_1 : \mathcal{N}(E'') \to \mathcal{B}(E')'$, then we would have defined a trace on $\mathcal{N}(E')$, by

$$\mathrm{tr}(T) = \mathrm{tr}(\eta(T')) = \langle \psi_1(T'), \mathrm{Id}_{E'}\rangle \qquad (T\in\mathcal{N}(E')).$$

This is impossible, as $\mathcal{N}(E') \neq E''\widehat{\otimes}E'$, so that $\mathcal{N}(E')' \subsetneq \mathcal{B}(E'')$ and thus $\mathrm{Id}_{E'}' = \mathrm{Id}_{E''} \notin \mathcal{N}(E')'$.

Lemma 5.6. *Let E be a Banach space. For $T_1, T_2 \in \mathcal{B}(E'')$ and $S\in\mathcal{B}(E')$, we have $\eta(T_1\circ T_2\circ S') = \eta(T_1\circ\mathcal{Q}(T_2\circ S'))$.*

Proof. For $T_1, T_2 \in \mathcal{B}(E'')$, $S\in\mathcal{B}(E')$, $x\in E$ and $\mu\in E'$, we have

$$\begin{aligned}\langle (T_2\circ S'\circ\kappa_E)(x), \mu\rangle &= \langle \kappa_{E'}(\mu), (T_2\circ S'\circ\kappa_E)(x)\rangle = \langle (\kappa_E'\circ S''\circ T_2'\circ\kappa_{E'})(\mu), x\rangle \\ &= \langle \kappa_E(x), \eta(T_2\circ S')(\mu)\rangle = \langle (\mathcal{Q}(T_2\circ S')\circ\kappa_E)(x), \mu\rangle.\end{aligned}$$

We hence see that $T_2\circ S'\circ\kappa_E = \mathcal{Q}(T_2\circ S')\circ\kappa_E$. Thus we have

$$\begin{aligned}\eta(T_1\circ\mathcal{Q}(T_2\circ S')) &= \kappa_E'\circ\mathcal{Q}(T_2\circ S')'\circ T_1'\circ\kappa_{E'} = (T_2\circ S'\circ\kappa_E)'\circ T_1'\circ\kappa_{E'} \\ &= \kappa_E'\circ S''\circ T_2'\circ T_1'\circ\kappa_{E'} = \eta(T_1\circ T_2\circ S'),\end{aligned}$$

as required. □

The maps ψ_1 and ψ_2 allow us to study the topological centres of $\mathcal{N}_\alpha(E)''$, as they give us a concrete way of getting at interesting subalgebras of $\mathcal{N}_\alpha(E)''$.

Lemma 5.7. *Let E be a Banach space and $T\in\mathcal{B}(E'')$. Then the following are equivalent:*

1. *$\mathcal{Q}(T) = T$.*
2. *$T\in\mathcal{B}(E')^a$.*
3. *$\mathcal{Q}(T)\circ R = T\circ R$ for each $R\in\mathcal{F}(E')^a$.*

The following are also equivalent:

(a) *$T(E'')\subseteq\kappa_E(E)$.*
(b) *$\mathcal{Q}(R)\circ T = R\circ T$ for each $R\in\mathcal{B}(E'')$.*
(c) *$\mathcal{Q}(R)\circ T = R\circ T$ for each $R\in\mathcal{F}(E'')$.*

Proof. (1)⇔(2) is clear. Then, setting $R = \kappa_{E'}(\mu)\otimes\Lambda\in\mathcal{F}(E')^a$, we have

$$\mathcal{Q}(T)\circ R = \kappa_{E'}(\mu)\otimes\mathcal{Q}(T)(\Lambda), \qquad T\circ R = \kappa_{E'}(\mu)\otimes T(\Lambda),$$

so that we clearly have (1)⇔(3).

For the second equivalence, we clearly have (b)⇒(c). If (a) holds, then we can find $T_0 \in \mathcal{B}(E'', E)$ with $\kappa_E \circ T_0 = T$. We can verify that $\kappa_E'' \circ \kappa_E = \kappa_{E''} \circ \kappa_E$. Then, for $R \in \mathcal{B}(E'')$,

$$\mathcal{Q}(R)T = \kappa_{E'}' R'' \kappa_E'' \kappa_E T_0 = \kappa_{E'}' \kappa_{E''} R \kappa_E T_0 = R \kappa_E T_0 = RT,$$

so that (a)⇒(b). Finally, if (c) holds but (a) does not, then for some $\Lambda \in E''$, $M \in \kappa_E(E)^\circ \subseteq E'''$, $\langle M, T(\Lambda)\rangle = 1$, say. Let $R = M \otimes \Lambda \in \mathcal{F}(E'')$, so that $\eta(R) = \Lambda \otimes \kappa_E'(M) = 0$, as $M \in \kappa_E(E)^\circ$. Thus $\mathcal{Q}(R) \circ T = 0$, but $R(T(\Lambda)) = \Lambda\langle M, T(\Lambda)\rangle = \Lambda \neq 0$. This contradiction shows that (c)⇒(a). □

For a Banach space E and a tensor norm α, define the following subsets of $\mathcal{B}_\alpha(E'')$:

$$Z_1^0(E,\alpha) = \{T' : T \in \mathcal{B}_{\alpha^t}(E'),\ T \circ \kappa_E' \circ S'' = \kappa_E' \circ T'' \circ S''\ (S \in \mathcal{N}_\alpha(E)')\},$$
$$Z_2^0(E,\alpha) = \{T \in \mathcal{B}_\alpha(E'') : T(E'') \subseteq \kappa_E(E),\ T \circ S' \in \mathcal{W}(E)^{aa}\ (S \in \mathcal{N}_\alpha(E)')\}.$$

Proposition 5.8. *Let E be a Banach space, let α be a tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then*

$$\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{A}'')) \subseteq Z_1^0(E,\alpha), \qquad \theta_1(\mathfrak{Z}_t^{(2)}(\mathcal{A}'')) \subseteq Z_2^0(E,\alpha).$$

Furthermore,

$$\psi_2(T) \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'') \qquad (T \in \mathcal{F}(E'') \cap Z_1^0(E,\alpha)),$$
$$\psi_1(T) \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'') \qquad (T \in \mathcal{F}(E'') \cap Z_2^0(E,\alpha)).$$

Proof. By the homomorphism properties of θ_1, we see that

$$\theta_1(\Phi) \circ \theta_1(\Psi) = \mathcal{Q}(\theta_1(\Phi)) \circ \theta_1(\Psi) \qquad (\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}''), \Psi \in \mathcal{A}''),$$
$$\theta_1(\Psi) \circ \theta_1(\Phi) = \mathcal{Q}(\theta_1(\Psi)) \circ \theta_1(\Phi) \qquad (\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}''), \Psi \in \mathcal{A}'').$$

Then, as $\theta_1 \circ \psi_1$ is the identity on $\mathcal{F}(E'')$, setting $\Psi = \psi_1(R)$ for $R \in \mathcal{F}(E'')$, we have

$$\theta_1(\Phi) \circ R = \mathcal{Q}(\theta_1(\Phi)) \circ R \qquad (\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}''), R \in \mathcal{F}(E'')),$$
$$R \circ \theta_1(\Phi) = \mathcal{Q}(R) \circ \theta_1(\Phi) \qquad (\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}''), R \in \mathcal{F}(E'')).$$

So Lemma 5.7 immediately gives us

$$\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{A}'')) \subseteq \mathcal{B}(E')^a, \qquad \theta_1(\mathfrak{Z}_t^{(2)}(\mathcal{A}'')) \subseteq \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E)\}.$$

Recall that $\theta_1(\mathcal{A}'') \subseteq \mathcal{B}_\alpha(E'')$, so that, for example, $\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{A}'')) \subseteq \mathcal{B}_{\alpha^t}(E')^a$.

Furthermore, for $R = M \otimes \Lambda \in E''' \otimes E''$, $S \in \mathcal{A}'$ and $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$, let $T = \eta(\theta_1(\Phi))$ so that $\theta_1(\Phi) = T'$, so that

$$\begin{aligned}\langle \Phi \Box \psi_1(R), S\rangle &= \langle \Phi, \eta(R \circ S')\rangle = \langle \Phi, \Lambda \otimes \kappa_E'(S''(M))\rangle = \langle \theta_1(\Phi)(\Lambda), \kappa_E'(S''(M))\rangle \\ &= \langle (T'' \circ \kappa_{E'} \circ \kappa_E' \circ S'')(M), \Lambda\rangle = \langle (\kappa_{E'} \circ T \circ \kappa_E' \circ S'')(M), \Lambda\rangle \\ &= \langle \Lambda, (T \circ \kappa_E' \circ S'')(M)\rangle \\ \langle \Phi \Diamond \psi_1(R), S\rangle &= \langle \psi_1(R), \eta(\theta_1(\Phi)) \circ S\rangle = \langle \psi_1(R), T \circ S\rangle = \operatorname{tr}(\eta(R \circ S' \circ T')) \\ &= \operatorname{tr}\big(\eta\big(T''(S''(M)) \otimes \Lambda\big)\big) = \langle \Lambda, (\kappa_E' \circ T'' \circ S'')(M)\rangle.\end{aligned}$$

Thus we have $T \circ \kappa_E' \circ S'' = \kappa_E' \circ T'' \circ S''$, so that $\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{A}'')) \subseteq Z_1^0(E,\alpha)$.

For $S \in \mathcal{A}'$ and $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, letting $T \in \mathcal{B}(E'', E)$ be such that $\kappa_E \circ T = \theta_1(\Phi)$, we have

$$\eta(\theta_1(\Phi) \circ S') = \kappa_E' \circ S'' \circ (\kappa_E \circ T)' \circ \kappa_{E'} = \kappa_E' \circ S'' \circ T'.$$

For $R = \Lambda \otimes \mu \in E'' \otimes E'$, we hence have

$$\begin{aligned}
\langle \psi_1(R') \Box \Phi, S \rangle &= \langle \psi_1(R'), \eta(\theta_1(\Phi) \circ S') \rangle = \langle \psi_1(R'), \kappa_E' \circ S'' \circ T' \rangle \\
&= \operatorname{tr}(\kappa_E' \circ S'' \circ T' \circ R) = \langle \Lambda, (\kappa_E' \circ S'' \circ T')(\mu) \rangle \\
\langle \psi_1(R') \Diamond \Phi, S \rangle &= \langle \Phi, R \circ S \rangle = \langle \Phi, S'(\Lambda) \otimes \mu \rangle = \langle (\kappa_E \circ T \circ S')(\Lambda), \mu \rangle \\
&= \langle \mu, (T \circ S')(\Lambda) \rangle = \langle (S'' \circ T')(\mu), \Lambda \rangle.
\end{aligned}$$

Thus we have $\kappa_{E'} \circ \kappa_E' \circ S'' \circ T' = S'' \circ T'$. By Lemma 5.9 below, this is if and only if $\kappa_E \circ T \circ S' \in \mathcal{B}(E')^a$. By Lemma 5.10 below, we have

$$\mathcal{B}(E')^a \cap \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E)\} = \mathcal{W}(E)^{aa},$$

which implies that $\theta_1(\mathfrak{Z}_t^{(2)}(\mathcal{A}'')) \subseteq Z_2^0(E, \alpha)$.

Suppose that $R = \Lambda \otimes \mu \in \mathcal{F}(E')$ is such that $R' \in Z_1^0(E, \alpha)$, so that for $S \in \mathcal{A}'$, we have $R \circ \kappa_E' \circ S'' = \kappa_E' \circ R'' \circ S''$; that is

$$(S''' \circ \kappa_E'')(\Lambda) \otimes \mu = (S''' \circ \kappa_{E''})(\Lambda) \otimes \mu.$$

Thus, for $\Phi \in \mathcal{A}''$, we have

$$\begin{aligned}
\langle \psi_2(R') \Box \Phi, S \rangle &= \langle \psi_2(R'), \eta(\theta_1(\Phi) \circ S') \rangle = \operatorname{tr}(R \circ \eta(\theta_1(\Phi) \circ S')) \\
&= \langle \Lambda, \eta(\theta_1(\Phi) \circ S')(\mu) \rangle = \langle \Lambda, (\kappa_E' \circ S'' \circ \theta_1(\Phi)' \circ \kappa_{E'})(\mu) \rangle \\
&= \langle (S''' \circ \kappa_E'')(\Lambda), (\theta_1(\Phi)' \circ \kappa_{E'})(\mu) \rangle \\
&= \langle (S''' \circ \kappa_{E''})(\Lambda), (\theta_1(\Phi)' \circ \kappa_{E'})(\mu) \rangle \\
&= \langle (S'' \circ \theta_1(\Phi)' \circ \kappa_{E'})(\mu), \Lambda \rangle = \langle (\theta_1(\Phi) \circ S')(\Lambda), \mu \rangle \\
&= \langle \Phi, S'(\Lambda) \otimes \mu \rangle = \langle \Phi, R \circ S \rangle = \langle \psi_2(R') \Diamond \Phi, S \rangle.
\end{aligned}$$

Thus $\psi_2(Z_1^0(E, \alpha) \cap \mathcal{F}(E'')) \subseteq \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$.

Similarly, for $R = M \otimes \kappa_E(x) \in \mathcal{F}(E'') \cap Z_2^0(E, \alpha)$ and $S \in \mathcal{A}'$, we have $R \circ S' \in \mathcal{W}(E)^{aa}$, which is if and only if $S''(M) \otimes \kappa_E(x) \in \mathcal{W}(E)^{aa}$. This is if and only if $S''(M) = \kappa_{E'}(\mu)$ for some $\mu \in E'$. Then, for $\Phi \in \mathcal{A}''$, we have

$$\begin{aligned}
\langle \Phi \Diamond \psi_1(R), S \rangle &= \langle \psi_1(R), \eta(\theta_1(\Phi)) \circ S \rangle = \operatorname{tr}\left(\eta\big(R \circ S' \circ \mathcal{Q}(\theta_1(\Phi))\big)\right) \\
&= \operatorname{tr}\left(\kappa_E' \circ \eta(\theta_1(\Phi))'' \circ (\kappa_{E'}(\mu) \otimes \kappa_E(x))' \circ \kappa_{E'}\right) \\
&= \operatorname{tr}\left(\kappa_E' \circ \eta(\theta_1(\Phi))'' \circ (\kappa_E(x) \otimes \kappa_{E'}(\mu))\right) \\
&= \langle \kappa_E(x), (\kappa_E' \circ \eta(\theta_1(\Phi))'' \circ \kappa_{E'})(\mu) \rangle \\
&= \langle \eta(\theta_1(\Phi))(\mu), x \rangle = \langle (\kappa_E' \circ \theta_1(\Phi)' \circ \kappa_{E'})(\mu), x \rangle \\
&= \langle \theta_1(\Phi)(\kappa_E(x)), \mu \rangle = \langle \Phi, \kappa_E(x) \otimes \mu \rangle = \langle \Phi, \eta(S''(M) \otimes \kappa_E(x)) \rangle \\
&= \langle \Phi, \eta(R \circ S') \rangle = \langle \Phi \Box \psi_1(R), S \rangle.
\end{aligned}$$

Thus $\psi_1(\mathcal{F}(E'') \cap Z_2^0(E, \alpha)) \subseteq \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. $\square$

Lemma 5.9. *Let $S \in \mathcal{B}(E')$ and $T \in \mathcal{B}(E'', E)$. Then $\kappa_{E'} \circ \kappa_E' \circ S'' \circ T' = S'' \circ T'$ if and only if $\kappa_E \circ T \circ S' \in \mathcal{B}(E')^a$.*

Proof. We have that $\kappa_E \circ T \circ S' \in \mathcal{B}(E')^a$ if and only if $\mathcal{Q}(\kappa_E \circ T \circ S') = \kappa_E \circ T \circ S'$. Now, for $\Lambda \in E''$ and $\mu \in E'$, we have

$$\begin{aligned}\langle \mathcal{Q}(\kappa_E \circ T \circ S')(\Lambda), \mu\rangle &= \langle \Lambda, \eta(\kappa_E \circ T \circ S')(\mu)\rangle = \langle \Lambda, (\kappa_E' \circ S'' \circ T' \circ \kappa_E' \circ \kappa_{E'})(\mu)\rangle \\ &= \langle \Lambda, (\kappa_E' \circ S'' \circ T')(\mu)\rangle = \langle (\kappa_{E'} \circ \kappa_E' \circ S'' \circ T')(\mu), \Lambda\rangle,\end{aligned}$$

and also

$$\langle (\kappa_E \circ T \circ S')(\Lambda), \mu\rangle = \langle \mu, (T \circ S')(\Lambda)\rangle = \langle (S'' \circ T')(\mu), \Lambda\rangle.$$

Thus $\kappa_E \circ T \circ S' \in \mathcal{B}(E')^a$ if and only if $S'' \circ T' = \kappa_{E'} \circ \kappa_E' \circ S'' \circ T'$, as required. $\square$

Note that the above proof (and the lemma below) shows that

$$Z_2^0(E, \alpha) = \{T \in \mathcal{B}_\alpha(E'') : T(E'') \subseteq \kappa_E(E), \quad T \circ S' \in \mathcal{B}(E')^a \qquad (S \in \mathcal{N}_\alpha(E)')\}.$$

Lemma 5.10. *For a Banach space E and a tensor norm α, we have*

$$Z_1^0(E, \alpha) \cap Z_2^0(E, \alpha) = (W(E) \cap \mathcal{B}_\alpha(E))^{aa},$$
$$\mathcal{B}(E')^a \cap \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E)\} = \mathcal{W}(E)^{aa}.$$

Proof. Firstly, for $T \in \mathcal{B}(E')$, suppose that $T'(E'') \subseteq \kappa_E(E)$. Then we can find $T_0 \in \mathcal{B}(E'', E)$ with $\kappa_E \circ T_0 = T'$. Then, for $x \in E$ and $\mu \in E'$, we have

$$\langle \mu, T_0(\kappa_E(x))\rangle = \langle T'(\kappa_E(x)), \mu\rangle = \langle T(\mu), x\rangle,$$

so that $(T_0 \circ \kappa_E)' = T$. Furthermore, $(T_0 \circ \kappa_E)''(E'') = T'(E'') \subseteq \kappa_E(E)$, so that by Theorem 3.8, $(T_0 \circ \kappa_E) \in \mathcal{W}(E)$. Thus we have the second equality.

Now suppose that $T' \in Z_1^0(E, \alpha) \cap Z_2^0(E, \alpha)$, so that we immediately have $T = R'$ for some $R \in \mathcal{W}(E)$. Then $R'' \in \mathcal{B}_\alpha(E'')$, so that $R \in \mathcal{B}_\alpha(E)$, by Proposition 3.6.

Conversely, let $R \in \mathcal{W}(E) \cap \mathcal{B}_\alpha(E)$. Then, for $S \in \mathcal{B}(E')$, we have

$$R' \circ \kappa_E' \circ S'' = (\kappa_E \circ R)' \circ S'' = (R'' \circ \kappa_E)' \circ S'' = \kappa_E' \circ R''' \circ S'',$$

so that $R' \in Z_1^0(E, \alpha)$. We clearly have that $R'' \in Z_2^0(E, \alpha)$, completing the proof. $\square$

In some special cases, we can say more than the above proposition.

Theorem 5.11. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Suppose that $\mathcal{A}' \subseteq \mathcal{W}(E')$. Then*

$$Z_1^0(E, \alpha) = \mathcal{B}_{\alpha^t}(E')^a, \qquad Z_2^0(E, \alpha) = \kappa_E \circ \mathcal{B}_\alpha(E'', E),$$

where $\kappa_E \circ \mathcal{B}(E'', E) = \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E)\}$.

