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# Genus distribution of graphs under surgery: adding edges and splitting vertices

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Abstract. Our concern is deriving genus distributions of graphs obtained by surgical operations on graphs whose genus distribution is known. One operation in focus here is adding an edge. The other is splitting a vertex, for which the inverse operation is edge-contraction. Our main result is this Splitting Theorem: Let $G$ be a graph and $w$ a 4-valent vertex of $G$. Let $H_1$, $H_2$, and $H_3$ be the three graphs into which $G$ can be split at $w$, so that the two new vertices of each split are 3-valent. Then $2\mathrm{gd}(G) = \mathrm{gd}(H_1) + \mathrm{gd}(H_2) + \mathrm{gd}(H_3)$.

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## 1. Introduction

Counting the imbeddings of a graph in various surfaces is an enumerative branch of topological graph theory. Our main result here relates the sum of the genus distributions of all the splits of a graph at a designated vertex to the genus distribution of the graph itself.

Graphs may have self-loops and multiple adjacencies. An edge of a graph is conceptualized intuitively via its topological model, as the continuous image of the unit interval $[0, 1]$, which is 1-1 everywhere for a proper edge, and 1-1 everywhere except at 0 and 1 for a self-loop. The images of small neighborhoods of 0 and 1 are called *edge-ends*. Every edge has two edge-ends, but a self-loop has only one endpoint.

All imbeddings of concern here are in orientable surfaces. Two imbeddings of a graph are equivalent if they induce the same rotation system on the graph. Given a rotation $\rho_w$ at any vertex $w$ of a graph $G$ and a subgraph $H \subseteq G$ such that $w \in V_H$, the *induced rotation* at $w$ in $H$ is the unique rotation that can be obtained by deleting from $\rho_w$ the edge-ends of every edge in $E_G - E_H$. Moreover, given an imbedding $\iota : G \to S$, the *induced imbedding* of $H$ is the imbedding all of whose rotations are induced by the rotation system corresponding to $\iota$.

We use the abbreviation *fb-walk* to refer to a face-boundary walk. In much of the discussion here, we visualize an fb-walk as a topological entity, lying a little off the graph itself, slightly into the interior of its face, and *not* simply as a sequence of edges in the graph.

Throughout this paper, graphs are connected and imbeddings are cellular and oriented, unless the alternative is inferrable from context. Discussion presumes familiarity with topological graph theory. The usage here follows [GT87] and [BWGT09]. ([BL95], [MT01], and [Whi01] provide additional background, each in somewhat different terminology from here.)

Recent work on counting imbeddings of a graph in a minimum-genus surface includes [BGGS00], [GRS07], [GG08], and [KV02]. Results on counting imbeddings in all orientable surfaces or in all surfaces have been achieved by [CGR94], [CLW06], [FGS89], [GF87], [GRT89], [KL93], [KL94], [KS02], [McG87], [Mul99], [Sta90], [Sta91a], [Sta91b], [Tes00], [VW07], [WL06], and [WL08], and others. Complementary work on counting maps on a given surface is given by [CD01], [Jac87], [JV90], [JV01], and by many others.

### 1.1. Genus distribution.

The *genus distribution* of a graph $G$ is the sequence

$$\mathrm{gd}(G) = \langle g_0(G), \quad g_1(G), \quad g_2(G), \quad \cdots \quad \rangle$$

in which $g_i(G)$ is the number of imbeddings of $G$ in the orientable surface $S_i$. The smallest and largest indices $i$ such that $g_i(G) > 0$ are the *minimum genus* $\gamma_{\min}(G)$ and the *maximum genus* $\gamma_{\max}(G)$ of the graph $G$. We recall that determination of $\gamma_{\min}(G)$ and $\gamma_{\max}(G)$ are NP-hard and P-hard, respectively. This implies that it is at least NP-hard to calculate genus distributions, which motivates the concentration of effort in calculating genus distibutions on interesting families of graphs, exactly as we do for minimum genus. We recall also that the total number of imbeddings equals the product of the numbers $(\deg(v) - 1)!$, taken over all vertices $v \in V_G$.

The effect of a surgical operation on the genus distribution of a graph depends on the incidence of the face-boundary walks incident on the vertices or edges where the surgery takes place. Accordingly, as in [PKG10], [GKP10], and [Gro10a], we specify roots and we refine the genus distribution according to the incidence of fb-walks on the roots. The results here can be used, as indicated in §5, with those three other papers on this topic to calculate previously unknown genus distributions of graphs in various kinds of recursively-constructed infinite sequences and to expedite calculations of some known genus distributions.

Although there is no hope (unless $P = NP$) of a polynomial-time general algorithm for genus distribution, the families of graphs of greatest interest are often amenable to recursively specification. This suggests a two-fold approach:

(1) Find recursions that correspond to various kinds of operations used to synthesize larger graphs from smaller graphs.
(2) Find useful ways to specify interesting families of graphs recursively.

