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A simple characterization of sets satisfying the Central Sets Theorem

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Abstract. Central subsets of a discrete semigroup S have very strong combinatorial properties which are a consequence of the Central Sets Theorem. Call a set a *C-set* if it satisfies the conclusion of the Central Sets Theorem. We obtain here a reasonably simple characterization of C-sets in an arbitrary discrete semigroup.

Contents

1. Introduction

In [2] Furstenberg defined a *central* subset of $\mathbb{N}$ in terms of notions from topological dynamics. (Specifically, a subset A of $\mathbb{N}$ is *central* if and only if there exist a dynamical system (X, T), points x and y of X, and a neighborhood U of y such that y is uniformly recurrent, x and y are proximal, and $A = \{n \in \mathbb{N} : T^n(x) \in U\}$. See [2] or [5] for the definitions of "dynamical system", "proximal", and "uniformly recurrent".) He showed that if $\mathbb{N}$ is divided into finitely many classes, then one of them must be central, and he proved the following theorem.

Theorem 1.1 (Furstenberg)**.** *Let C be a central subset of $\mathbb{N}$. Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let f_i be a sequence in $\mathbb{Z}$. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that:*

(1) *For all n, $\max H_n < \min H_{n+1}$.*

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(2) *For all* $F \in \mathcal{P}_f(\mathbb{N})$ *and all* $i \in \{1, 2, \ldots, l\}$,

$$\sum_{n \in F} \left(a_n + \sum_{t \in H_n} f_i(t) \right) \in C.$$

Proof. [2, Proposition 8.21]. □

Based on an idea of Vitaly Bergelson, it was shown that central subsets of $\mathbb{N}$ have a simple characterization in terms of the algebra of $\beta\mathbb{N}$, the Stone–Čech compactification of the discrete space $\mathbb{N}$. This characterization of central sets applies without change to an arbitrary discrete semigroup, and we take that as our definition. (We shall give a brief introduction to the algebra of βS for a discrete semigroup S at the end of this introduction.)

In this paper we shall always take S to be a discrete semigroup.

Definition 1.2. Let S be a discrete semigroup, let $K(\beta S)$ be the smallest two sided ideal of βS, and let $A \subseteq S$. Then A is *central* if and only if there is an idempotent in $K(\beta S) \cap c\ell A$.

Theorem 1.1 was already strong enough to derive several combinatorial consequences such as Rado's Theorem [8]. Subsequently, several incremental strengthenings were found. (See [1] for a listing of these.) What is currently the most general version of the Central Sets Theorem (for commutative semigroups) is the following. (There is also a version for noncommutative semigroups which we will present as Theorem 2.3.)

Theorem 1.3. *Let* $(S, +)$ *be a commutative semigroup and let* $\mathcal{T} = {}^{\mathbb{N}}S$*, the set of sequences in* S*. Let* C *be a central subset of* S*. There exist functions* $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ *and* $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ *such that:*

(1) *If* $F, G \in \mathcal{P}_f(\mathcal{T})$ *and* $F \subsetneq G$*, then* $\max H(F) < \min H(G)$.

(2) *Whenever* $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_m$*, and for each* $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$*, one has*

$$\sum_{i=1}^{m} \left(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t) \right) \in C.$$

Proof. [1, Theorem 2.2]. □

We introduce a name for sets satisfying the conclusion of the Central Sets Theorem.

Definition 1.4. Let $(S, +)$ be a commutative semigroup, let $C \subseteq S$, and let $\mathcal{T} = {}^{\mathbb{N}}S$. The set C is a *C-set* if and only if there exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that:

(1) If $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$.

(2) Whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has

$$\sum_{i=1}^{m}\left(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\right) \in C.$$

Central sets are important, and algebraically are easy to work with. However from a combinatorial viewpoint, C-sets are the objects that matter. In $(\mathbb{N}, +)$ and in many other semigroups, they are the objects that contain solutions to partition regular systems of homogeneous equations as well as the other myriads of properties that are a consequence of the Central Sets Theorem. See, for example [7, Theorem 2.8].

In [4] some combinatorial characterizations of central sets were obtained. One could not call them "simple" however because they all involved the assertion that some collection of sets was *collectionwise piecewise syndetic*, a notion with a quite complicated definition. In Section 2 of this paper, we shall obtain similar characterizations of C-sets, but we claim that these characterizations are indeed simple because they depend on the much simpler notion of J-sets.