Consequently, we have

$$\psi_2(T') \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'') \qquad (T \in \mathcal{F}(E')),$$
$$\psi_1(\kappa_E \circ T) \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'') \qquad (T \in \mathcal{F}(E'', E)).$$

When E is not reflexive, the two topological centres of $\mathcal{A}''$ are distinct, neither contains the other, and both strictly contain $\kappa_{\mathcal{A}}(\mathcal{A})$.

Proof. For $S \in \mathcal{A}'$, as $S \in \mathcal{W}(E')$, $S''(E''') \subseteq \kappa_{E'}(E')$. Hence we have

$$\kappa_{E'} \circ \kappa_E' \circ S'' = S'',$$

as κ'_E is a projection of E''' onto E'. We immediately have $Z_1^0(E,\alpha) = \mathcal{B}_{\alpha^t}(E')^a$. Similarly, for $S \in \mathcal{A}'$ and $T \in \kappa_E \circ \mathcal{B}(E'',E)$, we have that $T = \kappa_E \circ T_0$ for some $T_0 \in \mathcal{B}(E'',E)$. As $\kappa_{E'} \circ \kappa'_E \circ S'' = S''$, for $\mu \in E'$ and $\Lambda \in E''$, we have

$$\begin{aligned}\langle (T_0 \circ S' \circ \kappa_E)''(\Lambda), \mu\rangle &= \langle \Lambda, (\kappa'_E \circ S'' \circ T'_0)(\mu)\rangle = \langle (S'' \circ T'_0)(\mu), \Lambda\rangle \\ &= \langle \mu, (T_0 \circ S')(\Lambda)\rangle = \langle (\kappa_E \circ T_0 \circ S')(\Lambda), \mu\rangle.\end{aligned}$$

Thus $\kappa_E \circ T_0 \circ S' = (T_0 \circ S' \circ \kappa_E)'' \in \mathcal{W}(E)^{aa}$, so that $T \in Z_2^0(E,\alpha)$. We conclude that $Z_2^0(E,\alpha) = \kappa_E \circ \mathcal{B}_\alpha(E'',E)$.

Suppose that E is not reflexive, so that $\mathcal{A}$ is not Arens regular. Let $\Lambda \in E''$ and $\mu \in E'$ be nonzero, and let $T_1 = \Lambda \otimes \mu \in \mathcal{F}(E')$, so that $\psi_2(T'_1) \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$. Suppose that $\psi_2(T'_1) = \kappa_{\mathcal{A}}(T)$ for some $T \in \mathcal{A}$, so that $T'_1 = \theta_1(\psi_2(T'_1)) = T''$, which is a contradiction. Also, $\theta_1(\psi_2(T'_1)) = T'_1 \notin Z_2^0(E,\alpha)$, so that $\psi_2(T'_1) \notin \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. Thus the first topological centre strictly contains $\kappa_{\mathcal{A}}(\mathcal{A})$ and is not contained in the second topological centre.

Similarly, let $M \in \kappa_E(E)^\circ \subseteq E'''$ and $x \in E$ be nonzero, and let $T_2 = M \otimes x \in \mathcal{F}(E'',E)$, so that $\psi_1(\kappa_E \circ T_2) \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. Again, we see that $\psi_1(\kappa_E \circ T_2) \notin \kappa_{\mathcal{A}}(\mathcal{A})$, and that $\psi_1(\kappa_E \circ T_2) \notin \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$, so that the second topological centre strictly contains $\kappa_{\mathcal{A}}(\mathcal{A})$ and is not contained in the first topological centre. $\square$

The above certainly applies when $\alpha = \varepsilon$, as then $\mathcal{A}' = \mathcal{A}(E)' = \mathcal{I}(E') \subseteq \mathcal{W}(E')$ (by Corollary 3.15). Similarly, it applies to the (right) p-nuclear operators. This is clear for the right p-nuclear operators by Theorem 3.33. For the p-nuclear operators, as $\mathcal{N}_p(E)$ is a quotient of $E'\widehat{\otimes}_{g_p}E$, we see that $\mathcal{N}_p(E)'$ is a subspace of $\mathcal{B}_{g'_p}(E')$. By the discussion before Theorem 3.33, $T \in \mathcal{B}_{g'_p}(E')$ if and only if $T'\kappa_E \in \mathcal{P}_q(E,E'')$, which implies that $T'\kappa_E \in \mathcal{W}(E,E'')$. Thus $T = \kappa'_E\kappa_{E'}T = (T'\kappa_E)'\kappa_{E'}$ is also weakly-compact, as required.

However, the above does not apply when $\alpha = \pi$ in the interesting case of when E is not reflexive, for when E has the approximation property, $\mathcal{A} = \mathcal{N}(E) = E'\widehat{\otimes}E$, and so $\mathcal{A}' = \mathcal{B}(E') \neq \mathcal{W}(E')$. We shall see later (Corollary 5.27) that this is a real problem, and not just an artifact of the method of proof.

The key to extending the above theorem is to look at the map θ_1.

Proposition 5.12. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Let*

$$I_1 = \ker\theta_1 \subseteq \mathcal{A}'', \qquad I_2 = \ker(\mathcal{Q}\circ\theta_1) \subseteq \mathcal{A}''.$$

Then I_1 is a closed ideal for either Arens product, and I_2 is a closed ideal in $(\mathcal{A}'',\Diamond)$. Furthermore, we have

$$\mathcal{A}''\Box I_1 = I_1\Diamond\mathcal{A}'' = I_2\Diamond\mathcal{A}'' = \{0\}.$$

In particular, $I_1\Box I_1 = I_1\Diamond I_1 = I_2\Diamond I_2 = \{0\}$. For $i = 1,2$, we have

$$I_1\Box\psi_i(E'''\widehat{\otimes}_\alpha E'') = \psi_i(E'''\widehat{\otimes}_\alpha E'')\Diamond I_1 = I_2\Box\psi_i(\kappa_{E'}(E')\widehat{\otimes}_\alpha\kappa_E(E)) = \{0\}.$$

Proof. By the homomorphism properties of θ_1, we see that I_1 is a closed ideal in $\mathcal{A}''$, with respect to either Arens product. By (the proof of) Proposition 2.5, for $T,S \in \mathcal{B}(E'')$, we have $\mathcal{Q}(T)\circ\mathcal{Q}(S) = \mathcal{Q}(\mathcal{Q}(T)\circ S)$, and so, for $\Phi \in I_2$ and $\Psi \in \mathcal{A}''$, we have

$$\mathcal{Q}(\theta_1(\Phi\Diamond\Psi)) = \mathcal{Q}(\mathcal{Q}(\theta_1(\Phi)) \circ \theta_1(\Psi)) = 0,$$
$$\mathcal{Q}(\theta_1(\Psi\Diamond\Phi)) = \mathcal{Q}(\mathcal{Q}(\theta_1(\Psi)) \circ \theta_1(\Phi)) = \mathcal{Q}(\theta_1(\Psi)) \circ \mathcal{Q}(\theta_1(\Phi)) = 0,$$

so that I_2 is a closed ideal in $(\mathcal{A}'', \Diamond)$.

For $S \in \mathcal{A}'$, $\Psi \in \mathcal{A}''$ and $\Phi \in I_1$, we have

$$\langle \Psi\Box\Phi, S\rangle = \langle \Psi, \eta(\theta_1(\Phi) \circ S')\rangle = 0, \qquad \langle \Phi\Diamond\Psi, S\rangle = \langle \Psi, \eta(\theta_1(\Phi)) \circ S\rangle = 0.$$

Thus we see that $\Psi\Box\Phi = \Phi\Diamond\Psi = 0$. Similarly, for $\Phi \in I_2$ and $\Psi \in \mathcal{A}''$, we have

$$\langle \Phi\Diamond\Psi, S\rangle = \langle \Psi, \eta(\theta_1(\Phi)) \circ S\rangle = 0 \qquad (S \in \mathcal{A}'),$$

so that $\Phi\Diamond\Psi = 0$ for each $\Psi \in \mathcal{A}''$.

For $u \in E''' \otimes E''$ and $S \in \mathcal{A}'$, we have that $\psi_i(u) \cdot S$ and $S \cdot \psi_i(u)$ are in $E'' \otimes E'$, for $i = 1, 2$. Thus, for $\Phi \in I_1$ and $i = 1, 2$, we have

$$\langle \Phi\Box\psi_i(u), S\rangle = \langle \Phi, \psi_i(u) \cdot S\rangle = 0 = \langle \Phi, S \cdot \psi_i(u)\rangle = \langle \psi_i(u)\Diamond\Phi, S\rangle.$$

Similarly, for $u = \kappa_{E'}(\mu) \otimes \kappa_E(x) \in E''' \otimes E''$, $S \in \mathcal{A}'$, $\Phi \in I_2$ and $i = 1, 2$, we have

$$\begin{aligned}\langle \Phi\Box\psi_i(u), S\rangle &= \langle \Phi, S \circ (\kappa_E(x) \otimes \mu)\rangle = \langle (\theta_1(\Phi)' \circ \kappa_{E'} \circ S)(\mu), \kappa_E(x)\rangle \\ &= \langle (\eta(\theta_1(\Phi)) \circ S)(\mu), x\rangle = 0.\end{aligned}$$

□

Theorem 5.13. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Suppose that (E', α) and (E'', α) are both a Gröthendieck pair (in particular, this holds if α is accessible). Then the two topological centres of $\mathcal{A}''$ strictly contain $\kappa_{\mathcal{A}}(\mathcal{A})$. Suppose that the two sets*

$$\overline{\mathrm{lin}}\{S \cdot \Phi : S \in \mathcal{A}',\ \Phi \in \mathcal{A}''\}, \qquad \overline{\mathrm{lin}}\{\Phi \cdot S : S \in \mathcal{A}',\ \Phi \in \mathcal{A}''\}$$

are distinct, and neither contains the other. Then the topological centres are distinct and neither contains the other.

Proof. By the Theorem 5.11, we may suppose that $\mathcal{A}' \not\subseteq \mathcal{W}(E')$. Furthermore, by continuity, we may suppose that $\mathcal{W}(E') \cap \mathcal{A}'$ is not dense in $\mathcal{A}'$. By Proposition 5.3, we have

$$\{S \cdot \Phi : S \in \mathcal{A}',\ \Phi \in \mathcal{A}''\} + \{\Phi \cdot S : S \in \mathcal{A}',\ \Phi \in \mathcal{A}''\} \subseteq \mathcal{I}(E') \subseteq \mathcal{W}(E').$$

Note also that $\phi_1(E'' \otimes E') = \mathcal{F}(E') \subseteq \mathcal{W}(E')$. Consequently, by the Hahn–Banach theorem, we can find a nonzero $\Phi \in \mathcal{A}''$ so that

$$\begin{aligned}&\langle \Phi, \phi_1(u)\rangle = 0 && (u \in E'' \otimes E'),\\ &\langle \Phi, S \cdot \Psi\rangle = \langle \Phi, \Psi \cdot S\rangle = 0 && (\Psi \in \mathcal{A}'', S \in \mathcal{A}').\end{aligned}$$

Then $\theta_1(\Phi) = 0$ so that $\Phi \in I_1$ (and hence $\Phi \notin \kappa_{\mathcal{A}}(\mathcal{A})$), and thus, for $\Psi \in \mathcal{A}''$ and $S \in \mathcal{A}'$, we have

$$\langle \Phi\Box\Psi, S\rangle = \langle \Phi, \Psi \cdot S\rangle = 0 = \langle \Phi\Diamond\Psi, S\rangle,$$

as $\Phi\Diamond\mathcal{A}'' = \{0\}$. Hence $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$. Similarly, we have

$$\langle \Psi\Diamond\Phi, S\rangle = \langle \Phi, S \cdot \Psi\rangle = 0 = \langle \Psi\Box\Phi, S\rangle,$$

as $\mathcal{A}''\Box\Phi = \{0\}$, so that $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$.

Define

$$X_1 = \overline{\mathrm{lin}}\{S \cdot \Phi : S \in \mathcal{A}', \Phi \in \mathcal{A}''\}, \qquad X_2 = \overline{\mathrm{lin}}\{\Phi \cdot S : S \in \mathcal{A}', \Phi \in \mathcal{A}''\}.$$

When $X_1 \not\subseteq X_2$, we can find a nonzero $\Phi \in \mathcal{A}''$ with $\theta_1(\Phi) = 0$, $\langle \Phi, \lambda \rangle = 0$ for each $\lambda \in X_2$, and $\langle \Phi, S_0 \cdot \Phi_0 \rangle \neq 0$ for some $S_0 \in \mathcal{A}'$ and $\Phi_0 \in \mathcal{A}''$. As above, we see that $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, but we have $\langle \Phi_0 \Diamond \Phi, S_0 \rangle = \langle \Phi, S_0 \cdot \Phi_0 \rangle \neq 0$, while $\langle \Phi_0 \Box \Phi, S_0 \rangle = 0$ as $\mathcal{A}'' \Box \Phi = \{0\}$. Thus $\Phi \notin \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$. Similarly, when $X_2 \not\subseteq X_1$, we have $\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \not\subseteq \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. □

We shall show later, in Corollary 5.27, that we cannot hope to completely remove the second condition in the above theorem.

To conclude, in slightly less than full generality, we have the following.

Theorem 5.14. *Let E be a Banach space which is not reflexive, let α be an accessible tensor norm and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. The topological centres of $\mathcal{A}''$ both strictly contain $\kappa_{\mathcal{A}}(\mathcal{A})$, and are both strictly contained in $\mathcal{A}''$. When $\mathcal{A}' \subseteq \mathcal{W}(E')$, the topological centres are distinct and neither contains the other.* □

5.1. When the dual space has the bounded approximation property. To say more about the topological centres of $\mathcal{N}_\alpha(E)''$, we need to impose some conditions on the Banach space E. Following Grosser, we shall now study the case when E' has the bounded approximation property. It turns out that this is an important special case which makes up, in some sense, for the fact that E is not assumed to be reflexive. For example, in [Gro87], the concept of *Arens semiregularity* is studied.

Definition 5.15. Let $\mathcal{A}$ be a Banach algebra. A *multiplier* on $\mathcal{A}$ is a pair (L, R) of maps in $\mathcal{B}(\mathcal{A})$ such that

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \qquad aL(b) = R(a)b \qquad (a, b \in \mathcal{A}).$$

The collection of multipliers on $\mathcal{A}$ is denoted by $\mathcal{M}(\mathcal{A})$.

Definition 5.16. Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity, let $\Xi \in \mathcal{A}''$ be a mixed identity, and let

$$D(\Xi) = \{L''(\Xi) : (L, R) \in \mathcal{M}(\mathcal{A})\}.$$

Then $\mathcal{A}$ is Arens semiregular if and only if the Arens products coincide on $D(\Xi)$, for each mixed identity Ξ (see [Gro84]).

In [Gro87], Grosser shows that $\mathcal{A}(E)$ (when E' has the bounded approximation property) is Arens semiregular when $\mathcal{I}(E') = \mathcal{N}(E')$. He also demonstrates (see [Gro87, Section 4]) that when E' has the bounded approximation property, we have $\mathcal{A}(E)'' = \mathcal{B}(E'') \oplus \ker \theta_1$.

This last property can be generalised to the α-nuclear case, and we shall see that, when E' has the bounded approximation property, we can completely identify the topological centres of $\mathcal{N}_\alpha(E)''$, at least when α is accessible.

Throughout this section, E will be a Banach space such that E' has the bounded approximation property. For a tensor norm α, as E has the bounded approximation property as well, $\mathcal{A} = \mathcal{N}_\alpha(E) = E' \widehat{\otimes}_\alpha E$ and so $\mathcal{A}' = \mathcal{B}_{\alpha'}(E')$. Suppose that α is accessible (so that α' is also accessible). As in Proposition 4.3, we see that $\phi_1 : \mathcal{N}_{\alpha'}(E') = E'' \widehat{\otimes}_{\alpha'} E' \to \mathcal{A}'$ is an isomorphism onto its range, and so $\theta_1 : \mathcal{A}'' \to \mathcal{B}_\alpha(E'')$ is surjective. When E' has the metric approximation property, or α' is totally accessible, ϕ_1 is actually an isometry onto its range, and so θ_1 is a quotient operator (metric surjection).

Theorem 5.17. *Let E be a Banach space such that E' has the bounded approximation property. Let α be an accessible tensor norm, and $\mathcal{A} = \mathcal{N}_\alpha(E)$. There exists a homomorphism, which is also an isomorphism onto its range, $\psi_1 : (\mathcal{B}_\alpha(E''), \circ) \to (\mathcal{A}'', \Box)$ such that $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{B}_\alpha(E'')}$. There also exists a bounded homomorphism $\psi_2 : (\mathcal{B}_\alpha(E''), \star) \to (\mathcal{A}'', \Diamond)$ such that $\theta_1 \circ \psi_2 = \mathcal{Q}$. For $i = 1, 2$ and $T \in \mathcal{A}$, we have $\psi_i(T'') = \kappa_{\mathcal{A}}(T)$. When E' has the metric approximation property, ψ_1 can be chosen to be an isometry and ψ_2 can be chosen to be norm-decreasing. Furthermore, these maps extend the maps defined in Theorem* 5.4*, when they are restricted to $\mathcal{F}(E'')$.*

Proof. As in Example 4.4, as E' has the bounded approximation property, we can find $\Xi \in \mathcal{I}(E')' = \mathcal{A}(E)''$ with $\theta_1(\Xi) = \mathrm{Id}_{E''}$. As α is accessible, by the Gröthendieck Composition theorem, for $T \in \mathcal{B}_\alpha(E'')$ and $S \in \mathcal{B}_{\alpha'}(E') = \mathcal{A}'$, we have that $S' \in \mathcal{B}_{\breve{\alpha}}(E'')$, so that $T \circ S' \in \mathcal{I}(E'')$, and hence $\eta(T \circ S') \in \mathcal{I}(E')$. Similarly, $\eta(T) \in \mathcal{B}_\alpha(E')$, and so $\eta(T) \circ S = \eta(S' \circ T) \in \mathcal{I}(E')$. Define, for $i = 1, 2$, $\psi_i : \mathcal{B}_\alpha(E'') \to \mathcal{A}''$ by

$$\langle \psi_1(T), S \rangle = \langle \Xi, \eta(T \circ S') \rangle, \qquad \langle \psi_2(T), S \rangle = \langle \Xi, \eta(T) \circ S \rangle$$

for $T \in \mathcal{B}_\alpha(E'')$ and $S \in \mathcal{B}_{\alpha'}(E')$. Then we have

$$|\langle \psi_1(T), S \rangle| \leq \|\Xi\| \|\eta(T \circ S')\|_\pi \leq \|\Xi\| \|T \circ S'\|_\pi \leq \|\Xi\| \|T\|_\alpha \|S\|_{\alpha'},$$

so that $\|\psi_1\| \leq \|\Xi\|$. Similarly, $\|\psi_2\| \leq \|\Xi\|$. As we form Ξ from a bounded approximate identity for $\mathcal{A}(E)$, by results in [GW93], we see that the smallest we can make $\|\Xi\|$ is the bound for which E' has the bounded approximation property. In particular, if E' has the metric approximation property, then ψ_1 and ψ_2 can be constructed to be norm-decreasing.