The present paper is mostly concerned with the first aspect. The second aspect is illustrated, for instance, by [Gro10b], which develops a recursive specification of the 3-regular outerplanar graphs and uses it to construct a recursion for their genus distributions.

## 2. Joining two vertices

Let $(G, u, v)$ be a double-rooted (connected) graph with both roots $u$ and $v$ of degree 2. The result of joining the roots $u$ and $v$ by an edge $e$ is denoted $G + e$. We seek to derive the genus distribution $\mathrm{gd}(G + e)$ from some form of *partitioned genus distribution* of $(G, u, v)$.

**2.1. Recombinant strands.** For any graph imbedding $\iota : (G, u, v) \to S_i$, there is a set of four imbeddings of $G + e$ that induce $\iota$. Each of these four is said to be an *imbedding resulting from inserting* edge $e$.

Since $u$ is 2-valent, the fb-walk incident on one side of $u$ in the imbedding $\iota : G \to S$ may be a different fb-walk from the fb-walk on the other side of $u$; or the same fb-walk might be incident on both sides of $u$. Likewise, there may be two fb-walks incident on $v$ in the imbedding $\iota : G \to S$, or just one such fb-walk. The genera of the surfaces for the four imbeddings resulting from adding edge $e$ depend on the number of fb-walks incident at both roots and on whether one or two fb-walks incident at one root are also incident at the other root.

On whichever side of root $u$ or of root $v$ an edge-end of $e$ is inserted, it breaks the fb-walk on that side into a *strand*. If both ends of edge $e$ are inserted along the same fb-walk of $\iota : G \to S$, then that walk is broken into two strands. No matter how many fb-walks of $\iota : G \to S$ remain intact, there will be two strands to be *combined* into fb-walks of an imbedding of the resultant graph $G + e$, using new segments of walk along edge $e$. This construction of fb-walks in an imbedding of $G+e$, resulting from insertion of edge $e$ into an imbedding of $G$, is a simplest instance of the general method of *recombinant strands*.

Figure 2.1 illustrates the recombination of strands using the inserted edge $e$ and of the sides of $u$ and $v$ on which the fb-walk is not broken. There may be altogether a maximum of four different fb-walks incident on $u$ and $v$, and a minimum of one fb-walk that is twice incident on both roots. The drawings should be understood as *rotation projections* in the sense of [GT87] — that is, they represent imbeddings of $G + e$.

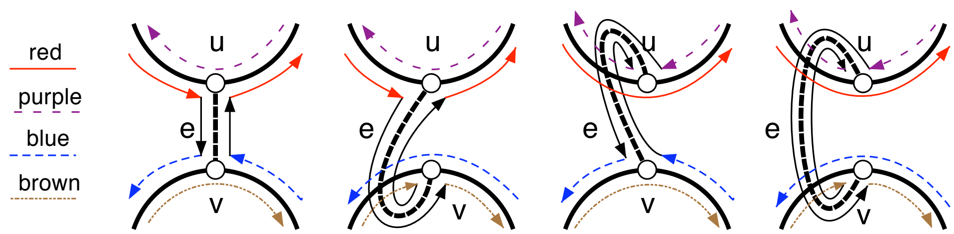


Figure 2.1: Four ways to insert edge $e$ into imbedding $\iota : G \to S$.

Four different graphic representations are used in the figure, in order not to inappropriately conflate two different fb-walks in any of the possible cases. Thus two or more different graphics occur along the same fb-walk in the cases with fewer than four distinct fb-walks. Since there is no universally understood terminology for the different graphics used in Figure 2.1, we assign the names *red* and *purple* to the fb-walks we observe locally at $u$, and the names *blue* and *brown* to what we observe locally at $v$; to compensate for the absence of color print, we provide a legend at the left. The thin black arcs on either side of inserted edge $e$ represent the edge-steps along edge $e$ that occur in the indicated imbedding of $G + e$.

**2.2. Partitioning the genus distribution.** When calculating the genus distribution of $G + e$, the genus distribution of $G$ is partitioned into nine *partial distributions*, also called *partials*. When some of these partials are null-valued, it simplifies the calculations.

**Cases *dd*, *dd′* and *dd″*.** We first consider the circumstance where the fb-walk on one side of root $u$ differs from the fb-walk on the other side, and the same is true at root $v$. (Mnemonic: *dd* for "different-different".)

- $dd_i(G, u, v)$ is the number of imbeddings of $G$ into the surface $S_i$ in which no two of the red, purple, blue, and brown fb-walks coincide.
- $dd'_i(G, u, v)$ is the number of imbeddings $G \to S_i$ in which the red and purple walks are distinct, and the blue and brown walks are distinct, but exactly one of the fb-walks at $u$ (i.e., red or purple) coincides with one of the fb-walks at $v$ (i.e., blue or brown).
- $dd''_i(G, u, v)$ is the number of imbeddings $G \to S_i$ in which the red and purple walks are distinct, and the blue and brown walks are distinct, and one of the fb-walks at $u$ coincides with one of the fb-walks at $v$, and the other fb-walk at $u$ coincides with the other fb-walk at $v$.