Definition 1.5. Let $(S, +)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$. A set $A \subseteq S$ is a *J-set* if and only if whenever $F \in \mathcal{P}_f(\mathcal{T})$, there exist $d \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $d + \sum_{t \in H} f(t) \in A$.

If S is not commutative, then the definition of J-sets is somewhat more complicated, but still much simpler than the statement of the noncommutative Central Sets Theorem.

One of the reasons that J-sets are interesting is that, if S is a discrete commutative semigroup, then every subset of S with positive upper density is a J-set [7, Theorem 6.10]. This fact, of course, generalizes Szemerédi's Theorem [9].

We conclude this introduction now with our promised presentation of some details about the algebraic structure of βS, and one more result that we will use later.

Given a discrete semigroup $(S, \cdot)$, we take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. The operation $\cdot$ extends to βS in such a way that $(\beta S, \cdot)$ is a right topological semigroup, meaning that for each $p \in \beta S$. the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous. Further, if $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. Any compact Hausdorff right topological semigroup T has a smallest two sided ideal, $K(T)$ which is the union of all of the minimal left ideals of T as well as the union of all of the minimal right ideals of T. Minimal left ideals of T are closed. Given $A \subseteq S$, $c\ell(A) = \overline{A} = \{p \in \beta S : A \in p\}$. Thus $A \subseteq S$ is central if and only if there is an idempotent $p \in K(\beta S)$ such that $A \in p$. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if

$\{x \in S : x^{-1}A \in q\} \in p$ where $x^{-1}A = \{y \in S : xy \in A\}$. If the operation is written additively, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$ where $-x + A = \{y \in S : x + y \in A\}$. Notice that, while in this case we write the operation on βS additively, $(\beta S, +)$ is very unlikely to be commutative. See [5] for an elementary introduction to the algebra of βS and for any unfamiliar details.

2. Characterizing C-sets

In this section we extend the notion of C-sets to arbitrary semigroups, and derive a simple characterization of such sets.

The notation in the following definition does not reflect all of the variables upon which it depends. In a noncommutative semigroup, when we write $\prod_{t\in F} x_t$ we mean the product in increasing order of indices.

Definition 2.1. Let $(S, \cdot)$ be a semigroup.

(a) For $m \in \mathbb{N}$,

$$\mathcal{I}_m = \{\big(H(1), H(2), \ldots, H(m)\big) : \text{each } H(j) \in \mathcal{P}_f(\mathbb{N}) \text{ and for any } j \in \{1, 2, \ldots, m-1\},\ \max H(j) < \min H(j+1)\}.$$

(b) $\mathcal{T} = {}^{\mathbb{N}}S$.

(c) Given $m \in \mathbb{N}$, $a \in S^{m+1}$, $H \in \mathcal{I}_m$, and $f \in \mathcal{T}$,

$$x(m, a, H, f) = \Big(\prod_{j=1}^{m} \big(a(j) \cdot \prod_{t\in H(j)} f(t)\big)\Big) \cdot a(m+1)\,.$$

(d) $J(S) = \{p \in \beta S : (\forall A \in p)\ A \text{ is a } J\text{-set}\}$.

Definition 2.2. Let $(S, \cdot)$ be a semigroup.

(a) $A \subseteq S$ is a *J-set* if and only if for each $F \in \mathcal{P}_f(\mathcal{T})$ there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that for each $f \in F$, $x(m, a, H, f) \in A$.

(b) $A \subseteq S$ is a *C-set* if and only if there exist $m : \mathcal{P}_f(\mathcal{T}) \to \mathbb{N}$, $\alpha \in \times_{F\in\mathcal{P}_f(\mathcal{T})} S^{m(F)+1}$, and $H \in \times_{F\in\mathcal{P}_f(\mathcal{T})} \mathcal{I}_{m(F)}$ such that:

(i) If $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then

$$\max\big(H(F)\big(m(F)\big)\big) < \min\big(H(G)(1)\big).$$

(ii) Whenever $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_n$, and for each $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$, one has

$$\prod_{i=1}^{n} x(m(G_i), \alpha(G_i), H(G_i), f_i) \in A.$$

As in the commutative case, we see that central sets are C-sets. As we remark before the proof of Theorem 2.8, the converse of this statement is not true.

Theorem 2.3. *Let $(S, \cdot)$ be a semigroup and let A be a central set in S. Then A is a C-set.*

Proof. [1, Corollary 3.10]. □

It is not immediately obvious that if S is commutative, then the definitions of C-set given in Definitions 1.4 and 2.2(b) agree or that the definitions of J-set given in Definitions 1.5 and 2.2(a) agree. But, by [6, Lemma 2.4], these definitions do agree. There is a simple algebraic characterization of C-sets.