For $T \in \mathcal{F}(E'')$ and $S \in \mathcal{A}'$, we have

$$\langle \psi_1(T), S \rangle = \langle \Xi, \eta(T \circ S') \rangle = \mathrm{tr}(\eta(T \circ S')),$$
$$\langle \psi_2(T), S \rangle = \langle \Xi, \eta(T) \circ S \rangle = \mathrm{tr}(\eta(T) \circ S),$$

so that the maps ψ_i extend those defined in Theorem 5.4.

For $\Lambda \in E'', \mu \in E'$ and $T \in \mathcal{B}_\alpha(E'')$, we have

$$\eta(T \circ \phi_1(\Lambda \otimes \mu)') = \eta(\kappa_{E'}(\mu) \circ T(\Lambda)) = \phi_1(T(\Lambda) \otimes \mu).$$

Thus we have

$$\langle \theta_1(\psi_1(T))(\Lambda), \mu \rangle = \langle \Xi, \eta(T \circ \phi_1(\Lambda \otimes \mu)') \rangle = \langle \Xi, \phi_1(T(\Lambda) \otimes \mu) \rangle = \langle T(\Lambda), \mu \rangle,$$

as $\theta_1(\Xi) = \mathrm{Id}_{E''}$. Thus $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{B}_\alpha(E'')}$. As θ_1 is norm-decreasing, we see that ψ_1 is an isomorphism onto its range, and an isometry when E' has the metric approximation property (for a suitably chosen Ξ). Similarly, we have

$$\langle \theta_1(\psi_2(T))(\Lambda), \mu \rangle = \langle \Xi, \eta(T) \circ \phi_1(\Lambda \otimes \mu) \rangle = \langle \Lambda, \eta(T)(\mu) \rangle,$$

so that $\theta_1 \circ \psi_2 = \mathcal{Q}$.

For $T \in \mathcal{A} = E'\widehat{\otimes}_\alpha E$, we have $T \in \mathcal{B}_\alpha(E)$, so that $T'' \in \mathcal{B}_\alpha(E'')$. Suppose that $T = \mu \otimes x$. Then, for $S \in \mathcal{B}_{\alpha'}(E')$, we have

$$\begin{aligned}\langle \psi_1(T''), S\rangle &= \langle \Xi, \eta(T'' \circ S')\rangle = \langle \Xi, S \circ T'\rangle = \langle \Xi, \kappa_E(x) \otimes S(\mu)\rangle \\ &= \langle \kappa_E(x), S(\mu)\rangle = \langle S, T\rangle = \langle \kappa_{\mathcal{A}}(T), S\rangle, \\ \langle \psi_2(T''), S\rangle &= \langle \Xi, T' \circ S\rangle = \langle \Xi, S'(\kappa_E(x)) \otimes \mu\rangle \\ &= \langle S'(\kappa_E(x)), \mu\rangle = \langle S(\mu), x\rangle = \langle S, T\rangle = \langle \kappa_{\mathcal{A}}(T), S\rangle.\end{aligned}$$

By linearity and continuity, we see that $\psi_i(T'') = \kappa_{\mathcal{A}}(T)$ for $T \in \mathcal{A}$ and $i = 1, 2$.

By Lemma 5.6, for $T_1, T_2 \in \mathcal{B}(E'')$ and $S \in \mathcal{B}(E')$, we have $\eta(T_1 \circ T_2 \circ S') = \eta(T_1 \circ \mathcal{Q}(T_2 \circ S'))$. Then, for $T_1, T_2 \in \mathcal{B}_\alpha(E'')$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}\langle \psi_1(T_1)\Box\psi_1(T_2), S\rangle &= \langle \psi_1(T_1), \eta(\theta_1(\psi_1(T_2)) \circ S')\rangle = \langle \psi_1(T_1), \eta(T_2 \circ S')\rangle \\ &= \langle \Xi, \eta(T_1 \circ \mathcal{Q}(T_2 \circ S'))\rangle = \langle \Xi, \eta(T_1 \circ T_2 \circ S')\rangle \\ &= \langle \psi_1(T_1 \circ T_2), S\rangle.\end{aligned}$$

We see that $\psi_1 : (\mathcal{B}_\alpha(E''), \circ) \to (\mathcal{A}'', \Box)$ is a homomorphism. Similarly, we have

$$\begin{aligned}\langle \psi_2(T_1 \star T_2), S\rangle &= \langle \psi_2(\mathcal{Q}(T_1) \circ T_2), S\rangle = \langle \Xi, \eta(\mathcal{Q}(T_1) \circ T_2) \circ S\rangle \\ &= \langle \Xi, \eta(S' \circ \mathcal{Q}(T_1) \circ T_2)\rangle = \langle \Xi, \kappa_E' \circ T_2' \circ \eta(T_1)'' \circ S'' \circ \kappa_{E'}\rangle \\ &= \langle \Xi, \kappa_E' \circ T_2' \circ \kappa_{E'} \circ \eta(T_1) \circ S\rangle = \langle \Xi, \eta(T_2) \circ \eta(T_1) \circ S\rangle \\ &= \langle \Xi, \eta(T_2) \circ \eta(S' \circ T_1)\rangle = \langle \psi_2(T_2), \eta(S' \circ T_1)\rangle \\ &= \langle \psi_2(T_2), \eta(T_1) \circ S\rangle = \langle \psi_2(T_1)\Diamond\psi_2(T_2), S\rangle.\end{aligned}$$

We see that $\psi_2 : (\mathcal{B}_\alpha(E''), \star) \to (\mathcal{A}'', \Diamond)$ is a homomorphism. □

There is evidently some choice in the construction of ψ_1 and ψ_2, as we are free to choose a mixed identity $\Xi \in \mathcal{A}(E)''$. However, below we shall see that this is unimportant as far as the study of topological centres go.

As $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{B}_\alpha(E'')}$, we have that $\psi_1 \circ \theta_1$ is a projection of $\mathcal{A}''$ onto $\psi_1(\mathcal{B}_\alpha(E''))$. Thus we can write

$$\mathcal{A}'' = \mathcal{B}_\alpha(E'') \oplus \ker\theta_1 = \mathcal{B}_\alpha(E'') \oplus I_1,$$

with reference to Proposition 5.12.

For a Banach space E (such that E' has the bounded approximation property) and an accessible tensor norm α, define

$$\begin{aligned}Z_1(E, \alpha) &= \{T' : T \in \mathcal{B}_{\alpha^t}(E'),\ T \circ S \in \mathcal{N}_{\alpha'}(E'), \\ &\qquad\qquad \kappa_{E'} \circ T \circ \kappa_E' \circ S'' = T'' \circ S'' \quad (S \in \mathcal{B}_{\alpha'}(E'))\}, \\ Z_2(E, \alpha) &= \{T \in \mathcal{B}_\alpha(E'') : T(E'') \subseteq \kappa_E(E),\ T \circ S' \in \mathcal{N}_{\alpha'}(E')^a \quad (S \in \mathcal{B}_{\alpha'}(E'))\}, \\ X_1(E, \alpha) &= \overline{\mathrm{lin}}\{\eta(T \circ S') : S \in \mathcal{B}_{\alpha'}(E'),\ T \in \mathcal{B}_\alpha(E'')\} \subseteq \mathcal{B}_{\alpha'}(E'), \\ X_2(E, \alpha) &= \overline{\mathrm{lin}}\{T \circ S : S \in \mathcal{B}_{\alpha'}(E'),\ T \in \mathcal{B}_{\alpha^t}(E')\} \subseteq \mathcal{B}_{\alpha'}(E').\end{aligned}$$

By the Gröthendieck Composition Theorem, we see that X_1 and X_2 are subsets of $\overline{\mathcal{I}(E')}$, where the closure is taken with respect to the norm on $\mathcal{B}_{\alpha'}(E')$. Notice that in the definition of Z_1, for $T \in \mathcal{B}_{\alpha^t}(E')$ and $S \in \mathcal{B}_{\alpha'}(E')$, $T \circ S \in \mathcal{I}(E')$ by another application of the Gröthendieck Composition Theorem. If a reflexive Banach space can be interpolated somewhere (compare with the proof of Theorem 3.31) then we may even conclude that $T \circ S \in \mathcal{N}(E')$, so that as $\alpha' \leq \pi$, we automatically have that $T \circ S \in \mathcal{N}_{\alpha'}(E')$. A similar remark holds for $Z_2(E, \alpha)$.

Theorem 5.18. *Let E be a Banach space such that E' has the bounded approximation property, let α be an accessible tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then*

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \{\psi_2(T) + \Phi : T \in Z_1(E, \alpha),\ \Phi \in X_1(E, \alpha)^\circ\}.$$

Proof. Let $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$, so that we can write $\Phi = \Phi_0 + \psi_1(T')$ for some $\Phi_0 \in I_1$ and $T \in Z_1^0(E, \alpha)$, by Proposition 5.8, and the discussion above. Similarly, for $\Psi \in \mathcal{A}''$, let $\Psi = \Psi_0 + \psi_1(R)$, for some $\Psi_0 \in I_1$ and $R \in \mathcal{B}_\alpha(E'')$. Then we have, with reference to Proposition 5.12,

$$\begin{aligned} \Phi \Box \Psi &= \Phi \Box \psi_1(R) = \Phi_0 \Box \psi_1(R) + \psi_1(T' \circ R), \\ \Phi \Diamond \Psi &= \psi_1(T') \Diamond \Psi = \psi_1(T') \Diamond \Psi_0 + \psi_1(T') \Diamond \psi_1(R). \end{aligned}$$

Setting $\Psi_0 = 0$, we have

$$\Phi_0 \Box \psi_1(R) + \psi_1(T' \circ R) = \psi_1(T') \Diamond \psi_1(R) \qquad (R \in \mathcal{B}_\alpha(E'')), \tag{1}$$

and so we also have

$$\psi_1(T') \Diamond \Psi_0 = 0 \qquad (\Psi_0 \in I_1).$$

For $S \in \mathcal{B}_{\alpha'}(E') = \mathcal{A}'$, we thus have $\langle \Psi_0, S \cdot \psi_1(T') \rangle = \langle \Psi_0, T \circ S \rangle = 0$ for each $\Psi_0 \in I_1$. By the Hahn–Banach theorem, this holds if and only if

$$T \circ S \in \phi_1(E'' \widehat{\otimes}_{\alpha'} E') = \mathcal{N}_{\alpha'}(E') \qquad (S \in \mathcal{B}_{\alpha'}(E')),$$

as $\phi_1(E'' \widehat{\otimes}_{\alpha'} E') = \mathcal{N}_{\alpha'}(E')$ is a closed subspace of $\mathcal{A}'$, by Proposition 3.29, given that E' has the bounded approximation property. As $\mathcal{N}_{\alpha'}(E') \subseteq \mathcal{W}(E')$, we thus have that $\kappa_{E'} \circ \kappa_E' \circ T'' \circ S'' = T'' \circ S''$, and so we see that

$$Z_1^0(E, \alpha) \cap \{T' : T \in \mathcal{B}_{\alpha^t}(E'),\ T \circ S \in \mathcal{N}_{\alpha'}(E')\ (S \in \mathcal{B}_{\alpha'}(E')\} = Z_1(E, \alpha).$$

Then, for $S \in \mathcal{B}_{\alpha'}(E') = \mathcal{A}'$ and $R \in \mathcal{B}_\alpha(E'')$, we have

$$\begin{aligned} \langle \Phi_0 \Box \psi_1(R) + \psi_1(T' \circ R), S \rangle &= \langle \Phi_0, \eta(R \circ S') \rangle + \langle \Xi, \eta(T' \circ R \circ S') \rangle \\ &= \langle \Phi_0, \eta(R \circ S') \rangle + \langle \psi_1(T'), \eta(R \circ S') \rangle, \end{aligned}$$

as $\eta(T' \circ R \circ S') = \eta(R \circ S') \circ T = \eta(\eta(R \circ S')') \circ T = \eta(T' \circ \eta(R \circ S')')$. We also know that $T \circ \kappa_E' \circ S'' = \kappa_E' \circ T'' \circ S''$, so that

$$\begin{aligned} \langle \psi_1(T') \Diamond \psi_1(R), S \rangle &= \langle \psi_1(R), T \circ S \rangle = \langle \Xi, \eta(R \circ S' \circ T') \rangle \\ &= \langle \Xi, \kappa_E' \circ T'' \circ S'' \circ R' \circ \kappa_{E'} \rangle = \langle \Xi, T \circ \kappa_E' \circ S'' \circ R' \circ \kappa_{E'} \rangle \\ &= \langle \Xi, T \circ \eta(R \circ S') \rangle = \langle \psi_2(T'), \eta(R \circ S') \rangle. \end{aligned}$$

By Equation (1), we see that

$$\langle \Phi_0, \eta(R \circ S') \rangle = \langle \psi_2(T') - \psi_1(T'), \eta(R \circ S') \rangle \qquad (R \in \mathcal{B}_\alpha(E''), S \in \mathcal{B}_{\alpha'}(E')).$$

Thus, for $S \in X_1(E, \alpha)$, we have

$$\langle \Phi, S \rangle = \langle \psi_1(T') + \Phi_0, S \rangle = \langle \psi_2(T'), S \rangle,$$

and so $\Phi - \psi_2(T') \in X(E, \alpha)^\circ$. Hence $\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \subseteq \psi_2(Z_1(E, \alpha)) + X_1(E, \alpha)^\circ$.

Conversely, for $T' \in Z_1(E,\alpha)$ and $\Phi \in X_1(E,\alpha)^\circ$, for $\Psi_0 \in I_1$, $R \in \mathcal{B}_\alpha(E'')$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}\langle(\Phi+\psi_2(T'))\Box(\Psi_0+\psi_1(R)),S\rangle &= \langle(\Phi+\psi_2(T'))\Box\psi_1(R),S\rangle \\ &= \langle\Phi+\psi_2(T'),\psi_1(R).S\rangle \\ &= \langle\Phi+\psi_2(T'),\eta(R\circ S')\rangle \\ &= \langle\psi_2(T'),\eta(R\circ S')\rangle = \langle\Xi,T\circ\eta(R\circ S')\rangle.\end{aligned}$$

As $\mathcal{N}_{\alpha'}(E') \subseteq X_1(E,\alpha)$, we have that $\Phi \in I_1$, and as $T' \in Z_1(E,\alpha)$, we have $T\circ S \in \mathcal{N}_{\alpha'}(E')$, so that

$$\begin{aligned}\langle(\Phi+\psi_2(T'))\Diamond(\Psi_0+\psi_1(R)),S\rangle &= \langle\psi_2(T')\Diamond(\Psi_0+\psi_1(R)),S\rangle \\ &= \langle\Psi_0+\psi_1(R),T\circ S\rangle = \langle\psi_1(R),T\circ S\rangle \\ &= \langle\Xi,\eta(R\circ S'\circ T')\rangle = \langle\Xi,T\circ\eta(R\circ S')\rangle,\end{aligned}$$

again using the fact that $T \in Z_1(E,\alpha)$. Hence $\Phi+\psi_2(T') \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$, and we have

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \psi_2(Z_1(E,\alpha)) + X(E,\alpha)^\circ,$$

as required. □

Theorem 5.19. *Let E be a Banach space such that E' has the bounded approximation property, let α be an accessible tensor norm, and let $\mathcal{A}=\mathcal{N}_\alpha(E)$. Then*

$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \{\psi_1(T)+\Phi : T\in Z_2(E,\alpha),\ \Phi\in X_2(E,\alpha)^\circ\}.$$

Proof. Let $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. With reference to Proposition 5.8, we can write $\Phi = \Phi_0+\psi_1(T)$ for some $\Phi_0 \in I_1$ and $T \in Z_2^0(E,\alpha)$. Similarly, for $\Psi \in \mathcal{A}''$, let $\Psi = \Psi_0+\psi_1(R)$, for some $\Psi_0 \in I_1$ and $R \in \mathcal{B}_\alpha(E'')$. Then we have, with reference to Proposition 5.12,

$$\begin{aligned}\Psi\Box\Phi &= (\Psi_0+\psi_1(R))\Box\psi_1(T) = \Psi_0\Box\psi_1(T)+\psi_1(R\circ T), \\ \Psi\Diamond\Phi &= \psi_1(R)\Diamond(\Phi_0+\psi_1(T)) = \psi_1(R)\Diamond\Phi_0+\psi_1(R)\Diamond\psi_1(T).\end{aligned}$$

Setting $\Psi_0 = 0$ gives us

$$\psi_1(R\circ T) = \psi_1(R)\Diamond\Phi_0+\psi_1(R)\Diamond\psi_1(T) \qquad (R\in\mathcal{B}_\alpha(E'')), \tag{2}$$

and thus also that $\Psi_0\Box\psi_1(T)=0$ for each $\Psi_0 \in I_1$. Again, this holds if and only if, for each $S \in \mathcal{B}_{\alpha'}(E')$, we have $\psi_1(T)\cdot S = \eta(T\circ S') \in \mathcal{N}_{\alpha'}(E')$.

As $T \in Z_2^0(E,\alpha)$, for $S \in \mathcal{B}_{\alpha'}(E')$, we have $T\circ S' \in \mathcal{B}(E')^a$, so that

$$\langle\psi_1(R)\Diamond\psi_1(T),S\rangle = \langle\Xi,\eta(T\circ S'\circ\mathcal{Q}(R))\rangle = \langle\Xi,\eta(R)\circ\eta(T\circ S')\rangle.$$

Then, as $T=\kappa_E\circ T_0$ for some $T_0\in\mathcal{B}(E'',E)$, we have

$$\begin{aligned}\eta(R\circ T\circ S') &= \kappa_E'\circ S''\circ T'\circ R'\circ\kappa_{E'} = \kappa_E'\circ S''\circ T_0'\circ\kappa_E'\circ R'\circ\kappa_{E'} \\ &= \kappa_E'\circ S''\circ T_0'\circ\kappa_E'\circ\kappa_{E'}\circ\eta(R) = \eta(T\circ S')\circ\eta(R).\end{aligned}$$

Thus we get

$$\langle\psi_1(R\circ T),S\rangle = \langle\Xi,\eta(R\circ T\circ S')\rangle = \langle\Xi,\eta(T\circ S')\circ\eta(R)\rangle.$$

Now, for $S \in \mathcal{B}_{\alpha'}(E')$, we have $\eta(T \circ S') = \phi_1(u)$ for some $u \in E''\widehat{\otimes}_{\alpha'}E'$. By Equation (2), we have

$$\begin{aligned}\langle \psi_1(R)\Diamond\Phi_0, S\rangle &= \langle \Phi_0, \eta(R)\circ S\rangle = \langle \psi_1(R\circ T) - \psi_1(R)\Diamond\psi_1(T), S\rangle \\ &= \langle \Xi, \eta(T\circ S')\circ\eta(R) - \eta(R)\circ\eta(T\circ S')\rangle \\ &= \langle \Xi, \phi_1(u)\circ\eta(R) - \eta(R)\circ\phi_1(u)\rangle = 0.\end{aligned}$$

Thus $\Phi_0 \in X_2(E,\alpha)^\circ$, and we see that $\mathfrak{Z}_t^{(2)}(\mathcal{A}'') \subseteq \psi_1(Z_2(E,\alpha)) + X_2(E,\alpha)^\circ$.