Theorem 2.4. *Let S be an infinite semigroup and let $A \subseteq S$. Then A is a C-set if and only if there is an idempotent $p \in J(S) \cap \overline{A}$.*

Proof. [1, Theorem 3.8]. □

Recall that ω is the first infinite ordinal and each ordinal is the set of its predecessors. In particular $0 = \emptyset$ and for $n \in \mathbb{N}$, $n = \{0, 1, \ldots, n-1\}$.

Definition 2.5. (a) If f is a function and $\text{domain}(f) = n \in \omega$, then for all x, $f^\frown x = f \cup \{(n, x)\}$.
(b) Let T be a set of functions whose domains are members of ω. For each $f \in T$, $B_f(T) = \{x : f^\frown x \in T\}$.

The necessity part of the following, moderately technical, lemma is the key to our characterization of C-sets.

Lemma 2.6. *Let S be an infinite semigroup and let $p \in \beta S$. Then p is an idempotent if and only if for each $A \in p$ there is a nonempty set T of functions such that:*

(1) *For all $f \in T$, $\text{domain}(f) \in \omega$ and $\text{range}(f) \subseteq A$.*
(2) *For all $f \in T$, $B_f(T) \in p$.*
(3) *For all $f \in T$ and all $x \in B_f(T)$, $B_{f^\frown x}(T) \subseteq x^{-1}B_f(T)$.*

Proof. Sufficiency. Let $A \in p$ and let T be as guaranteed. We need to show that $\{x \in S : x^{-1}A \in p\} \in p$. To this end, pick $f \in F$. Then $B_f(T) \in p$ so it suffices to show that $B_f(T) \subseteq \{x \in S : x^{-1}A \in p\}$. Let $x \in B_f(T)$. Then $B_{f^\frown x}(T) \in p$ and $B_{f^\frown x}(T) \subseteq x^{-1}B_f(T) \subseteq x^{-1}A$ so $x^{-1}A \in p$ as required.

Necessity. Let $A \in p$. Given any $B \in p$, let $B^\star = \{x \in B : x^{-1}B \in p\}$. By [5, Lemma 4.14], $B^\star \in p$ and if $x \in B^\star$, then $x^{-1}B^\star \in p$. We define inductively for $n \in \omega$, $T_n = \{f \in T : \text{domain}(f) = n\}$ and for $f \in T_n$, $B_f = B_f(T)$. Of course $T_0 = \{\emptyset\}$. We let $B_\emptyset = A^\star$.

Now let $n \in \omega$ and assume that we have defined T_k for $k \leq n$ and have defined B_f for $f \in T_k$ such that:

(i) T_k is a set of functions with domain k and range contained in A.
(ii) If $f \in T_k$ and $x \in B_f$, then $B_f \in p$ and $x^{-1}B_f \in p$.
(iii) If $k < n$, $f \in T_k$, and $x \in B_f$, then $B_{f^\frown x} = (x^{-1}B_f)^\star$.

These hypotheses are satisfied at $n = 0$. Let

$$T_{n+1} = \{f^\frown x : f \in T_n \text{ and } x \in B_f\}.$$

Now let $g \in T_{n+1}$, let $f = g_{|n}$, and let $x = g(n)$ (so $g = f^\frown x$). Then $x^{-1}B_f \in p$. Let $B_g = (x^{-1}B_f)^\star$. $\square$

Statements (c) and (d) of the following theorem are our sought after simple characterizations of C-sets. While the definition of C-sets for arbitrary semigroups is considerably more complicated than that for commutative semigroups, the proof of the following theorem is no different from what it would be if we assumed that S was commutative.

Theorem 2.7. *Let S be an infinite semigroup and let $A \subseteq S$. Statements* (a), (b), *and* (c) *are equivalent and are implied by statement* (d). *If S is countable, then all four statements are equivalent.*

(a) *A is a C-set.*

(b) *There is a nonempty set T of functions such that:*

  (i) *For all $f \in T$, $\mathrm{domain}(f) \in \omega$ and $\mathrm{range}(f) \subseteq A$.*

  (ii) *For all $f \in T$ and all $x \in B_f(T)$, $B_{f^\frown x}(T) \subseteq x^{-1}B_f(T)$.*