Conversely, for $T \in Z_2(E,\alpha)$ and $S \in \mathcal{B}_{\alpha'}(E')$, we have $\eta(T\circ S') = \phi_1(u)$ for some $u \in E''\widehat{\otimes}_{\alpha'}E'$, and that $T\circ S' = \eta(T\circ S')'$. Thus, for $\Phi_0 \in X_2(E,\alpha)^\circ$, $\Psi_0 \in I_1$ and $R \in \mathcal{B}_\alpha(E'')$, we have

$$\begin{aligned}\langle (\Psi_0 + \psi_1(R))\Box(\Phi_0 + \psi_1(T)), S\rangle &= \langle \Psi_0\Box\psi_1(T), S\rangle + \langle \psi_1(R\circ T), S\rangle \\ &= \langle \Psi_0, \eta(T\circ S')\rangle + \langle \Xi, \eta(R\circ T\circ S')\rangle \\ &= \langle \Psi_0, \phi_1(u)\rangle + \langle \Xi, \eta(T\circ S')\circ\eta(R)\rangle \\ &= \langle \Xi, \phi_1(u)\circ\eta(R)\rangle = \mathrm{tr}(\phi_1(u)\circ\eta(R)),\end{aligned}$$

by using the same calculation as above, given that $T(E'') \subseteq \kappa_E(E)$. Similarly, as $\Phi_0 \in X_2(E,\alpha)^\circ$, we have

$$\begin{aligned}\langle (\Psi_0 + \psi_1(R))\Diamond(\Phi_0 + \psi_1(T)), S\rangle &= \langle \psi_1(R)\Diamond\Phi_0, S\rangle + \langle \psi_1(R)\Diamond\psi_1(T), S\rangle \\ &= \langle \Phi_0, \eta(R)\circ S\rangle + \langle \psi_1(T), \eta(R)\circ S\rangle \\ &= \langle \Xi, \eta(T\circ S'\circ\mathcal{Q}(R))\rangle = \langle \Xi, \eta(\phi_1(u)'\circ\mathcal{Q}(R))\rangle \\ &= \langle \Xi, \eta(R)\circ\phi_1(u)\rangle = \mathrm{tr}(\phi_1(u)\circ\eta(R)).\end{aligned}$$

Consequently, $\Phi_0 + \psi_1(T) \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, and so we conclude that

$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \psi_1(Z_2(E,\alpha)) + X_2(E,\alpha)^\circ,$$

as required. □

These results (that is, the definitions of $Z_i(E,\alpha)$ and $X_i(E,\alpha)$, for $i = 1, 2$) might seem overly complicated. However, the next couple of corollaries (and their proofs) will show that, in the general case, we cannot remove any of the conditions.

Corollary 5.20. *Let E be a Banach space such that E' has the bounded approximation property, and let $\mathcal{A} = \mathcal{N}_2(E)$, the 2-nuclear operators. Then we have*

$$\begin{aligned}\mathfrak{Z}_t^{(1)}(\mathcal{A}'') &= \psi_2\big(\mathcal{P}_2(E'')\cap\mathcal{B}(E')^a\big) + \mathcal{N}_{d_2}(E')^\circ, \\ \mathfrak{Z}_t^{(2)}(\mathcal{A}'') &= \psi_1\big(\kappa_E\circ\mathcal{P}_2(E'',E)\big) + \mathcal{N}_{d_2}(E')^\circ, \\ \mathfrak{Z}_t^{(1)}(\mathcal{A}'')\cap\mathfrak{Z}_t^{(2)}(\mathcal{A}'') &= \psi_1(\mathcal{P}_2(E)^{aa}) + \mathcal{N}_{d_2}(E')^\circ.\end{aligned}$$

Here we treat $\mathcal{N}_{d_2}(E')$ as a subspace of $\mathcal{B}_{d_2}(E') = \mathcal{A}'$, so that $\mathcal{N}_{d_2}(E')^\circ$ is a subspace of $\mathcal{B}_{d_2}(E')' = \mathcal{A}''$.

Proof. We have that $\mathcal{A} = E'\widehat{\otimes}_{g_2}E$, so that $\mathcal{A}' = \mathcal{B}_{d_2}(E') = \mathcal{P}_2(E, E'')$, where, as in the comment after Theorem 5.11, the isometric isomorphism between $\mathcal{B}_{d_2}(E')$ and $\mathcal{P}_2(E, E'')$ is $S \mapsto S'\circ\kappa_E$. Furthermore, we see that $S \in \mathcal{B}_{d_2}(E')$ implies that $S' \in \mathcal{B}_{d_2^t}(E'') = \mathcal{B}_{g_2}(E'') = \mathcal{P}_2(E'')$, which in turn implies that $S'\circ\kappa_E \in \mathcal{P}_2(E'', E)$. Thus $S \in \mathcal{B}_{d_2}(E')$ if and only if $S' \in \mathcal{P}_2(E'')$.

We shall show that $X_1(E,g_2)$ is the closure of $\mathcal{N}(E') = E''\widehat{\otimes}E'$, with respect to the norm $\|\cdot\|_{d_2}$ on $\mathcal{B}_{d_2}(E')$. Indeed, let $S\in\mathcal{B}_{d_2}(E')$, so that $S'\circ\kappa_E\in\mathcal{P}_2(E,E'')$, and let $T\in\mathcal{B}_{g_2}(E'')=\mathcal{P}_2(E'')$. Then, by Proposition 3.35, we see that $T\circ S'\circ\kappa_E\in\mathcal{N}(E,E'')$, so that $\eta(T\circ S')=\kappa_E'\circ S''\circ T'\circ\kappa_{E'}\in\mathcal{N}(E')$, as required. Clearly $X_1(E,\alpha)$ contains $\mathcal{F}(E')$, so we see that $X_1(E,g_2)=\overline{\mathcal{N}(E')}$. An analogous calculation shows that $X_2(E,g_2)=\overline{\mathcal{N}(E')}$.

We now note that $\mathcal{N}_{d_2}(E')=E''\widehat{\otimes}_{d_2}E'$, so that from Proposition 3.28, it follows that the closure of $\mathcal{F}(E')$ (and hence $\mathcal{N}(E')$) in $\mathcal{B}_{d_2}(E')$ is simply $\mathcal{N}_{d_2}(E')$.

Now let $S,T\in\mathcal{B}_{d_2}(E')$, which is equivalent to $S',T'\in\mathcal{P}_2(E'')$. As S is weakly-compact, we have that $\kappa_{E'}\circ\kappa_E'\circ S''=S''$, so that $T''\circ S''=T''\circ\kappa_{E'}\circ\kappa_E'\circ S''=\kappa_{E'}\circ T\circ\kappa_E'\circ S''$. As above, $T\circ S$ is nuclear, so that certainly $T\circ S\in\mathcal{N}_{d_2}(E')$. Thus $Z_1(E,g_2)=\mathcal{P}_2(E'')\cap\mathcal{B}(E')^a=\mathcal{B}_{d_2}(E')^a$.

Let $T\in\mathcal{B}_{g_2}(E'')=\mathcal{P}_2(E'')$, and let $S\in\mathcal{B}_{d_2}(E')$, so that $S'\in\mathcal{P}_2(E'')$. Suppose that $T(E'')\subseteq\kappa_E(E)$, so that for some $T_1\in\mathcal{B}(E'',E)$, $T=\kappa_E\circ T_1$. By Theorem 3.33, and the fact that any closed subspace of a Hilbert space is 1-complemented, we see that $T_1\in\mathcal{P}_2(E'',E)$. Thus T_1S' is nuclear, by another application of Proposition 3.35. Then

$$\eta(TS')=\kappa_E'S''T'\kappa_{E'}=\kappa_E'S''T_1'\kappa_E'\kappa_{E'}=\kappa_E'S''T_1'=(T_1S'\kappa_E)'$$

so that as S is weakly-compact,

$$\mathcal{Q}(TS')=T_1''S'''\kappa_E''=T_1''\kappa_{E''}\kappa_{E'}'S'''\kappa_E''=\kappa_ET_1(\kappa_{E'}S)'\kappa_E''=TS'\kappa_{E'}'\kappa_E''=TS'.$$

Hence $TS'\in\mathcal{B}(E)^{aa}$ and as $T_1S'\kappa_E\in\mathcal{N}(E)$, we see that $TS'\in\mathcal{N}(E)^{aa}$ so that certainly $T\circ S'\in\mathcal{N}_{d_2}(E')^a$. Consequently, $Z_2(E,g_2)=\kappa_E\circ\mathcal{P}_2(E,E'')$.

Finally, it follows from Lemma 5.10 that

$$\big(\mathcal{P}_2(E'')\cap\mathcal{B}(E')^a\big)\cap\big(\kappa_E\circ\mathcal{P}_2(E'',E)\big)\subseteq\mathcal{W}(E)^{aa}.$$

Then using Proposition 3.6, we have that $T\in\mathcal{P}_2(E)$ if and only if $T''\in\mathcal{P}_2(E'')$; notice also that $T\in\mathcal{P}_2(E)$ implies that $T\in\mathcal{W}(E)$. It hence follows that

$$\big(\mathcal{P}_2(E'')\cap\mathcal{B}(E')^a\big)\cap\big(\kappa_E\circ\mathcal{P}_2(E'',E)\big)\subseteq\mathcal{P}_2(E)^{aa}.$$

The proof is hence complete if we can show that ψ_1 and ψ_2 agree on $P_2(E)^{aa}$. Let $\Xi\in\mathcal{A}(E)''$ be a mixed identity, let $T\in\mathcal{P}_2(E)$ and let $S\in\mathcal{B}_{d_2}(E')$ (that is, $S'\in\mathcal{P}_2(E'')$) so that ST' and $T'S$ are both nuclear. Consequently

$$\langle\psi_1(T'')-\psi_2(T''),S\rangle=\langle\Xi,S\circ T'-T'\circ S\rangle=\langle\mathrm{Id}_{E''},S\circ T'-T'\circ S\rangle,$$

so that an application of Lemma 5.23 below completes the proof. □

Corollary 5.21. *Let E be a Banach space such that E' has the Radon–Nikodým property and the bounded approximation property, and let $\mathcal{A}=\mathcal{N}_2(E)$. Then*

$$\begin{aligned}\mathfrak{Z}_t^{(1)}(\mathcal{A}'')&=\psi_2\big(\mathcal{P}_2(E'')\cap\mathcal{B}(E')^a\big),\\ \mathfrak{Z}_t^{(2)}(\mathcal{A}'')&=\psi_1\big(\kappa_E\circ\mathcal{P}_2(E'',E)\big),\\ \mathfrak{Z}_t^{(1)}(\mathcal{A}'')\cap\mathfrak{Z}_t^{(2)}(\mathcal{A}'')&=\kappa_{\mathcal{A}}(\mathcal{A}).\end{aligned}$$

Proof. Apply the above with Proposition 3.36, which shows that $\mathcal{N}_{d_2}(E')=\mathcal{B}_{d_2}(E')$ under our conditions on E. We claim that $\mathcal{N}_2(E)=\mathcal{P}_2(E)$. Let $T\in\mathcal{P}_2(E)$, so that $T'\in\mathcal{B}_{d_2}(E')=\mathcal{N}_2^\tau(E')=E''\widehat{\otimes}_{d_2}E'$. We wish to show that $T\in\mathcal{N}_2(E)=E'\widehat{\otimes}_{g_2}E$, which will follow if $T'\in\kappa_E(E)\widehat{\otimes}_{d_2}E'$ which is a closed subspace of $E''\widehat{\otimes}_{d_2}E'$, by [Rya02, Corollary 6.25]. By the Hahn–Banach Theorem, it is sufficient to show that if $S \in (E''\widehat{\otimes}_{d_2}E')' = \mathcal{P}_2(E'')$ vanishes on $\kappa_E(E)\widehat{\otimes}_{d_2}E'$, then $\langle S, T'\rangle = 0$ as well. Now, if S vanishes on $\kappa_E(E)\widehat{\otimes}_{d_2}E'$, then $S\kappa_E = 0$. As T is weakly-compact, there exists $T_0 \in \mathcal{P}_2(E'', E)$ such that $T'' = \kappa_E T_0$. By using Proposition 3.35, we see that

$$\langle S, T'\rangle = \mathrm{tr}(ST'') = \mathrm{tr}(S\kappa_E T_0) = 0,$$

as required. Hence $\mathcal{P}_2(E) = \mathcal{A}$, and so $\psi_1(\mathcal{P}_2(E)^{aa}) = \kappa_{\mathcal{A}}(\mathcal{A})$, completing the proof. □

The interested reader can try to generalise the above results to p-summing operators. In the case $p \neq 2$, we find that the more complicated characterisation of p-summing operators means that we can, for a general Banach space E, say surprisingly little. Obviously more could be said for specific examples of Banach spaces.

Corollary 5.22. *Let E be a Banach space such that E' has the bounded approximation property, and let $\mathcal{A} = \mathcal{A}(E)$. Then we have*

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \{\psi_2(T') : T \in \mathcal{B}(E'),\ T \circ S \in \mathcal{N}(E')\ (S \in \mathcal{I}(E'))\},$$
$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \{\psi_1(T) : T \in \mathcal{B}(E''),\ T(E'') \subseteq \kappa_E(E),\ T \circ S' \in \mathcal{N}(E'')\ (S \in \mathcal{I}(E'))\},$$
$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \psi_1(\mathcal{W}(E)^{aa}) = \psi_2(\mathcal{W}(E)^{aa}).$$

Proof. We see that $\mathcal{A}'' = \mathcal{B}(E'')\oplus I_1$, $\mathcal{A}' = \mathcal{B}_\pi(E') = \mathcal{I}(E')$, and $\mathcal{B}_\alpha(E'') = \mathcal{B}(E'')$. It is then clear that $X_1(E,\varepsilon) = X_2(E,\varepsilon) = \mathcal{I}(E')$, and so $X_1(E,\varepsilon)^\circ = X_2(E,\varepsilon)^\circ = \{0\}$. Then, for $S \in \mathcal{I}(E')$, as $\mathcal{I}(E') \subseteq \mathcal{W}(E')$, we have that $\kappa_{E'} \circ \kappa_E' \circ S'' = S''$, and so, for $T \in \mathcal{B}(E'')$, we have $\kappa_{E'} \circ T \circ \kappa_E' \circ S'' = T'' \circ \kappa_{E'} \circ \kappa_E' \circ S'' = T'' \circ S''$. Thus

$$Z_1(E,\varepsilon) = \{T' : T \in \mathcal{B}(E'),\ T \circ S \in \mathcal{N}(E')\ (S \in \mathcal{I}(E'))\},$$

which gives the result for $\mathfrak{Z}_t^{(1)}(\mathcal{A}'')$.

Similarly, for $T \in \mathcal{B}(E'')$ with $T = \kappa_E \circ T_0$ for some $T_0 \in \mathcal{B}(E'', E)$, and $S \in \mathcal{I}(E')$, $\mu \in E'$ and $\Lambda \in E''$, we have

$$\begin{aligned}\langle \Lambda, (\kappa_E' \circ S'' \circ T' \circ \kappa_{E'})(\mu)\rangle &= \langle (\kappa_{E'} \circ \kappa_E' \circ S'' \circ T_0' \circ \kappa_E' \circ \kappa_{E'})(\mu), \Lambda\rangle \\ &= \langle (S'' \circ T_0')(\mu), \Lambda\rangle = \langle \mu, (T_0 \circ S')(\Lambda)\rangle \\ &= \langle (T \circ S')(\Lambda), \mu\rangle.\end{aligned}$$

Thus $\eta(T \circ S')' = T \circ S'$, and so we have

$$Z_2(E,\varepsilon) = \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E),\ T \circ S' \in \mathcal{N}(E'')\ (S \in \mathcal{I}(E'))\},$$

as required.

We apply Lemma 5.10 and Theorem 3.31 to see that

$$\{T' : T \in \mathcal{B}(E'),\ T'(E'') \subseteq \kappa_E(E),\ T \circ S, S \circ T \in \mathcal{N}(E')\ (S \in \mathcal{I}(E'))\} = \mathcal{W}(E)^{aa}.$$

To complete the proof, we need to show that $\psi_1(T'') = \psi_2(T'')$ for $T \in \mathcal{W}(E)$. This follows exactly as in the case for the 2-nuclear operators. □

The following was known to Grosser (see [Gro89]) although he seems to have been unaware of Theorem 3.9, and so does not use the following simple factorisation argument.

Lemma 5.23. *Let E be Banach space such that E' has the approximation property. Let $T \in \mathcal{W}(E')$ and $S \in \mathcal{I}(E')$. Then $T \circ S, S \circ T \in \mathcal{N}(E') = E''\widehat{\otimes}E'$ and $\langle \mathrm{Id}_{E''}, T \circ S\rangle = \langle \mathrm{Id}_{E''}, S \circ T\rangle$.*

Proof. We follow the proof of Theorem 3.31, and again use Theorem 3.9. As T is weakly-compact, we can find a reflexive Banach space F, $T_1 \in \mathcal{B}(E', F)$ and $T_2 \in \mathcal{B}(F, E')$ so that $T = T_2 \circ T_1$. Then, as F is reflexive, $\mathcal{N}(E', F) = \mathcal{I}(E', F)$, so that $T_1 \circ S \in \mathcal{N}(E', F)$. Similarly, $T_2' \circ S' \in \mathcal{N}(E'', F')$, so that $S'' \circ T_2'' \in \mathcal{N}(F'', E''')$. As F is reflexive, we identify F'' with F, and we have that $T_2'' = \kappa_{E'} \circ T_2$. Thus $S'' \circ T_2'' = S'' \circ \kappa_{E'} \circ T_2 = \kappa_{E'} \circ S \circ T_2 \in \mathcal{N}(F, E''')$. As $S \in \mathcal{W}(E')$, we have that $S \circ T_2 = \kappa_E' \circ \kappa_{E'} \circ S \circ T_2 \in \mathcal{N}(F, E')$.

Immediately, we have $T \circ S = T_2 \circ T_1 \circ S \in \mathcal{N}(E')$ and $S \circ T = S \circ T_2 \circ T_1 \in \mathcal{N}(E')$. We also have, as $S \circ T_2 \in E''\widehat{\otimes}E'$ and $T_1 \circ S \in E''\widehat{\otimes}E'$,

$$\langle \mathrm{Id}_{E''}, S \circ T\rangle = \mathrm{tr}(S \circ T_2 \circ T_1) = \mathrm{tr}(T_1 \circ S \circ T_2) = \mathrm{tr}(T_2 \circ T_1 \circ S) = \langle \mathrm{Id}_{E''}, T \circ S\rangle,$$

as required. □

Note that $Z_1(E, \varepsilon) = \mathcal{B}(E')^a$ if and only if $\mathcal{N}(E') = \mathcal{I}(E')$. When $E = F'$ for some Banach space F, we have that $\kappa_E \circ \kappa_F' \in Z_2(E, \varepsilon)$ if and only if $\kappa_E \circ \kappa_F' \circ S' \in \mathcal{N}(E'')$ for each $S \in \mathcal{I}(E')$. This is if and only if

$$\eta(\kappa_E \circ \kappa_F' \circ S') = \kappa_E' \circ S'' \circ \kappa_F'' \circ \kappa_E' \circ \kappa_{E'} = \kappa_{F'}' \circ S'' \circ \kappa_F'' \in \mathcal{N}(E')$$

for each $S \in \mathcal{I}(E')$. In particular,

$$\kappa_{F'}' \circ S''' \circ \kappa_F'' = (S'' \circ \kappa_{F'})' \circ \kappa_F'' = (\kappa_{F'} \circ S)' \circ \kappa_F'' = S' \in \mathcal{N}(F'') \quad (S \in \mathcal{I}(F')).$$

Thus we have that $\kappa_E \circ \kappa_F' \in Z_2(E, \varepsilon)$ implies that $S = \eta(S') \in \mathcal{N}(F')$ for each $S \in \mathcal{I}(F')$, that is, $\mathcal{I}(F') = \mathcal{N}(F')$. We see that we cannot, in general, remove any of the conditions which define $Z_1(E, \varepsilon)$ and $Z_2(E, \varepsilon)$.

Example 5.24. The above considerations all apply to l^1 whose dual, l^∞, has the metric approximation property, but not the Radon–Nikodým property. It is folklore that $\mathcal{N}(l^\infty) \neq \mathcal{I}(l^\infty)$, which can be seen by studying operators on $C(X)$ spaces, as detailed in [DU77] (and performed explicitly in [Daw(2)04]). Thus $Z_1(l^1, \varepsilon)$ and $Z_2(l^1, \varepsilon)$ are nontrivial, in the sense just described. However, it is well-known (see [LL03]) that $\mathcal{A}(l^1)$ is the unique, closed, two-sided ideal in $\mathcal{B}(l^1)$, so that $\mathcal{A}(l^1) = \mathcal{W}(l^1)$, and hence $\mathcal{A} = \mathcal{A}(l^1)$ satisfies

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \kappa_{\mathcal{A}}(\mathcal{A}).$$

This is an example of a rather concrete Banach space not dealt with in [DL04].

Corollary 5.25. *Let E be a Banach space such that E' has the bounded approximation property, and let $\mathcal{A} = \mathcal{N}(E) = E'\widehat{\otimes}E$. Then we have*

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \{\psi_2(T'') : T \in \mathcal{I}(E) \cap \mathcal{A}(E)\} + \mathcal{I}(E')^\circ,$$
$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \{\psi_1(T'') : T \in \mathcal{I}(E) \cap \mathcal{A}(E)\} + \mathcal{I}(E')^\circ.$$

Proof. We have $\alpha = \pi$ so that $\mathcal{A}'' = \mathcal{B}_\pi(E'') \oplus I_1 = \mathcal{I}(E'') \oplus I_1$. We also have $\mathcal{A}' = \mathcal{B}(E')$ and $\mathcal{N}_{\alpha'}(E') = \mathcal{N}_\varepsilon(E') = \mathcal{A}(E')$. Thus we have

$$\begin{aligned} X_1(E,\pi) &= \overline{\operatorname{lin}}\{\eta(T\circ S') : S\in\mathcal{B}(E'),\ T\in\mathcal{I}(E'')\} = \overline{\mathcal{I}(E')}, \\ X_2(E,\pi) &= \overline{\operatorname{lin}}\{T\circ S : S\in\mathcal{B}(E'),\ T\in\mathcal{I}(E')\} = \overline{\mathcal{I}(E')}, \\ Z_1(E,\pi) &= \{T' : T\in\mathcal{I}(E'),\ T\circ S\in\mathcal{A}(E'), \\ &\qquad\qquad \kappa_{E'}\circ T\circ\kappa_E'\circ S'' = T''\circ S''\ (S\in\mathcal{B}(E'))\}, \\ Z_2(E,\pi) &= \{T\in\mathcal{I}(E'') : T(E'')\subseteq\kappa_E(E),\ T\circ S'\in\mathcal{A}(E')^a\ (S\in\mathcal{B}(E'))\}. \end{aligned}$$

Note that $\overline{\mathcal{I}(E')}$ means closure with respect to the topology on $\mathcal{B}(E')$.

Letting $S = \mathrm{Id}_{E'}$ in the expression for $Z_1(E,\pi)$ above yields $Z_1(E,\pi)\subseteq\mathcal{I}(E')\cap\mathcal{A}(E')$ and that $T\in Z_1(E,\pi)$ implies that $\kappa_{E'}\circ T\circ\kappa_E' = T''$. For $M\in\kappa_E(E)^\circ$, we have $\kappa_E'(M) = 0$, so that $T''(M) = 0$. Thus a Hahn–Banach argument tells us that $T'(E'')\subseteq\kappa_E(E)$. As in the proof of Lemma 5.10, we have $\mathcal{B}(E')^a\cap(\kappa_E\circ\mathcal{B}(E'',E)) = \mathcal{B}(E)^{aa}$, so that $T\in\mathcal{I}(E)^a$. Thus we have $Z_1(E,\pi) = \{T'' : T\in\mathcal{I}(E), T'\in\mathcal{A}(E')\}$, noting that for $T\in\mathcal{I}(E)$, we have $\kappa_{E'}\circ T'\circ\kappa_{E'} = T'''$. Now, by [Rya02, Proposition 5.55], we know that $T'\in\mathcal{A}(E')$ if and only if $T\in\mathcal{A}(E)$. Thus

$$Z_1(E,\pi) = \{T'' : T\in\mathcal{I}(E)\cap\mathcal{A}(E)\},$$

as required.

For $T\in Z_2(E,\pi)$, we similarly see that $T\in\mathcal{A}(E')^a$ and that $T\in\mathcal{W}(E)^{aa}$, as before. Thus we can again conclude that

$$Z_2(E,\pi) = \{T'' : T\in\mathcal{I}(E)\cap\mathcal{A}(E)\},$$

as required. □

The space $\mathcal{I}(E)\cap\mathcal{A}(E)$ is easily seen to a closed subspace of $\mathcal{I}(E)$; indeed, let (T_n) be a sequence in $\mathcal{I}(E)\cap\mathcal{A}(E)$ with $\|T_n - T\|_\pi\to 0$ for some $T\in\mathcal{I}(E)$. Then $\|T_n - T\|\leq\|T_n - T\|_\pi\to 0$, so that $T\in\mathcal{A}(E)$. Note that, when E is not reflexive, $\mathcal{I}(E')\subseteq\mathcal{W}(E')\subsetneq\mathcal{B}(E')$, so that $X_1(E,\pi) = X_2(E,\pi)$ is a nontrivial subspace of $\mathcal{N}(E)'$.

Example 5.26. Again, it is folklore that when $E = C([0,1])$, we can find $T\in\mathcal{I}(E)\cap\mathcal{A}(E)$ with $T\notin\mathcal{N}(E)$, and so that E' has the bounded approximation property. Hence the conditions in the above theorem are not vacuous, as we do not have $\mathcal{N}(E) = \mathcal{I}(E)\cap\mathcal{A}(E)$.

Corollary 5.27. *Let E be a Banach space such that E' has the bounded approximation property and $\mathcal{N}(E') = \mathcal{I}(E')$. Then we can identify $\mathcal{A}(E)''$ with $\mathcal{B}(E'')$, and we have*

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}(E)'') = \mathcal{B}(E')^a, \qquad \mathfrak{Z}_t^{(2)}(\mathcal{A}(E)'') = \kappa_E\circ\mathcal{B}(E'',E).$$

Furthermore, we have $\mathcal{N}(E)'' = \mathcal{B}(E')'$ and

$$\mathfrak{Z}_t^{(1)}(\mathcal{N}(E)'') = \mathfrak{Z}_t^{(2)}(\mathcal{N}(E)'') = \kappa_{\mathcal{N}(E)}(\mathcal{N}(E)) + \ker\theta_1.$$

In particular $\mathfrak{Z}_t^{(1)}(\mathcal{N}(E)'')\cap\mathfrak{Z}_t^{(2)}(\mathcal{N}(E)'')$ strictly contains $\kappa_{\mathcal{N}(E)}(\mathcal{N}(E))$.

Proof. For $\mathcal{A}(E)''$, the result was first shown in [DL04], and follows immediately from Corollary 5.22, given that $\mathcal{N}(E') = \mathcal{I}(E') = E''\widehat{\otimes}E'$.

For $\mathcal{N}(E)''$, we have that $\mathcal{N}(E) = E'\widehat{\otimes}E$ and $\mathcal{N}(E)' = \mathcal{B}(E')$. Then $\overline{\mathcal{I}(E')} = \overline{\mathcal{N}(E')} = \mathcal{A}(E') = E''\widehat{\otimes}_\varepsilon E'$ in $\mathcal{B}(E')$. These agree with the image of ϕ_1, so that $\overline{\mathcal{I}(E')}^\circ = \ker\theta_1 = \mathcal{A}(E')^\circ$. As E is infinite-dimensional, $\ker\theta_1 \neq \{0\}$. For $T \in \mathcal{I}(E)\cap\mathcal{A}(E)$, we have $T' \in \mathcal{N}(E')$, so that by Proposition 3.13, $T \in \mathcal{N}(E) \subseteq \mathcal{A}(E)$. Hence $\mathcal{I}(E)\cap\mathcal{A}(E) = \mathcal{N}(E)$. Clearly ψ_1 and ψ_2 agree on $\mathcal{N}(E)$, so we are done. □

Example 5.28. Following [DL04], consider c_0, so that $c_0' = l^1$, as a separable dual space, has the Radon–Nikodým property, and thus we have $\mathcal{I}(l^1) = \mathcal{N}(l^1) = l^\infty\widehat{\otimes}l^1$, as l^∞ has the metric approximation property. Thus the above corollary holds, and we have $\mathcal{A}(c_0)'' = \mathcal{B}(l^\infty)$. By Corollary 5.22, we have that

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}(c_0)'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}(c_0)'') = \mathcal{W}(c_0)^{aa}.$$

Again, $\mathcal{B}(c_0)$ contains only one proper, closed two-sided ideal, namely $\mathcal{A}(c_0)$. In particular, $\mathcal{A}(c_0) = \mathcal{W}(c_0)$, so (as in the l^1 case) we again have, for $\mathcal{A} = \mathcal{A}(c_0)$, that $\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \kappa_{\mathcal{A}}(\mathcal{A})$.

We can also apply the above corollary to $\mathcal{N}(c_0) = l^1\widehat{\otimes}c_0$ to see that

$$\mathfrak{Z}_t^{(1)}(\mathcal{N}(c_0)'') = \mathfrak{Z}_t^{(2)}(\mathcal{N}(c_0)'').$$

We have that $\phi_1 : l^\infty\check{\otimes}l^1 \to \mathcal{B}(l^1) = \mathcal{N}(c_0)'$ is an isometry onto its range, which is $\mathcal{A}(l^1)$, so that

$$\ker\theta_1 = \{\Phi \in \mathcal{N}(c_0)'' = \mathcal{B}(l^1)' : \langle \Phi, S\rangle = 0 \ (S \in \mathcal{A}(l^1))\}$$

is large with respect to $\mathcal{N}(c_0)''$.

Example 5.29. Let P be Pisier's space, as constructed in [Pis83], so that $\mathcal{A}(P) = \mathcal{N}(P)$. Applying Theorem 5.14, we see that the topological centres of $\mathcal{A}(P)''$ are distinct and neither contains the other. Hence this also holds for $\mathcal{N}(P)''$, and we conclude that, in general, we cannot say that the topological centres of the bidual of the nuclear operators are equal.

Example 5.30. Again, following [DL04], consider J, the James space, which was defined in [Jam51]. Let c_{00} be the vector space of sequences of complex numbers which are eventually zero, and for $x = (x_n) \in c_{00}$, let

$$\|x\|_J = \sup\left\{\left(\sum_{i=1}^n |x_{r_i} - x_{r_{i+1}}|^2 + |x_{r_{n+1}} - x_{r_1}|^2\right)^{1/2}\right\},$$

where the supremum is taken over all integers n and increasing sequences of integers $(r_i)_{i=1}^{n+1}$. We can show that $\|\cdot\|_J$ is a norm; let J be the completion of $(c_{00}, \|\cdot\|_J)$. We can show that J is $\{x \in c_0 : \|x\|_J < \infty\}$. Then, as shown in [Jam51], J is isometric with J'', but $J''/\kappa_J(J)$ is isomorphic to $\mathbb{C}$. The standard unit vector basis (e_n) is a basis for J.

So, for some $\Lambda_0 \in J''$, the map $J \oplus \mathbb{C} \to J''; (x,\alpha) \mapsto \kappa_J(x) + \alpha\Lambda_0$ is an isomorphism. Let $M_0 \in \kappa_J(J)^\circ \subseteq J'''$ be such that $\langle M_0, \Lambda_0\rangle = 1$, so that we may define a projection $P : J'' \to \kappa_J(J)$ by $P(\Phi) = \Phi - \langle M_0, \Phi\rangle\Lambda_0$ for $\Phi \in J''$. We may check that the maps

$$\mathcal{B}(J) \oplus J' \to \mathcal{B}(J'); \quad (T, \mu) \mapsto T' + \Lambda_0 \otimes \mu,$$
$$\mathcal{B}(J) \oplus J \to \kappa_J \circ \mathcal{B}(J'', J); \quad (T, x) \mapsto P \circ T'' + M_0 \otimes \kappa_J(x)$$

are isomorphisms.

By [DU77, Chapter VII], J' has the Radon–Nikodým property, so that $\mathcal{N}(J') = \mathcal{I}(J')$. As J has a basis, it has the bounded approximation property (and thus J'' has the bounded approximation property, so that J' also does). We can thus again apply the above corollaries, and so we have $\mathcal{A}(J)'' = \mathcal{B}(J'')$. Thus we have, given the identifications above,

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}(J)'') = \mathcal{B}(J')^a \cong \mathcal{B}(J) \oplus J',$$
$$\mathfrak{Z}_t^{(2)}(\mathcal{A}(J)'') = \kappa_J \circ \mathcal{B}(J'', J) \cong \mathcal{B}(J) \oplus J,$$
$$\mathfrak{Z}_t^{(1)}(\mathcal{A}(J)'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}(J)'') = \mathcal{W}(J)^{aa}.$$

It is reasonably simple to show that $\mathcal{W}(J)$ is a maximal closed ideal in $\mathcal{B}(J)$ (in fact, it is the unique maximal closed ideal in $\mathcal{B}(J)$, as shown by Laustsen in [Lau02]) and that $\mathcal{W}(J)$ has co-dimension one in $\mathcal{B}(J)$. As summarised in [LL03, Section 3], $\mathcal{A}(J) = \mathcal{K}(J)$ is not equal to $\mathcal{W}(J)$, so that

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}(J)'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}(J)'') \neq \kappa_{\mathcal{A}(J)}(\mathcal{A}(J)),$$

a fact shown in [DL04].

We can apply the above to study $\mathcal{N}(J') = J''\widehat{\otimes}J'$. We have $\mathcal{N}(J')' = \mathcal{B}(J'')$ and so $\ker\theta_1 = \mathcal{A}(J'')^\circ$, and

$$\mathfrak{Z}_t^{(1)}(\mathcal{N}(J')'') = \mathfrak{Z}_t^{(2)}(\mathcal{N}(J')'') = \kappa_{\mathcal{N}(J')}(\mathcal{N}(J')) + \mathcal{A}(J'')^\circ.$$

Now, we have $\mathcal{A}(J)' = \mathcal{N}(J')$ and $\mathcal{A}(J)'' = \mathcal{B}(J'')$, so that $\kappa'_{\mathcal{A}(J)} : \mathcal{N}(J')'' \to \mathcal{N}(J')$ is an projection. Hence we can write

$$\begin{aligned}\mathcal{N}(J')'' = \mathcal{B}(J'')' &= \kappa_{\mathcal{N}(J')}(\mathcal{N}(J')) \oplus \ker\kappa'_{\mathcal{A}(J)} = \kappa_{\mathcal{N}(J')}(\mathcal{N}(J')) \oplus \kappa_{\mathcal{A}(J)}(\mathcal{A}(J))^\circ \\ &= \kappa_{\mathcal{N}(J')}(\mathcal{N}(J')) \oplus (\mathcal{A}(J)^{aa})^\circ.\end{aligned}$$

Notice that as $\kappa_{\mathcal{A}(J)}(\mathcal{A}(J)) = \mathcal{A}(J)^{aa} \subseteq \mathcal{A}(J'')$, we have $\mathcal{A}(J'')^\circ \subseteq (\mathcal{A}(J)^{aa})^\circ$, and so we have

$$\mathcal{N}(J')''/\mathfrak{Z}_t^{(1)}(\mathcal{N}(J')'') = (\mathcal{A}(J)^{aa})^\circ/\mathcal{A}(J'')^\circ.$$

5.2. When the integral and nuclear operators coincide. We now drop the requirement that E' have the bounded approximation property. Motivated by the fact that, for many Banach spaces E, we have $\mathcal{A}(E)' = \mathcal{I}(E') = \mathcal{N}(E')$, we might consider studying the case when $\mathcal{N}_\alpha(E)' = \mathcal{N}_{\alpha'}(E')$. However, this seems too strong a condition (for example, it seems unlikely that it is ever true for $\alpha = \pi$). This said, we can again use the Gröthendieck Composition theorem to show that, when $\mathcal{A} = \mathcal{N}_\alpha(E)$ for an accessible α, we have $\mathcal{A}'' \cdot \mathcal{A}' + \mathcal{A}' \cdot \mathcal{A}'' \subseteq \mathcal{I}(E')$. Thus the case when $\mathcal{N}(E') = \mathcal{I}(E')$ should be interesting to study, and it is certainly not a vacuous condition to impose upon E, by the following lemma.

For a Banach space E, recall that $E^{[n]}$ is the nth iterated dual of E, so that $E^{[1]} = E'$ etc.

Lemma 5.31. *Let E be a Banach space such that $E^{[n]}$ has the Radon–Nikodým property for some $n \in \mathbb{N}$. Then $\mathcal{I}(E^{[m]}) = \mathcal{N}(E^{[m]})$ for each $1 \le m \le n$.*

Proof. By Theorem 3.18, if E' has the Radon–Nikodým property, then $\mathcal{N}(E') = \mathcal{I}(E')$.

Suppose that F is a Banach space such that $\mathcal{N}(F'') = \mathcal{I}(F'')$. For $T \in \mathcal{I}(F')$, we have $T' \in \mathcal{I}(F'') = \mathcal{N}(F'')$, and so $T = \eta(T') = \kappa_F' \circ T' \circ \kappa_{F'} \in \mathcal{N}(F')$. Thus, by induction, if $\mathcal{N}(E^{[n+1]}) = \mathcal{I}(E^{[n+1]})$ for some $n \in \mathbb{N}$, then $\mathcal{N}(E^{[m]}) = \mathcal{I}(E^{[m]})$ for each $1 \leq m \leq n+1$. We are thus done by another application of Theorem 3.18. $\square$

Example 5.32. Let JT be the James Tree Space (defined in [Jam74]), so that each even dual of JT has the Radon–Nikodým property, but each odd dual does not (see [DU77, Chapter VII, Section 5]). Thus, by the above lemma, $\mathcal{I}(JT') = \mathcal{N}(JT')$ while JT' does not have the Radon–Nikodým property.

Let E be a Banach space and α be a tensor norm. With reference to Proposition 4.3, we treat ϕ_1 as a map $E''\widehat{\otimes}_{\alpha'}E' \to \mathcal{N}_{\alpha'}(E') \subseteq \mathcal{N}_\alpha(E)' \subseteq \mathcal{B}_{\alpha'}(E')$. Then $\theta_1 : \mathcal{N}_\alpha(E)'' \to \mathcal{B}_\alpha(E'')$ actually maps into

$$\mathcal{N}_{\alpha'}(E')' = (\ker J_{\alpha'})^\circ = \{T \in \mathcal{B}_\alpha(E'') : \langle T, u\rangle = 0 \ (u \in E''\widehat{\otimes}_{\alpha'}E', J_{\alpha'}(u) = 0)\}.$$

The next lemma tells us that, in this case, $(\ker J_{\alpha'})^\circ$ is a right ideal in $(\mathcal{B}_\alpha(E''), \circ)$ and a left ideal in $(\mathcal{B}_\alpha(E''), \star)$.