  (iii) *For all $F \in \mathcal{P}_f(T)$, $\bigcap_{f\in F} B_f(T)$ is a J-set.*

(c) *There is a downward directed family $\langle C_F\rangle_{F\in I}$ of subsets of A such that:*

  (i) *For all $F \in I$ and all $x \in C_F$, there exists $G \in I$ such that $C_G \subseteq x^{-1}C_F$.*

  (ii) *For each $\mathcal{F} \in \mathcal{P}_f(I)$, $\bigcap_{F\in\mathcal{F}} C_F$ is a J-set.*

(d) *There is a decreasing sequence $\langle C_n\rangle_{n=1}^\infty$ of subsets of A such that:*

  (i) *For all $n \in \mathbb{N}$ and all $x \in C_n$, there exists $m \in \mathbb{N}$ such that $C_m \subseteq x^{-1}C_n$.*

  (ii) *For all $n \in \mathbb{N}$, C_n is a J-set.*

Proof. To see that (a) implies (b) pick an idempotent $p \in J(S)$ such that $A \in p$. Pick a set T of functions as guaranteed by Lemma 2.6. Conclusions (i) and (ii) hold directly. Given $F \in \mathcal{P}_f(T)$, one has that for each $f \in F$, $B_f(T) \in p$, so $\bigcap_{f\in F} B_f(T) \in p$ and so $\bigcap_{f\in F} B_f(T)$ is a J-set.

To see that (b) implies (c), let T be as guaranteed by (b). Let $I = \mathcal{P}_f(T)$ and for $F \in I$, let $C_F = \bigcap_{f\in F} B_f(T)$. Given $\mathcal{F} \in \mathcal{P}_f(I)$, if $G = \bigcup\mathcal{F}$, then $\bigcap_{F\in\mathcal{F}} C_F = C_G$ which is therefore a J-set. To verify (i), let $F \in I$ and let $x \in C_F$. Let $G = \{f^\frown x : f \in F\}$. For each $f \in F$, $B_{f^\frown x}(T) \subseteq x^{-1}B_f(T)$ and so $C_G \subseteq x^{-1}C_F$.

To see that (c) implies (a), let $\langle C_F\rangle_{F\in I}$ be as guaranteed by (c). Let $M = \bigcap_{F\in I} \overline{C}_F$. By [5, Theorem 4.20], M is a subsemigroup of βS. Let $\mathcal{R} = \{B \subseteq S : B \text{ is a } J\text{-set}\}$. By [6, Theorem 2.14], if $B \cup C \in \mathcal{R}$, then $B \in \mathcal{R}$ or $C \in \mathcal{R}$, and consequently by [5, Theorem 3.11] there is some $p \in \beta S$ such that $\{C_F : F \in I\} \subseteq p$. Therefore $M \cap J(S) \neq \emptyset$ and by [1, Theorem 3.5], $J(S)$ is an ideal of βS and so $M \cap J(S)$ is a compact subsemigroup of βS. Thus there is an idempotent $p \in M \cap J(S)$.

It is trivial that (d) implies (c). Assume now that S is countable. We shall show that (b) implies (d). So let T be as guaranteed by (b). Then

T is countable so enumerate T as $\{f_n : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$, let $C_n = \bigcap_{k=1}^n B_{f_k}(T)$. Then each C_n is a J-set. Let $n \in \mathbb{N}$ and let $x \in C_n$. Pick $m \in \mathbb{N}$ such that $\{f_k{}^\frown x : k \in \{1, 2, \ldots, n\}\} \subseteq \{f_1, f_2, \ldots, f_m\}$. Then $C_m \subseteq x^{-1}C_n$. $\square$

We now present an illustration of the use of Theorem 2.7. In [3] the set A in the following theorem was defined, and shown to be a C-set directly. The proof given below is significantly shorter than the one in [3]. We were interested in A because A has zero Banach density and hence is not central.

For $x \in \mathbb{N}$ we denote by $\mathrm{supp}(x)$ the unique subset of ω such that

$$x = \sum_{t \in \mathrm{supp}(x)} 2^t.$$

Theorem 2.8. *For $n \in \mathbb{N}$, let $a_n = \min\{t \in \mathbb{N} : (\frac{2^n-1}{2^n})^t \leq \frac{1}{2}\}$ and let $s_n = \sum_{i=1}^n a_i$. (So $s_1 = 1$ and $s_2 = 4$.) Let $b_0 = 0$, let $b_1 = 1$, and for $n \in \mathbb{N}$ and $t \in \{s_n, s_n + 1, s_n + 2, \ldots, s_{n+1} - 1\}$, let $b_{t+1} = b_t + n + 1$. For $k \in \omega$, let $B_k = \{b_k, b_k + 1, b_k + 2, \ldots, b_{k+1} - 1\}$. Let*

$$A = \{x \in \mathbb{N} : (\forall k \in \omega)(B_k \setminus \mathrm{supp}(x) \neq \emptyset)\}.$$

Then A is a C-set.

Proof. For $n \in \mathbb{N}$, let

$$C_n = \{x \in \mathbb{N} : \min \mathrm{supp}(x) \geq b_n \text{ and } (\forall k \in \omega)(B_k \setminus \mathrm{supp}(x) \neq \emptyset)\}.$$

Then $C_0 = A$. We claim that $\langle C_n \rangle_{n=0}^\infty$ satisfies Theorem 2.7(d). So let $n \in \mathbb{N}$ and let $x \in C_n$. Pick m such that $b_m > \max \mathrm{supp}(x)$. Then $C_m \subseteq -x + C_n$.

Now let $n \in \mathbb{N}$. We show that C_n is a J-set. So let $F \in \mathcal{P}_f(\mathcal{T})$. Let $r = |F|$ and pick k such that $b_{k+1} - b_k > r$. Pick $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in F$, $\sum_{t \in H} f(t) \in \mathbb{Z}2^{b_k}$. (Choose an infinite subset C of $\mathbb{N}$ such that for all $s, t \in C$ and all $f \in F$, $f(s) \equiv f(t) \pmod{2^{b_k}}$. Then pick $H \subseteq C$ such that $|H| = 2^{b_k}$.) Pick $c \in \mathbb{N}2^{b_k}$ such that for all $f \in F$, $c + \sum_{t \in H} f(t) > 0$.

Let $l = \max \bigcup \{\mathrm{supp}(c + \sum_{t \in H} f(t)) : f \in F\}$ and pick j such that $l < b_j$. Pick $r_0 \in B_k$ such that $B_k \setminus \mathrm{supp}(2^{r_0} + c + \sum_{t \in H} f(t)) \neq \emptyset$ for each $f \in F$. Inductively for $i \in \{1, 2, \ldots, j - k\}$, pick $r_i \in B_{k+i}$ such that $B_{k+i} \setminus \mathrm{supp}(2^{r_i} + \sum_{t=0}^{i-1} 2^{r_t} + c + \sum_{t \in H} f(t)) \neq \emptyset$ for each $f \in F$. Let $d = c + \sum_{i=0}^{j-k} 2^{r_i}$. Then for each $f \in F$, $d + \sum_{t \in H} f(t) \in C_n$. $\square$

We conclude with a corollary establishing the existence of many semigroups that have subsets which are C-sets but not central sets. (See [7] for an examination of this phenomenon.) By [5, Exercise 1.7.3 and Corollary 4.22], if S and T are semigroups, $h : S \to T$ is a surjective homomorphism, and $\widetilde{h} : \beta S \to \beta T$ is the continuous extension of h, then $\widetilde{h}$ is a surjective homomorphism and so $\widetilde{h}[K(\beta S)] = K(\beta T)$. We see now that a corresponding statement about $J(S)$ and $J(T)$ holds. Assertions (6) and (7) in the next theorem were certainly known to anyone who cared to think about the question, but we don't believe they have been published.

Theorem 2.9. *Let S and T be semigroups, let $h : S \to T$ be a surjective homomorphism, and let $\widetilde{h} : \beta S \to \beta T$ be the continuous extension of h.*

(1) *If A is a J-set in S, then $h[A]$ is a J-set in T.*
(2) *If A is a J-set in T, then $h^{-1}[A]$ is a J-set in S.*
(3) *$\widetilde{h}[J(S)] = J(T)$.*
(4) *If A is a C-set in S, then $h[A]$ is a C-set in T.*
(5) *If A is a C-set in T, then $h^{-1}[A]$ is a C-set in S.*
(6) *If A is a central set in S, then $h[A]$ is a central set in T.*
(7) *If A is a central set in T, then $h^{-1}[A]$ is a central set in S.*

Proof. (1) Let A be a J-set in S. To see that $h[A]$ is J-set in T, let $F \in \mathcal{P}_f\big({}^{\mathbb{N}}T\big)$ be given. Pick $k : T \to S$ such that for all $x \in T$, $h\big(k(x)\big) = x$. (That is $k(x)$ choses an element of $h^{-1}[\{x\}]$.) Let $G = \{k \circ f : f \in F\}$. Then $G \in \mathcal{P}_f\big({}^{\mathbb{N}}S\big)$ so pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that for each $f \in F$, $x(m, a, H, k \circ f) \in A$. Define $b \in T^{m+1}$ by, for $j \in \{1, 2, \ldots, m+1\}$, $b(j) = h\big(a(j)\big)$. Then for each $f \in F$, $x(m, b, H, f) \in h[A]$.