Lemma 5.33. *Let E be a Banach space and α be a tensor norm. Then $(\ker J_{\alpha'})^\circ$ is a right ideal in $(\mathcal{B}_\alpha(E''), \circ)$. Furthermore, for $T \in (\ker J_{\alpha'})^\circ$ and $S \in \mathcal{B}_{\alpha^t}(E')^a$, we have $S \circ T \in (\ker J_{\alpha'})^\circ$.*

Proof. Let $T \in (\ker J_{\alpha'})^\circ$ and $u \in \ker J_{\alpha'}$. Let (u_n) be a sequence in $\mathcal{F}(E')$ such that $\sum_{n=1}^\infty u_n = u$ in $E''\widehat{\otimes}_{\alpha'}E'$. Let $S \in \mathcal{B}_\alpha(E'')$, and let $v = (S \otimes \mathrm{Id}_{E'})(u)$. Then, for $\mu \in E'$ and $\Lambda \in E''$, we have

$$\begin{aligned}\langle \Lambda, J_{\alpha'}(v)(\mu)\rangle &= \sum_{n=1}^\infty \langle \Lambda, J_{\alpha'}((S\otimes \mathrm{Id}_{E'})(u_n))(\mu)\rangle = \sum_{n=1}^\infty \langle J_{\alpha'}((S\otimes\mathrm{Id}_{E'})(u_n))'(\Lambda), \mu\rangle \\ &= \sum_{n=1}^\infty \langle S'(\kappa_{E'}(\mu)), J_{\alpha'}(u_n)'(\Lambda)\rangle = \langle S'(\kappa_{E'}(\mu)), J_{\alpha'}(u)'(\Lambda)\rangle = 0,\end{aligned}$$

as $J_{\alpha'}(u) = 0$. Thus $v \in \ker J_{\alpha'}$. We then have

$$\begin{aligned}\langle T\circ S, u\rangle &= \sum_{n=1}^\infty \langle T\circ S, u_n\rangle = \sum_{n=1}^\infty \mathrm{tr}(T\circ S\circ u_n') = \sum_{n=1}^\infty \mathrm{tr}\left(T \circ ((S\otimes \mathrm{Id}_{E'})(u_n))'\right) \\ &= \sum_{n=1}^\infty \langle T, (S\otimes\mathrm{Id}_{E'})(u_n)\rangle = \langle T, v\rangle = 0,\end{aligned}$$

as $T \in (\ker J_{\alpha'})^\circ$. Thus $T\circ S \in (\ker J_{\alpha'})^\circ$.

Similarly, for $T \in (\ker J_{\alpha'})^\circ$, $S \in \mathcal{B}_{\alpha^t}(E')$ and $u \in \ker J_{\alpha'}$, let $v = (\mathrm{Id}_{E''}\otimes S)(u)$. We can show that $v \in \ker J_{\alpha'}$, and similarly that

$$\langle S'\circ T, u\rangle = \langle T, (\mathrm{Id}_{E''}\otimes S)(u)\rangle = \langle T, v\rangle = 0,$$

so that $S' \circ T \in (\ker J_{\alpha'})^\circ$. $\square$

For a Banach space E and a tensor norm α, recall the following definitions:

$$Z_1^0(E,\alpha) = \{T' : T \in \mathcal{B}_{\alpha^t}(E'),\ T\circ\kappa_E'\circ S'' = \kappa_E' \circ T''\circ S'' \ (S \in \mathcal{N}_\alpha(E)')\},$$

$$Z_2^0(E,\alpha) = \{T \in \mathcal{B}_\alpha(E'') : T(E'') \subseteq \kappa_E(E),\ T\circ S' \in \mathcal{W}(E)^{aa} \ (S\in\mathcal{N}_\alpha(E)')\}.$$

Theorem 5.34. *Let E be a Banach space such that $\mathcal{N}(E') = \mathcal{I}(E')$, let α be an accessible tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then, for $i = 1, 2$, we have*

$$\mathfrak{Z}_t^{(i)}(\mathcal{A}'') = \theta_1^{-1}(Z_i^0(E,\alpha)).$$

Proof. For $\Phi \in \mathcal{A}''$ and $S \in \mathcal{A}' \subseteq \mathcal{B}_{\alpha'}(E')$, by Proposition 5.3, we have

$$\Phi \cdot S = \eta(\phi_1(\Phi) \circ S') \in \mathcal{N}(E'), \qquad S \cdot \Phi = \eta(\phi_1(\Phi)) \circ S \in \mathcal{N}(E'),$$

as $\mathcal{N}(E') = \mathcal{I}(E')$. Then, as $\alpha' \leq \pi$ on $E'' \otimes E'$, we clearly have that $\Phi \cdot S$ and $S \cdot \Phi$ are in $\mathcal{N}_{\alpha'}(E') \subseteq \mathcal{A}'$.

Then, for $\Phi, \Psi \in \mathcal{A}''$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}\langle \Phi \Box \Psi, S\rangle &= \langle \Phi, \eta(\theta_1(\Psi) \circ S')\rangle = \operatorname{tr}\big(\theta_1(\Phi) \circ \mathcal{Q}(\theta_1(\Psi) \circ S')\big),\\ \langle \Phi \Diamond \Psi, S\rangle &= \langle \Psi, \eta(\theta_1(\Phi)) \circ S\rangle = \operatorname{tr}\big(\theta_1(\Psi) \circ S' \circ \mathcal{Q}(\theta_1(\Phi))\big),\end{aligned}$$

as, for example, $\eta(\theta_1(\Psi) \circ S \in \mathcal{N}_{\alpha'}(E') = \phi_1(E''\widehat{\otimes}_{\alpha'} E')$. We thus see that $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$ if and only if

$$\operatorname{tr}\big(\theta_1(\Phi) \circ \mathcal{Q}(\theta_1(\Psi) \circ S')\big) = \operatorname{tr}\big(\theta_1(\Psi) \circ S' \circ \mathcal{Q}(\theta_1(\Phi))\big) \qquad (S \in \mathcal{A}', \Psi \in \mathcal{A}''),$$

and that $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$ if and only if

$$\operatorname{tr}\big(\theta_1(\Psi) \circ \mathcal{Q}(\theta_1(\Phi) \circ S')\big) = \operatorname{tr}\big(\theta_1(\Phi) \circ S' \circ \mathcal{Q}(\theta_1(\Psi))\big) \qquad (S \in \mathcal{A}', \Psi \in \mathcal{A}'').$$

Suppose that $\theta_1(\Phi) \in Z_1^0(E,\alpha)$, so that $\theta_1(\Phi) = \mathcal{Q}(\theta_1(\Phi))$. Taking adjoints, we also have

$$S''' \circ \kappa_E'' \circ \theta_1(\Phi) = S''' \circ \theta_1(\Phi)'' \circ \kappa_E'' \qquad (S \in \mathcal{A}').$$

Thus, for $S \in \mathcal{A}'$ and $\Psi \in \mathcal{A}''$, we have

$$\begin{aligned}\operatorname{tr}\big(\theta_1(\Phi) \circ \mathcal{Q}(\theta_1(\Psi) \circ S')\big) &= \operatorname{tr}\big(\kappa_{E'}' \circ \theta_1(\Psi)'' \circ S''' \circ \kappa_E'' \circ \theta_1(\Phi)\big)\\ &= \operatorname{tr}\big(\kappa_{E'}' \circ \theta_1(\Psi)'' \circ S''' \circ \theta_1(\Phi)'' \circ \kappa_E''\big),\end{aligned}$$

noting that $\eta(\theta_1(\Psi) \circ S') \in \mathcal{N}_{\alpha'}(E')$, a fact which allows us to alter the order of maps inside the trace. As $\eta(\theta_1(\Phi)) \circ S \in \mathcal{N}_{\alpha'}(E') \subseteq \mathcal{K}(E') \subseteq \mathcal{W}(E')$, we have $\kappa_{E'} \circ \kappa_E' \circ \eta(\theta_1(\Phi))'' = \eta(\theta_1(\Phi))''$. Thus we have

$$\begin{aligned}\operatorname{tr}\big(\kappa_{E'}' \circ \theta_1(\Psi)'' \circ S''' \circ \theta_1(\Phi)'' \circ \kappa_E''\big) &= \operatorname{tr}\big(\theta_1(\Psi)'' \circ S''' \circ \eta(\theta_1(\Phi))''' \circ \kappa_E'' \circ \kappa_{E'}'\big)\\ &= \operatorname{tr}\big(\theta_1(\Psi)'' \circ S''' \circ \eta(\theta_1(\Phi))'''\big)\\ &= \operatorname{tr}\big(\theta_1(\Psi) \circ S' \circ \mathcal{Q}(\theta_1(\Phi))\big).\end{aligned}$$

Hence $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$. Applying Proposition 5.8 allows us to conclude that

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \theta_1^{-1}(Z_1^0(E,\alpha)).$$

Similarly, suppose that $\theta_1(\Phi) \in Z_2^0(E,\alpha)$. Then $\theta_1(\Phi)(E'') \subseteq \kappa_E(E)$ and

$$\theta_1(\Phi) \circ S' \in \mathcal{B}(E')^a \qquad (S \in \mathcal{A}').$$

Let $T \in \mathcal{B}(E'', E)$ be such that $\kappa_E \circ T = \theta_1(\Phi)$. Then, for $S \in \mathcal{A}'$, we have $\kappa_E \circ T \circ S' = R_S'$ for some $R_S \in \mathcal{B}(E')$. As $R_S'(E'') \subseteq \kappa_E(E)$, by the argument used in Lemma 5.10, $R_S = R'$ where $R = T \circ S' \circ \kappa_E \in \mathcal{W}(E)$. Then $R_S = \eta(R_S') =$

$\eta(\theta_1(\Phi)\circ S')\in\mathcal{N}_{\alpha'}(E')$. In particular, $R_S\in\mathcal{W}(E')$ and so $\kappa_{E'}\circ\kappa_E'\circ R_S''=R_S''$, and so, for $\Psi\in\mathcal{A}''$, we have

$$\begin{aligned}\operatorname{tr}\big(\theta_1(\Psi)\circ\mathcal{Q}(\theta_1(\Phi)\circ S')\big)&=\operatorname{tr}\big(\theta_1(\Psi)\circ\theta_1(\Phi)\circ S'\big)=\operatorname{tr}\big(\theta_1(\Psi)\circ R_S'\big)\\&=\operatorname{tr}\big(R_S''\circ\theta_1(\Psi)'\big)=\operatorname{tr}\big(\kappa_{E'}\circ\kappa_E'\circ R_S''\circ\theta_1(\Psi)'\big)=\operatorname{tr}\big(\kappa_E'\circ R_S''\circ\theta_1(\Psi)'\circ\kappa_{E'}\big)\\&=\operatorname{tr}\big(\kappa_E'\circ S''\circ T'\circ\kappa_E'\circ\theta_1(\Psi)'\circ\kappa_{E'}\big)=\operatorname{tr}\big(R'\circ\eta(\theta_1(\Psi))\big)\\&=\operatorname{tr}\big(R_S\circ\eta(\theta_1(\Psi))\big)=\operatorname{tr}\big(\eta(\theta_1(\Psi))\circ R_S\big)=\operatorname{tr}\big(\theta_1(\Phi)\circ S'\circ\mathcal{Q}(\theta_1(\Psi))\big).\end{aligned}$$

Hence $\Phi\in\mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, and another application of Proposition 5.8 allows us to conclude that

$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'')=\theta_1^{-1}(Z_2^0(E,\alpha)).\qquad\square$$

Theorem 5.35. *Let E be a Banach space such that $\mathcal{N}(E')=\mathcal{I}(E')$, let α be a tensor norm, and let $\mathcal{A}=\mathcal{N}_\alpha(E)$. Suppose that E'' has the bounded approximation property. Then, for $i=1,2$, we have*

$$\mathfrak{Z}_t^{(i)}(\mathcal{A}'')=\theta_1^{-1}(Z_i^0(E,\alpha)).$$

Proof. As E'' has the bounded approximation property, so does E'. Thus, in the language of Proposition 5.3, (E',α) and (E'',α) are Gröthendieck pairs. The rest of the proof runs exactly as above. $\square$

We can then apply the same sort of arguments used in, for example, Theorem 5.11, to state some corollaries. Rather than do this, we state the most interesting case.

Corollary 5.36. *Let E be a Banach space such that $\mathcal{N}(E')=\mathcal{I}(E')$. Let $\mathcal{A}=\mathcal{A}(E)$, and let*

$$X=(\ker J_\pi)^\circ=\{T\in\mathcal{B}(E''):\langle T,u\rangle=0\ (u\in E''\widehat{\otimes}E',J_\pi(u)=0)\}.$$

Then $\theta_1:\mathcal{A}''\to X$ is an isometry, and, when we identify $\mathcal{A}''$ with X, we have

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'')=X\cap\mathcal{B}(E')^a,\qquad\mathfrak{Z}_t^{(2)}(\mathcal{A}'')=X\cap(\kappa_E\circ\mathcal{B}(E'',E)).$$

Proof. We have $\mathcal{A}'=\mathcal{I}(E')=\mathcal{N}(E')$ so that $\phi_1:E''\widehat{\otimes}E'\to\mathcal{A}'$ is a quotient map, and thus θ_1 is an isometry. The results now follow from the calculations done in the proof of Theorem 5.11, and the results of the above theorem. $\square$

Corollary 5.37. *Let E be a Banach space such that $\mathcal{N}(E')=\mathcal{I}(E')$, and let $\mathcal{A}=\mathcal{N}_2(E)$. Then*

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'')=\theta_1^{-1}(\mathcal{P}_2(E'')\cap\mathcal{B}(E')^a),\quad\mathfrak{Z}_t^{(2)}(\mathcal{A}'')=\theta_1^{-1}(\kappa_E\circ\mathcal{P}_2(E'',E)).$$

Suppose that E' has the Radon–Nikodým property (this applies, in particular, when E' is separable), and let

$$X=(\ker J_{d_2})^\circ=\{T\in\mathcal{P}_2(E''):\langle T,u\rangle=0\ (u\in E''\widehat{\otimes}_{d_2}E',J_{d_2}(u)=0)\}.$$

Then $\theta_1:\mathcal{A}''\to X$ is an isometry.

Proof. As before, in this case, $\mathcal{A}'\subseteq\mathcal{W}(E')$, so that by Theorem 5.11, $Z_1^0=\mathcal{B}_{d_2}(E')^a=\mathcal{P}_2(E'')\cap\mathcal{B}(E')^a$ and $Z_2^0=\kappa_E\circ\mathcal{B}_{g_2}(E'',E)=\kappa_E\circ\mathcal{P}_2(E'',E)$.

When E' has the Radon–Nikodým property, by Proposition 3.36, $\mathcal{A}'=\mathcal{B}_{d_2}(E')=\mathcal{N}_{d_2}(E')$, and thus $\phi_1:E''\widehat{\otimes}_{d_2}E'\to\mathcal{A}'$ is a quotient operator, and hence θ_1 is an isometry onto its range, as claimed. $\square$

In the case of the nuclear operators, we cannot say much more than the above theorem gives, as, in general, we have no good description of $\mathcal{N}(E)'$ (see Example 5.29). However, the next example shows that in special cases we can say more than we could before.

Example 5.38. By [FJ73] and Proposition 3.25, we can find a Banach space E_0 with the approximation property, such that $\mathcal{N}(E_0') = \mathcal{I}(E_0')$, and such that there exists $T_0 \in \mathcal{I}(E_0) \setminus \mathcal{N}(E_0)$ with $T_0' \in \mathcal{N}(E_0')$. Then let $\mathcal{A} = \mathcal{N}(E_0) = E_0'\widehat{\otimes}E_0$, so that $\mathcal{A}' = \mathcal{B}(E_0')$, and we have

$$\begin{aligned} Z_1^0(E_0,\pi) &= \{T' : T \in \mathcal{I}(E_0'),\ T \circ \kappa_{E_0}' = \kappa_{E_0}' \circ T''\}, \\ Z_2^0(E_0,\pi) &= \{T \in \mathcal{I}(E_0'') : T(E_0'') \subseteq \kappa_{E_0}(E_0),\ T \in \mathcal{W}(E_0)^{aa}\} = \mathcal{I}(E_0)^{aa}. \end{aligned}$$

As argued before, for $T' \in Z_1^0(E_0,\pi)$, we have $T''(M) = 0$ for each $M \in \kappa_{E_0}(E_0)^\circ$, so that $T'(E_0'') \subseteq \kappa_{E_0}(E_0)$, and thus $T' \in \mathcal{W}(E_0)^{aa}$. Thus we conclude

$$\begin{gathered} Z_1^0(E_0,\pi) = \mathcal{I}(E_0)^{aa} = Z_2^0(E_0,\pi), \\ \mathfrak{Z}_t^{(1)}(\mathcal{N}(E_0)'') = \mathfrak{Z}_t^{(2)}(\mathcal{N}(E_0)'') = \theta_1^{-1}(\mathcal{I}(E)^{aa}). \end{gathered}$$

Examining the proof of Proposition 3.25, we see that $\mathcal{N}(E_0) \neq \mathcal{I}(E_0)$, so that we directly verify that $\mathcal{A}$ is not strongly Arens irregular. Of course, this fact also follows from Theorem 5.14. Finally, we note that $\phi_1 : E_0''\widehat{\otimes}_\varepsilon E_0' = \mathcal{A}(E_0') \to \mathcal{B}(E_0') = \mathcal{A}'$ certainly does not have dense range, so that θ_1 is not injective (thereby giving yet another way to show that $\mathcal{A}$ is not strongly Arens irregular).

5.3. Arens regularity of ideals of nuclear operators. We have so far not discussed when $\mathcal{N}_\alpha(E)$ is Arens regular. This is because we needed the above work to build up the necessary machinery.

Theorem 5.39. *Let E be a reflexive Banach space, let α be a tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Suppose that either α is accessible, or that E has the approximation property. Then $\mathcal{A}$ is Arens regular.*

Proof. The case when $\alpha = \varepsilon$ is well-known: see, for example, [You76, Theorem 3]. The case when $\alpha = \pi$ is [Dal00, Theorem 2.6.23], where the result is attributed to A. Ülger.

Suppose α is accessible. Then we simply apply Theorem 5.34. As E is reflexive, E' has the Radon–Nikodým property, and so $\mathcal{N}(E') = \mathcal{I}(E')$. Then, identifying E with E'', we have $\mathcal{W}(E) = \mathcal{B}(E)$, and so

$$Z_1^0(E,\alpha) = \mathcal{B}_\alpha(E) = Z_2^0(E,\alpha).$$

As the image of θ_1 is contained in $\mathcal{B}_\alpha(E)$, we immediately see that

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \theta^{-1}(\mathcal{B}_\alpha(E)) = \mathcal{A}'',$$

so that $\mathcal{A}$ is Arens regular.

When E has the approximation property, E and E' have the metric approximation property by Theorem 3.30. We then simply apply Theorem 5.35. ☐

Is it possible that for a nonaccessible α and a Banach space E which is reflexive, but lacks the approximation property, we have that $\mathcal{N}_\alpha(E)$ is not Arens regular? The scarcity of examples of such α or E makes this question hard to address.

6. Radicals of biduals of operator ideals

We now study the radical of $\mathcal{N}_\alpha(E)''$ for either Arens product. This is quite simple, essentially because of Proposition 5.12. Recall the definition of the radical of a Banach algebra $\mathcal{A}$,

$$\operatorname{rad}\mathcal{A} = \{a \in \mathcal{A} : e_{\mathcal{A}^\sharp} - ba \in \operatorname{Inv}\mathcal{A}^\sharp \ (b \in \mathcal{A}^\sharp)\}.$$

Here $\mathcal{A}^\sharp$ is the conditional unitisation of $\mathcal{A}$. The radical of $\mathcal{A}$ is the intersection of the kernels of the simple representations of $\mathcal{A}$, and hence represents the part of $\mathcal{A}$ which a certain representation theory cannot address. See [Dal00, Section 1.5] or [Pal94, Section 2.3] and [Pal94, Section 4] for further details about radicals of (Banach) algebras.

Lemma 6.1. *Let $\mathcal{A}$ be a nonunital Banach algebra. Then the following are equivalent:*

(1) *$a \in \operatorname{rad}\mathcal{A}$.*
(2) *For each $b \in \mathcal{A}$ and $\beta \in \mathbb{C}$, we have $c - bac - \beta ac = c - cba - \beta ca = ba + \beta a$ for some $c \in \mathcal{A}$.*
(3) *For each $b \in \mathcal{A}$ and $\beta \in \mathbb{C}$, we have $c - abc - \beta ac = c - cab - \beta ca = ab + \beta a$ for some $c \in \mathcal{A}$.*

Proof. An arbitrary element $b_0 \in \mathcal{A}^\sharp$ can be uniquely written as $b_0 = b + \beta e_{\mathcal{A}^\sharp}$ for some $b \in \mathcal{A}$ and $\beta \in \mathbb{C}$. Similarly, let $c_0 = c + \gamma e_{\mathcal{A}^\sharp} \in \mathcal{A}^\sharp$, so that

$$\begin{aligned}(e_{\mathcal{A}^\sharp} - b_0 a)c_0 &= c + \gamma e_{\mathcal{A}^\sharp} - bac - \gamma ba - \beta ac - \beta\gamma a,\\ c_0(e_{\mathcal{A}^\sharp} - b_0 a) &= c + \gamma e_{\mathcal{A}^\sharp} - cba - \gamma ba - \beta ca - \beta\gamma a.\end{aligned}$$

Thus $e_{\mathcal{A}^\sharp} - b_0 a \in \operatorname{Inv}\mathcal{A}^\sharp$ if and only if, for some $c \in \mathcal{A}$,

$$c - bac - ba - \beta ac - \beta a = 0 = c - cba - ba - \beta ca - \beta a.$$

The equivalence of (1) and (3) follows in an entirely analogous manner. $\square$

Recall the maps ψ_1 and ψ_2 defined in Theorem 5.4, and the sets I_1 and I_2 defined in Proposition 5.12.

Theorem 6.2. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then*

$$\operatorname{rad}(\mathcal{A}'', \Box) = I_1, \qquad \operatorname{rad}(\mathcal{A}'', \Diamond) = I_2.$$

In particular, when E is not reflexive, $\operatorname{rad}(\mathcal{A}'', \Box) \subsetneq \operatorname{rad}(\mathcal{A}'', \Diamond)$.

Proof. Suppose that $\Phi \not\in I_1$, so that $\theta_1(\Phi) \neq 0$. Then, for some $\Lambda \in E''$ and $M \in E'''$, we have $\langle M, \theta_1(\Phi)(\Lambda)\rangle = 1$. Let $R = M \otimes \Lambda \in \mathcal{F}(E'')$, and suppose that $\Phi \in \operatorname{rad}(\mathcal{A}'', \Box)$. Then, for some $\Psi \in \mathcal{A}''$, we have

$$\psi_1(R)\Box\Phi = \Psi - \Psi\Box\psi_1(R)\Box\Phi.$$

Applying θ_1, we have

$$R(\theta_1(\Phi)(\Lambda)) = \theta_1(\Psi)(\Lambda) - (\theta_1(\Psi) \circ R \circ \theta_1(\Phi))(\Lambda),$$

which is $\Lambda = \theta_1(\Psi)(\Lambda) - \theta_1(\Psi)(\Lambda)$, a contradiction, as $\Lambda \neq 0$.

Conversely, suppose that $\Phi \in I_1$. Fix $\beta \in \mathbb{C}$, and let $\Upsilon = \beta\Phi$. By Proposition 5.12, we have $\mathcal{A}''\Box\Phi = 0$, so for $\Psi \in \mathcal{A}''$, we have

$$\begin{aligned}\Upsilon - \Psi\Box\Phi\Box\Upsilon - \beta\Phi\Box\Upsilon &= \beta\Phi - \beta^2\Phi\Box\Phi = \beta\Phi,\\ \Upsilon - \Upsilon\Box\Psi\Box\Phi - \beta\Upsilon\Box\Phi &= \beta\Phi,\\ \Psi\Box\Phi + \beta\Phi &= \beta\Phi,\end{aligned}$$

which verifies condition (2) in the above lemma. Thus $\Phi \in \operatorname{rad}(\mathcal{A}'', \Box)$.

Similarly, suppose that, for $\Phi \in \mathcal{A}''$, we have $\mathcal{Q}(\theta_1(\Phi)) \neq 0$. Then, for some $\Lambda \in E''$, we have $\Lambda_0 := \mathcal{Q}(\theta_1(\Phi))(\Lambda) \neq 0$. Let $\mu \in E'$ be such that $\langle \Lambda_0, \mu\rangle = 1$, and set $R = \Lambda \otimes \mu \in \mathcal{F}(E')$. Suppose that $\Phi \in \operatorname{rad}(\mathcal{A}'', \Diamond)$, so that for some $\Psi \in \mathcal{A}''$, we have

$$\Phi\Diamond\psi_2(R') = \Psi - \Phi\Diamond\psi_2(R')\Diamond\Psi.$$

Applying θ_1, we have

$$\mathcal{Q}(\theta_1(\Phi)) \circ R' = \theta_1(\Psi) - \mathcal{Q}(\theta_1(\Phi)) \circ R' \circ \theta_1(\Psi),$$

where $\mathcal{Q}(\theta_1(\Phi)) \circ R' = \kappa_{E'}(\mu) \otimes \Lambda_0$, so that applying the above to Λ_0, we get

$$\langle \Lambda_0, \mu\rangle\Lambda_0 = \theta_1(\Psi)(\Lambda_0) - \langle \theta_1(\Psi)(\Lambda_0), \mu\rangle\Lambda_0.$$

Applying μ to this gives us, as $\langle \Lambda_0, \mu\rangle = 1$,

$$1 = \langle \theta_1(\Psi)(\Lambda_0), \mu\rangle - \langle \theta_1(\Psi)(\Lambda_0), \mu\rangle = 0,$$

a contradiction.

Conversely, suppose that $\Phi \in I_2$, so that $\mathcal{Q}(\theta_1(\Phi)) = 0$, and $\Phi\Diamond\mathcal{A}'' = \{0\}$. Then, for $\Psi \in \mathcal{A}''$ and $\beta \in \mathbb{C}$, let $\Upsilon = \beta\Phi$, so that we have

$$\begin{aligned}\Upsilon - \Phi\Diamond\Psi\Diamond\Upsilon - \beta\Phi\Diamond\Upsilon &= \Upsilon = \beta\Phi,\\ \Upsilon - \Upsilon\Diamond\Phi\Diamond\Psi - \beta\Upsilon\Diamond\Phi &= \Upsilon - \beta^2\Phi\Diamond\Phi = \beta\Phi,\\ \Phi\Diamond\Psi + \beta\Phi &= \beta\Phi,\end{aligned}$$

which verifies condition (3) in the above lemma. Thus $\Phi \in \operatorname{rad}(\mathcal{A}'', \Diamond)$. $\Box$

Corollary 6.3. *Let E be an infinite-dimensional Banach space with the approximation property. Then $\mathcal{N}(E)''$, with either Arens product, is not semisimple.*

Proof. We need to show that I_1 is not zero, as I_2 contains I_1. That is, we wish to show that $\theta_1 : \mathcal{N}(E)'' \to \mathcal{I}(E'')$ is not injective; that is, $\phi_1 : \mathcal{A}(E') \to \mathcal{N}(E)'$ does not have dense range, which in this case is equivalent to $\mathcal{N}(E)' = \mathcal{A}(E')$. As E has the approximation property, $\mathcal{N}(E)' = (E'\widehat{\otimes}E)' = \mathcal{B}(E')$, so we are done. $\Box$

Corollary 6.4. *Let E be a Banach space with $\mathcal{N}(E') = \mathcal{I}(E')$. Then $(\mathcal{A}(E)'', \Box)$ is semisimple while $(\mathcal{A}(E)'', \Diamond)$ is not semisimple.* $\Box$

7. Algebras of compact operators

We have not dealt yet with $\mathcal{K}(E)$, except when E has the approximation property, in which case $\mathcal{K}(E) = \mathcal{A}(E)$. In particular, we shall now generalise Theorem 5.14 and Theorem 5.39.

We can get surprisingly far in this process by just working with $\mathcal{A}(E)$. We first collect some general results about ideals and Arens products. Let $\mathcal{A}$ be a Banach algebra, and let $\mathcal{I}$ be a closed subalgebra in $\mathcal{A}$. Then $\mathcal{I}' = \mathcal{A}'/\mathcal{I}^\circ$, and so we can isometrically identify $\mathcal{I}''$ with $\mathcal{I}^{\circ\circ} \subseteq \mathcal{A}''$. We may check that the actions of $\mathcal{I}$ and

$\mathcal{I}''$ on $\mathcal{I}'$ respect the identification of $\mathcal{I}'$ with $\mathcal{A}'/\mathcal{I}^\circ$, and that the Arens products respect the identification of $\mathcal{I}''$ with $\mathcal{I}^{\circ\circ}$. We immediately see that

$$\mathfrak{Z}_t^{(i)}(\mathcal{A}'') \cap \mathcal{I}'' \subseteq \mathfrak{Z}_t^{(i)}(\mathcal{I}'') \qquad (i = 1, 2).$$

Proposition 7.1. *Let E be a Banach space. Then there is an isometry $\psi_1 : \mathcal{A}(E'') \to \mathcal{K}(E)''$ and a norm-decreasing map $\psi_2 : \mathcal{A}(E'') \to \mathcal{K}(E)''$ such that $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{A}(E'')}$ and $\theta_1 \circ \psi_2 = \mathcal{Q}$. Furthermore, $\psi_1 : \mathcal{A}(E'') \to (\mathcal{K}(E)'', \Box)$ and $\psi_2 : (\mathcal{A}(E''), \star) \to (\mathcal{K}(E)'', \Diamond)$ are homomorphisms.*

Proof. By the preceding discussion, we can use Theorem 5.4 to define homomorphisms $\psi_1 : \mathcal{A}(E'') \to (\mathcal{A}(E)'', \Box) \subseteq (\mathcal{K}(E)'', \Box)$ and $\psi_2 : (\mathcal{A}(E''), \star) \to (\mathcal{A}(E)'', \Diamond) \subseteq (\mathcal{K}(E)'', \Diamond)$.

Temporarily, let $\phi_2 : E''\widehat{\otimes}E' \to \mathcal{A}(E)'$ be the map ϕ_1, as applied to $\mathcal{A}(E)$, and let $\theta_2 = \phi_2'$. Clearly $\phi_1 : E''\widehat{\otimes}E' \to \mathcal{K}(E)'$ is such that, for $u \in E''\widehat{\otimes}E'$, $\phi_1(u)$ is equal to $\phi_2(u)$ when restricted to $\mathcal{A}(E) \subseteq \mathcal{K}(E)$. Hence θ_1 and θ_2 agree on $\mathcal{A}(E)^{\circ\circ}$, and so $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{A}(E'')}$ and $\theta_1 \circ \psi_2 = \mathcal{Q}$, as required. □

Proposition 7.2. *Let E be a Banach space. Then we have*

$$\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'')) \subseteq \mathcal{B}(E')^a, \qquad \theta_1(\mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'')) \subseteq \kappa_E \circ \mathcal{B}(E'', E).$$

Furthermore, we have

$$\begin{aligned} \psi_2(T') &\in \mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'') \qquad (T \in \mathcal{A}(E')^a), \\ \psi_1(T) &\in \mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'') \qquad (T \in \kappa_E \circ \mathcal{A}(E'', E)). \end{aligned}$$

As such, when E is not reflexive, the topological centres of $\mathcal{K}(E)''$ are distinct, neither contains the other, and both lie strictly between $\kappa_{\mathcal{K}(E)}(\mathcal{K}(E))$ and $\mathcal{K}(E)''$.

Proof. The calculations for $\theta_1(\mathfrak{Z}_t^{(i)}(\mathcal{K}(E)''))$, for $i = 1, 2$, follow exactly as for $\mathcal{A}(E)$, as in Proposition 5.8.

Let $T = \Lambda \otimes \mu \in \mathcal{F}(E')$ and let (x_α) be a bounded net in E which tends to Λ in the weak*-topology. Let $\lambda \in \mathcal{K}(E)'$, and let $S \in \mathcal{I}(E')$ be such that $\langle \lambda, R \rangle = \langle S, R \rangle$ for $R \in \mathcal{A}(E)$. Then we see that

$$\langle \psi_2(T'), \lambda \rangle = \mathrm{tr}(TS) = \langle \Lambda, S(\mu) \rangle = \lim_\alpha \langle S(\mu), x_\alpha \rangle = \lim_\alpha \langle \lambda, \mu \otimes x_\alpha \rangle.$$

Thus, for $K \in \mathcal{K}(E)$,

$$\begin{aligned} \langle \lambda \cdot \psi_2(T'), K \rangle &= \langle \psi_2(T'), K \cdot \lambda \rangle = \lim_\alpha \langle K \cdot \lambda, \mu \otimes x_\alpha \rangle \\ &= \lim_\alpha \langle \lambda, K'(\mu) \otimes x_\alpha \rangle = \langle \Lambda, SK'(\mu) \rangle = \langle \phi_1(S'(\Lambda) \otimes \mu), K \rangle. \end{aligned}$$

For $x \in E$ and $K \in \mathcal{K}(E)$, we have that

$$\begin{aligned} &\langle \lambda \cdot (\mu \otimes x), K \rangle = \langle \lambda, K'(\mu) \otimes x) \rangle = \langle SK'(\mu), x \rangle = \langle \phi_1(S'\kappa_E(x) \otimes \mu), K \rangle, \\ &\langle (\mu \otimes x) \cdot \lambda, K \rangle = \langle \lambda, \mu \otimes K(x) \rangle = \langle K''\kappa_E(x), S(\mu) \rangle = \langle \phi_1(\kappa_E(x) \otimes S(\mu)), K \rangle. \end{aligned}$$

As S is weakly-compact, there exists $S_0 \in \mathcal{B}(E''', E')$ such that $S'' = \kappa_{E'} \circ S_0$, and so, for $\Phi \in \mathcal{K}(E)''$,

$$\begin{aligned} \langle \psi_2(T')\Box\Phi, \lambda\rangle &= \lim_\alpha \langle \Phi\cdot\lambda, \mu\otimes x_\alpha\rangle = \lim_\alpha \langle \Phi, \lambda\cdot(\mu\otimes x_\alpha)\rangle \\ &= \lim_\alpha \langle \Phi, \phi_1(S'\kappa_E(x_\alpha)\otimes\mu)\rangle = \lim_\alpha \langle \theta_1(\Phi)S'\kappa_E(x_\alpha), \mu\rangle \\ &= \lim_\alpha \langle S_0\theta_1(\Phi)'\kappa_{E'}(\mu), x_\alpha\rangle = \langle \Lambda, S_0\theta_1(\Phi)'\kappa_{E'}(\mu)\rangle \\ &= \langle \theta_1(\Phi)S'(\Lambda), \mu\rangle = \langle \Phi, \phi_1(S'(\Lambda)\otimes\mu)\rangle = \langle \psi_2(T')\Diamond\Phi, \lambda\rangle. \end{aligned}$$

As λ and Φ were arbitrary, we see that $\psi_2(T') \in \mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'')$, as required.

Now let $T = M \otimes \kappa_E(x) \in \kappa_E \circ \mathcal{F}(E'', E)$, and let (μ_α) be a bounded net in E' tending to M in the weak*-topology. Let $\lambda \in \mathcal{K}(E)'$ and $S \in \mathcal{I}(E')$ be as before, so that

$$\begin{aligned} \langle \psi_1(T), \lambda\rangle &= \operatorname{tr}(\eta(TS')) = \operatorname{tr}\left(\kappa_{E'}'\kappa_{E''}\kappa_E(x)\otimes\kappa_E' S''(M)\right) = \langle S''(M), \kappa_E(x)\rangle \\ &= \langle M, S'\kappa_E(x)\rangle = \lim_\alpha \langle S(\mu_\alpha), x\rangle = \lim_\alpha \langle \lambda, \mu_\alpha\otimes x\rangle, \end{aligned}$$

again using that S is weakly-compact. Then, for $K \in \mathcal{K}(E)$, as K is weakly-compact,

$$\begin{aligned} \langle \psi_1(T)\cdot\lambda, K\rangle &= \lim_\alpha \langle \lambda\cdot K, \mu_\alpha\otimes x\rangle = \lim_\alpha \langle S(\mu_\alpha), K(x)\rangle \\ &= \langle M, S'\kappa_E K(x)\rangle = \langle \phi_1(\kappa_E(x)\otimes\kappa_E' S''(M)), K\rangle. \end{aligned}$$

Thus for $\Phi \in \mathcal{K}(E)''$,

$$\begin{aligned} \langle \Phi\Diamond\psi_1(T), \lambda\rangle &= \lim_\alpha \langle \lambda\cdot\Phi, \mu_\alpha\otimes x\rangle = \lim_\alpha \langle \Phi, \phi_1(\kappa_E(x)\otimes S(\mu_\alpha))\rangle \\ &= \lim_\alpha \langle \theta_1(\Phi)\kappa_E(x), S(\mu_\alpha)\rangle = \langle M, S'\theta_1(\Phi)\kappa_E(x)\rangle \\ &= \langle \theta_1(\Phi)\kappa_E(x), \kappa_E' S''(M)\rangle = \langle \Phi\Box\psi_1(T), \lambda\rangle. \end{aligned}$$

As λ and Φ were arbitrary, we see that $\psi_1(T) \in \mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'')$, as required. $\square$

We need a good description of $\mathcal{K}(E)'$, for which we use an idea from [FS75]. Let E be a Banach space and let $I \subseteq E'_{[1]}$ be a norming subset, that is

$$\|x\| = \sup\{|\langle \mu, x\rangle| : \mu \in I\} \qquad (x \in E).$$

For example, when E is separable, we can take I to be countable. Then let $\iota : E \to l^\infty(I)$ be the map

$$\iota(x) = \left(\langle \mu, x\rangle\right)_{\mu\in I} \in l^\infty(I),$$

so that ι is an isometry. Let $J : \mathcal{K}(E) \to \mathcal{K}(E, l^\infty(I))$ be given by $J(T) = \iota\circ T$ for $T \in \mathcal{K}(E)$, so that J is an isometry. As $l^\infty(I)'$ has the metric approximation property, we have

$$\mathcal{K}(E, l^\infty(I)) = \mathcal{A}(E, l^\infty(I)) = E'\check{\otimes} l^\infty(I),$$

so that $\mathcal{K}(E, l^\infty(I))' = \mathcal{I}(E', l^\infty(I)')$. Thus $J' : \mathcal{I}(E', l^\infty(I)') \to \mathcal{K}(E)'$ is a quotient operator, and $J'' : \mathcal{K}(E)'' \to \mathcal{I}(E', l^\infty(I)')'$ is an isometry onto its range.