(2) Let A be a J-set in T. To see that $h^{-1}[A]$ is J-set in S, let $F \in \mathcal{P}_f\big({}^{\mathbb{N}}S\big)$ be given and let $G = \{h \circ f : f \in F\}$, then $G \in \mathcal{P}_f\big({}^{\mathbb{N}}T\big)$ so pick $m \in \mathbb{N}$, $a \in T^{m+1}$, and $H \in \mathcal{I}_m$ such that for each $f \in F$, $x(m, a, H, h \circ f) \in A$. For each $j \in \{1, 2, \ldots, m+1\}$, pick $b(j) \in S$ such that $h\big(b(j)\big) = a(j)$. Then for each $f \in F$, $h\big(x(m, b, H, f)\big) = x(m, a, H, h \circ f) \in A$ so $x(m, b, H, f) \in h^{-1}[A]$.

(3) To see that $\widetilde{h}[J(S)] \subseteq J(T)$, let $p \in J(S)$ and let $A \in \widetilde{h}(p)$. Then $h^{-1}[A] \in p$ and so $h^{-1}[A]$ is a J-set in S and thus $A = h\big[h^{-1}[A]\big]$ is a J-set in T.

To see that $J(T) \subseteq \widetilde{h}[J(S)]$, let $p \in J(T)$. Let $\mathcal{A} = \{h^{-1}[A] : A \in p\}$. Then $\mathcal{A}$ is a set of subsets of S, each of which is a J-set in S, and $\mathcal{A}$ is closed under finite intersections. Further, by [6, Theorem 2.14], if the union of two sets is a J-set in S, one of them is also. Thus by [5, Theorem 3.11] one may pick $q \in J(S)$ such that $\mathcal{A} \subseteq q$. Then $\widetilde{h}(q) = p$.

(4) Let A be a C-set in S and pick by Theorem 2.4 an idempotent p in $J(S) \cap \overline{A}$. Since $\widetilde{h}$ is a homomorphism, $\widetilde{h}(p)$ is an idempotent om $J(T) \cap \overline{h[A]}$.

(5) Let A be a C-set in T and pick an idempotent p in $J(T) \cap \overline{A}$. Then $J(S) \cap \widetilde{h}^{-1}[\{p\}]$ is a compact subsemigroup of βS so there is an idempotent $q \in J(S) \cap \widetilde{h}^{-1}[\{p\}]$. Since $\widetilde{h}(q) = p$, $h^{-1}[A] \in q$.

(6) Let A be a central set in S and pick an idempotent p in $K(\beta S) \cap \overline{A}$. Then $\widetilde{h}(p)$ is an idempotent in $K(\beta T) \cap \overline{h[A]}$.

(7) Let A be a central set in T and pick an idempotent p in $K(\beta T) \cap \overline{A}$. Pick $r \in K(\beta S)$ such that $\widetilde{h}(r) = p$ and pick a minimal left ideal L of S

such that $r \in L$. Then $L \cap \widetilde{h}^{-1}[\{p\}]$ is a compact subsemigroup of βS so there is an idempotent $q \in L \cap \widetilde{h}^{-1}[\{p\}]$. Since $\widetilde{h}(q) = p$, $h^{-1}[A] \in q$. □

Corollary 2.10. *Let S and T be semigroups and let $h : S \to T$ be a surjective homomorphism. If A is a subset of T which is a C-set in T but not a central set in T, then $h^{-1}[A]$ is a C-set in S but not a central set in S.*

Proof. By Theorem 2.9(5), $h^{-1}[A]$ is a C-set in S. If $h^{-1}[A]$ were a central set in S, then one would have by Theorem 2.9(6) that $A = h\big[h^{-1}[A]\big]$ would be a central set in T. □

The set A produced in Theorem 2.8 was shown in [3] to not be a central set in $\mathbb{N}$. Thus, as a consequence of Corollary 2.10, any semigroup which can be mapped homomorphically onto $\mathbb{N}$ contains a C-set which is not central.

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