Lemma 7.3. *Let E be a Banach space and I, ι, J be as above. For each $\lambda \in \mathcal{K}(E)'$, there exists $S \in \mathcal{I}(E', l^\infty(I)')$ with $\|S\|_\pi = \|\lambda\|$ and $J'(S) = \lambda$. For $\Phi \in \mathcal{K}(E)''$, let $\Phi\cdot\lambda = J'(S_1)$ and $\lambda\cdot\Phi = J'(S_2)$ for some $S_1, S_2 \in \mathcal{I}(E', l^\infty(I)')$. Then we have*

$$\iota'\circ S_1 = \iota'\circ\kappa_{l^\infty(I)}'\circ S''\circ\theta_1(\Phi)'\circ\kappa_{E'}, \qquad \iota'\circ S_2 = \eta(\theta_1(\Phi))\circ\iota'\circ S.$$

Proof. As J isometrically identifies $\mathcal{K}(E)$ with a subspace of $\mathcal{A}(E, l^\infty(I))$, for $\lambda \in \mathcal{K}(E)'$, we can extend λ to a member of $\mathcal{A}(E, l^\infty(I))'$ by the Hahn–Banach theorem. This gives us the required $S \in \mathcal{I}(E', l^\infty(I)')$.

Let $R = \mu \otimes x \in \mathcal{K}(E)$ so that, for $T \in \mathcal{K}(E)$, we have

$$\begin{aligned}
\langle \lambda \cdot R, T\rangle &= \langle J'(S), R \circ T\rangle = \langle S, T'(\mu) \otimes \iota(x)\rangle = \langle (\kappa_{l^\infty(I)} \circ \iota)(x), (S \circ T')(\mu)\rangle \\
&= \langle \phi_1\big((S' \circ \kappa_{l^\infty(I)} \circ \iota)(x) \otimes \mu\big), T\rangle, \\
\langle R \cdot \lambda, T\rangle &= \langle J'(S), T \circ R\rangle = \langle S, \mu \otimes \iota(T(x))\rangle = \langle S(\mu), \iota(T(x))\rangle \\
&= \langle \phi_1\big(\kappa_E(x) \otimes (\iota' \circ S)(\mu)\big), T\rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\langle S_1(\mu), \iota(x)\rangle &= \langle S_1, J(\mu \otimes x)\rangle = \langle J'(S_1), R\rangle = \langle \Phi \cdot \lambda, R\rangle = \langle \Phi, \lambda \cdot R\rangle \\
&= \langle \Phi, \phi_1((S' \circ \kappa_{l^\infty(I)} \circ \iota)(x) \otimes \mu)\rangle = \langle (\theta_1(\Phi) \circ S' \circ \kappa_{l^\infty(I)} \circ \iota)(x), \mu\rangle \\
&= \langle (\kappa'_{l^\infty(I)} \circ S'' \circ \theta_1(\Phi)' \circ \kappa_{E'})(\mu), \iota(x)\rangle, \\
\langle S_2(\mu), \iota(x)\rangle &= \langle \lambda \cdot \Phi, R\rangle = \langle \Phi, R \cdot \lambda\rangle = \langle (\theta_1(\Phi) \circ \kappa_E)(x), (\iota' \circ S)(\mu)\rangle \\
&= \langle (\kappa'_E \circ \theta_1(\Phi)' \circ \kappa_{E'} \circ \iota' \circ S)(\mu), x\rangle,
\end{aligned}$$

as required. □

Proposition 7.4. *Let E be a Banach space and $\theta_1 : \mathcal{K}(E)'' \to \mathcal{B}(E'')$ be as before. Let*

$$I_1 = \ker \theta_1 \subseteq \mathcal{K}(E)'', \qquad I_2 = \ker(\mathcal{Q} \circ \theta_1) \subseteq \mathcal{K}(E)''.$$

Then I_1 is a closed ideal for either Arens product, and I_2 is a closed ideal in $(\mathcal{K}(E)'', \Diamond)$. Furthermore, we have

$$\mathcal{K}(E)'' \Box I_1 = I_1 \Diamond \mathcal{K}(E)'' = I_2 \Diamond \mathcal{K}(E)'' = \{0\}.$$

Proof. The first part follows exactly as in Proposition 5.12. Fix $\lambda \in \mathcal{K}(E)'$ and let $S \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S) = \lambda$. For $\Phi \in I_1$, let $S_1 \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S_1) = \Phi \cdot \lambda$, so that

$$\iota' \circ S_1 = \iota' \circ \kappa'_{l^\infty(I)} \circ S'' \circ \theta_1(\Phi)' \circ \kappa_{E'} = 0.$$

Hence we have, for $T \in \mathcal{K}(E)$,

$$\langle \Phi \cdot \lambda, T\rangle = \langle J'(S_1), T\rangle = \langle S_1, \iota \circ T\rangle = \operatorname{tr}(S_1 \circ T' \circ \iota') = \operatorname{tr}(\iota' \circ S_1 \circ T') = 0,$$

so that $\Phi \cdot \lambda = 0$, and hence

$$\langle \Psi \Box \Phi, \lambda\rangle = \langle \Psi, \Phi \cdot \lambda\rangle = 0 \qquad (\Psi \in \mathcal{K}(E)'').$$

As λ was arbitrary, we have $\mathcal{K}(E)'' \Box I_1 = \{0\}$. Similarly, let $S_2 \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S_2) = \lambda \cdot \Phi$, so that

$$\iota' \circ S_2 = \eta(\theta_1(\Phi)) \circ \iota' \circ S = 0.$$

Hence we have

$$\langle \lambda \cdot \Phi, T\rangle = \langle J'(S_2), T\rangle = \langle S_2, J(T)\rangle = \operatorname{tr}(T' \circ \iota' \circ S_2) = 0 \qquad (T \in \mathcal{K}(E)),$$

so that $\lambda \cdot \Phi = 0$. Thus $I_1 \Diamond \mathcal{K}(E)'' = \{0\}$.

Similarly, let $\Phi \in I_2$, so that $\eta(\theta_1(\Phi)) = 0$, and hence $\iota' \circ S_2 = 0$ when $J'(S_2) = \lambda \cdot \Phi$. Following the previous paragraph, we see that $I_2 \Diamond \mathcal{K}(E)'' = \{0\}$. □

As before, we now turn our attention to when we can use nuclear and not integral operators. This takes us to a result shown in [FS75].

Theorem 7.5. *Let E and F be Banach spaces such that E'' or F' has the Radon–Nikodým property. Define $V : E''\widehat{\otimes}F' \to \mathcal{K}(E,F)'$ by*

$$\langle V(\Phi\otimes\mu), T\rangle = \langle \Phi, T'(\mu)\rangle \qquad (\Phi\otimes\mu \in E''\widehat{\otimes}F', T\in\mathcal{K}(E,F)).$$

Then V is a quotient operator, and furthermore, for $\lambda\in\mathcal{K}(E,F)'$, there exists $u\in E''\widehat{\otimes}F'$ with $V(u)=\lambda$ and $\|u\|=\|\lambda\|$. Also, given $J_\pi : E''\widehat{\otimes}F' \to \mathcal{N}(E',F')\subseteq I(E',F')$, we have $\ker V\subseteq\ker J_\pi$.

Proof. This is [FS75, Theorem 1]. We will sketch the easier case, which is when E'' has the Radon–Nikodým property. Form I and $\iota : F\to l^\infty(I)$ in a similar way to above, and define $J:\mathcal{K}(E,F)\to\mathcal{A}(E,l^\infty(I))$ by $J(T)=\iota\circ T$ for $T\in\mathcal{K}(E,F)$. Then

$$(l^\infty(I)\check{\otimes}E')' = \mathcal{I}(l^\infty(I),E'') = \mathcal{N}(l^\infty(I),E'') = l^\infty(I)'\widehat{\otimes}E'',$$

as E'' has the Radon–Nikodým property and $l^\infty(I)$ is a dual space with the approximation property. By applying the swap map to both sides, we see that

$$\mathcal{K}(E,l^\infty(I))' = (E'\check{\otimes}l^\infty(I))' = E''\widehat{\otimes}l^\infty(I)'.$$

Thus $J' : E''\widehat{\otimes}l^\infty(I)'\to\mathcal{K}(E,F)'$. Hence we have the following diagram.

$$\begin{tikzcd} E''\widehat{\otimes}l^\infty(I)' \arrow[r, "J'"] \arrow[d, "\mathrm{Id}_{E''}\otimes\iota'"'] & \mathcal{K}(E,F)' . \\ E''\widehat{\otimes}F' \arrow[ur, "V"'] & \end{tikzcd}$$

We can verify that this diagram commutes, so as J is an isometry, J' is a quotient operator. As $\mathrm{Id}_{E''}\otimes\iota'$ is norm-decreasing, V must also be a quotient operator. We can then easily verify the other claims, and the case when F' has the Radon–Nikodým property follows in a similar manner. □

In particular, when E' or E'' has the Radon–Nikodým property, we have a quotient operator $V : E''\widehat{\otimes}E'\to\mathcal{K}(E)'$, and this respects the usual identification of $\mathcal{A}(E)'=\mathcal{I}(E')=\mathcal{N}(E')$. It is clear that V agrees with the map ϕ_1, and so $\theta_1 : \mathcal{K}(E)''\to\mathcal{B}(E'')$ is an isometry onto its range which contains $\mathcal{A}(E'')$ by Proposition 7.1. Indeed, we have

$$\theta_1(\mathcal{K}(E)'') = (\ker\phi_1)^\circ = \{T\in\mathcal{B}(E'') : \langle T,\tau\rangle = 0 \ (\tau\in E''\widehat{\otimes}E', \phi_1(\tau)=0)\}.$$

Theorem 7.6. *Let E be a Banach space such that E' or E'' has the Radon–Nikodým property. Then $\mathcal{K}(E)''$ is identified isometrically with $X=\theta_1(\mathcal{K}(E)'')\subseteq\mathcal{B}(E'')$ and we have*

$$\mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'') = X\cap\mathcal{B}(E')^a, \qquad \mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'') = X\cap(\kappa_E\circ\mathcal{B}(E'',E))$$
$$\mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'')\cap\mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'') = X\cap\mathcal{W}(E)^{aa}.$$

Proof. This follows exactly as in the $\mathcal{A}(E)$ case, Corollary 5.36. □

Notice that we cannot easily generalise Lemma 5.33 as we have no simple description of $\ker\phi_1$. Indeed, it is hard to say whether $\ker\phi_1$ is trivial or not.

Definition 7.7. Let E be a Banach space such that for each compact subset $K \subseteq E$ and each $\varepsilon > 0$, there exists $T \in \mathcal{K}(E)$ with $\|T(x) - x\| < \varepsilon$ for each $x \in K$. Then E has the *compact approximation property*. When we can control the norm of T, E has the *bounded compact approximation property* or the *metric compact approximation property*, as appropriate.

We might be tempted to suppose that ϕ_1 is injective when E' has the compact approximation property. This does not seem to be true in general, unlike the $\mathcal{A}(E)$ case.

The paper [GW93] is a good source of information on the compact approximation property, when applied to algebraic questions about $\mathcal{K}(E)$. We will come back to this, but for now, we need a definition from [GW93].

Definition 7.8. Let E be a Banach space. Then E' has the *$\mathcal{K}(E)^a$-approximation property* if, for each compact subset $K \subseteq E'$ and each $\varepsilon > 0$, there exists $T \in \mathcal{K}(E)$ such that $\|T'(\mu) - \mu\| \leq \varepsilon$ for each $\mu \in K$. Similarly, we have the idea of the *bounded $\mathcal{K}(E)^a$-approximation property.*

Thus the $\mathcal{K}(E)^a$-approximation property is stronger than E' having the compact approximation property, and [GW93, Example 4.3] shows that, in general, these properties do not coincide. In [GW93, Section 3], a sufficient condition on E is given for these properties to be the same, but given the lack of examples of Banach spaces without the (compact) approximation property, it is left open if this condition on E is common or not.

Then [GW93, Corollaries 2.6, 2.7] states that E' has the bounded $\mathcal{K}(E)^a$-approximation property if and only if $\mathcal{K}(E)$ has a bounded right approximate identity, or equivalently, a bounded approximate identity.

Proposition 7.9. *Let E be a Banach space such that E' or E'' has the Radon–Nikodým property, so that $\mathcal{K}(E)''$ is identified with a subalgebra of $\mathcal{B}(E'')$. When ϕ_1 is injective, E' has the metric $\mathcal{K}(E)^a$-approximation property. Conversely, when E' has the $\mathcal{K}(E)^a$-approximation property, we have that $\mathcal{B}(E)^{aa} \subseteq \mathcal{K}(E)''$ and that E' has the metric $\mathcal{K}(E)^a$-approximation property.*

Proof. Given the above, we see that E' has the bounded $\mathcal{K}(E)^a$-approximation property if and only if $\mathcal{K}(E)''$ has a mixed identity. As $\mathcal{K}(E)' = \phi_1(E''\widehat{\otimes}E')$, we see that ϕ_1 is injective if and only if ϕ_1 is an isometry $E''\widehat{\otimes}E' \to \mathcal{K}(E)'$, which is if and only if $\theta_1 : \mathcal{K}(E)'' \to \mathcal{B}(E'')$ is surjective. We can easily see that $\Xi \in \mathcal{K}(E)''$ is a mixed identity if and only if $\theta_1(\Xi) = \mathrm{Id}_{E''}$, in which case, as θ_1 is an isometry, we have that E' has the metric $\mathcal{K}(E)^a$-approximation property. We see immediately that when ϕ_1 is injective, E' has the metric $\mathcal{K}(E)^a$-approximation property.

Conversely, suppose that $\tau \in E''\widehat{\otimes}E'$ is such that $\phi_1(\tau) = 0$. We can find a representative $\tau = \sum_{n=1}^\infty \Lambda_n \otimes \mu_n$ with $\sum_{n=1}^\infty \|\Lambda_n\| < \infty$ and $\|\mu_n\| \to 0$ as $n \to \infty$. Let $S \in \mathcal{B}(E)$. Then $(S'(\mu_n))_{n=1}^\infty$ is a compact subset of E', so as E' has the $K(E)^a$-approximation property, for each $\varepsilon > 0$, there exists $R \in \mathcal{K}(E)$ with $\|S'(\mu_n) - R'(S'(\mu_n))\| < \varepsilon$ for each n. As $S \circ R \in \mathcal{K}(E)$, we hence have

$$\begin{aligned}
|\langle S'', \tau\rangle| &= |\langle S'', \tau\rangle - \langle \phi_1(\tau), S \circ R\rangle| \\
&= \left|\sum_{n=1}^\infty \langle \Lambda_n, S'(\mu_n) - R'(S'(\mu_n))\rangle\right| \leq \varepsilon \sum_{n=1}^\infty \|\Lambda_n\|.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we see that $\langle S'', \tau\rangle = 0$, and as $\tau \in \ker\phi_1$ was arbitrary, we see that $S'' \in \theta_1(\mathcal{K}(E)'')$, as required. Then $\mathrm{Id}_{E''} = \mathrm{Id}''_E \in \theta_1(\mathcal{K}(E)'')$, so again, E' has the metric $\mathcal{K}(E)^a$-approximation property. $\square$

This result is very similar to [GS88, Corollary 1.6], which in turn is an improvement of the argument used in [CJ85]. This gives yet another example of parallel development in this area, as the authors of [GW93] seem to have been unaware of these results.

The reason this is weaker than the corresponding result for $\mathcal{A}(E)$ is that we can easily show that $\mathcal{A}(E'') \subseteq \mathcal{A}(E)''$, but we do not know that $\mathcal{K}(E'') \subseteq \mathcal{K}(E)''$. Of course, when E is reflexive, this is not a problem.

Theorem 7.10. *Let E be a reflexive Banach space. Then $\mathcal{K}(E)$ is Arens regular, and $\mathcal{K}(E)''$ is identified, by θ_1, with an ideal in $\mathcal{B}(E)$. Furthermore, $\mathcal{K}(E)'' = \mathcal{B}(E)$ if and only if E has the compact approximation property.*

Proof. As E is reflexive, E' has the Radon–Nikodým property, and so $\phi_1 : E\widehat{\otimes}E' \to \mathcal{K}(E)'$ is a quotient operator, and $\theta_1 : \mathcal{K}(E)'' \to \mathcal{B}(E)$ is an isometry onto its range. We immediately see that $\mathcal{K}(E)$ is Arens regular (this is also shown in [Dal00, Theorem 2.6.23], and, in a more limited case, in [Pal85, Theorem 3]). The proof is complete by applying the above proposition. $\square$

Corollary 7.11. *Let E be a reflexive Banach space with the compact approximation property. Then E has the metric compact approximation property.* $\square$

Example 7.12. In [Wil92], Willis constructs a reflexive Banach space W which has the metric compact approximation property, but which does not have the approximation property. Thus we see that $\mathcal{K}(W)'' = \mathcal{B}(W)$, while $\mathcal{A}(W)''$ is, isometrically, a proper ideal in $\mathcal{B}(W)$. This example answers, in the affirmative, the question asked before Theorem 3 in [Pal85].

There do exist Banach spaces without the compact approximation property, for example, those constructed in [Sza78]. In general, however, we do not have a good supply of Banach spaces without the compact approximation property, a fact which explains the slightly hesitant approach taken in this section.

We finish this section by looking at the radicals of $\mathcal{K}(E)''$. This is simple, given the work we have already done.

Theorem 7.13. *Let E be a Banach space. Then we have*

$$\mathrm{rad}(\mathcal{K}(E)'', \Box) = I_1 = \ker\theta_1, \qquad \mathrm{rad}(\mathcal{K}(E)'', \Diamond) = I_2 = \ker(\mathcal{Q}\circ\theta_1).$$

Proof. Examining the proof of Theorem 6.2, we see that we only use properties of I_1, I_2, ψ_1 and ψ_2 which have been established for $\mathcal{K}(E)''$ in Proposition 7.4 and Proposition 7.1. Thus we simply use the same argument. $\square$

8. Conclusion

We have given a fairly complete classification of the topological centres arising from algebras of operators associated with tensor norms. In seems likely that given any concrete Banach space, the calculations to determine the topological centres can be carried out. However, we fall short of giving the complete range of behaviour of the topological centres, because of a lack of examples of "extreme" Banach spaces, namely, Banach spaces lacking the approximation property.

In recent years, building on the work of [GM93], there has been considerable interest and progress in the study of Banach spaces with a "small" space of operators (usually meaning that the strictly singular operators are of finite-codimension in the algebra of all operators). It is the author's opinion that a similar (although no doubt very hard) programme studying the known constructions of Banach spaces failing the approximation property could yield new constructions of spaces which fail the approximation property, and yet are more amenable to study than those currently known.

A more tentative line of enquiry would be to generalise the key idea of Section 7. Namely, we use the fact that if $\iota : E \to l^\infty(I)$ is an isometry, then $T \in \mathcal{B}(E)$ is compact if and only if $\iota \circ T$ is approximable. We might then define, for a tensor norm α, $T \in \mathcal{K}_\alpha(E)$ if and only if $\iota \circ T \in \mathcal{N}_\alpha(E, l^\infty(I))$. Does this lead to interesting algebras of operators, and are they amenable to study? Again, it seems likely that one would first need a good supply of Banach spaces without the approximation property.

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