**Cases *ds, sd, ds′*, and *sd′*.** We next consider the circumstance where a single fb-walk is twice incident at one root and two different fb-walks are incident at the other root. (Mnemonic: *ds* for "different-same", etc.) Cases *sd* is like case *ds*, except for a swap of the roles of the roots $u$ and $v$.

- $ds_i(G, u, v)$ is the number of imbeddings $G \to S_i$ in which the red and purple walks are distinct, and the blue and brown walks coincide, but neither the red walk nor the purple walk coincides with the fb-walk at $v$.
- $ds'_i(G, u, v)$ is the number of imbeddings $G \to S_i$ in which the red and purple walks are distinct, and the blue and brown walks coincide, and either the red walk or the purple walk coincides with the fb-walk at $v$.
- $sd_i(G, u, v)$ is the number of imbeddings $G \to S_i$ in which the blue and brown walks are distinct, and the red and purple walks coincide, but neither the blue walk nor the brown walk coincides with the fb-walk at vertex $u$.
- $sd'_i(G, u, v)$ is the number of imbeddings $G \to S_i$ in which the blue and brown walks are distinct, and the red and purple walks coincide, and either the blue walk or the brown walk coincides with the fb-walk at $u$.

**Cases *ss* and $\overline{ss}$.** The remaining circumstance is where the fb-walk on one side of root $u$ coincides with the fb-walk on the other side, and the same is true at root $v$. (Mnemonic: *ss* for "same-same".)

- $ss_i(G, u, v)$ is the number of imbeddings of $G$ into the surface $S_i$ in which the fb-walk at $u$ is different from the fb-walk at $v$.
- $\overline{ss}_i(G, u, v)$ is the number of imbeddings of $G$ into the surface $S_i$ in which the fb-walk at $u$ coincides with the fb-walk at $v$.

A complete set of partials for a graph $G$ is called a *partitioned genus distribution.* The sum of the partial distributions within a partitioned genus distribution of $G$ equals the genus distribution $\mathrm{gd}(G)$.

**Remark.** Different sets of partials can be used for different genus distribution problems. For instance, it is sufficient in certain problems to use single-root partials. Also, in a set of double-root partials, it is sometimes necessary to distinguish partials according to behavioral characteristics of the strands.

**2.3. Production rules for edge addition.** A *production (rule)* of the form

$$pq_i^x(G,u,v) \longrightarrow mg_i(G+e) \,+\, (4-m)g_{i+1}(G+e)$$

has the following interpretation: suppose that the imbedding $\iota : G \to S_i$ is included in the count by the partial $pq_i^x(G,u,v)$, where $x$ is either blank or a modifier, such as a prime or a double prime; then, of the four imbeddings that result from inserting edge $e$, exactly $m$ imbeddings are in the surface $S_i$, and the other $4-m$ imbeddings are in the surface $S_{i+1}$. We shall see how such productions enable us to calculate the genus distribution of $G+e$ from the partitioned genus distribution of $G$.

**Theorem 2.1.** *Let $(G,u,v)$ be a double-rooted graph with 2-valent co-roots. Then the following productions describe the relationship between its partitioned genus distribution and the genus distribution of the graph $G+e$ obtained by joining roots $u$ and $v$ with edge $e$.*

$$dd_i \longrightarrow 4g_{i+1} \tag{2.1}$$
$$dd_i' \longrightarrow g_i + 3g_{i+1} \tag{2.2}$$
$$dd_i'' \longrightarrow 2g_i + 2g_{i+1} \tag{2.3}$$
$$ds_i \longrightarrow 4g_{i+1} \tag{2.4}$$
$$ds_i' \longrightarrow 2g_i + 2g_{i+1} \tag{2.5}$$
$$sd_i \longrightarrow 4g_{i+1} \tag{2.6}$$
$$sd_i' \longrightarrow 2g_i + 2g_{i+1} \tag{2.7}$$
$$ss_i \longrightarrow 4g_{i+1} \tag{2.8}$$
$$\overline{ss}_i \longrightarrow 4g_i. \tag{2.9}$$

**Proof.** The productions for $dd$, $ds$, $sd$ and $ss$ all correspond to imbeddings in which a handle must be added for all four ways of inserting edge $e$, because the two roots do not lie on the same fb-walk.

Production $dd'$ corresponds to the case in which one of the two fb-walks (say, red) at $u$ coincides with one of the two fb-walks (say, blue) at $v$. Thus, edge $e$ could be drawn across the face without adding a handle. However, each of the other three ways of inserting edge $e$ places the two edge-ends of $e$ in different faces, so a new handle is needed. In production $dd''$, there are two ways to place both ends of edge $e$ in the same face and two to place them in different faces; in the former case, the edge can be drawn on $S_i$, and in the latter case, a handle must be added.

For productions $ds'$ and $sd'$, the edge $e$ can be drawn on $S_i$ when its end at the root with two incident fb-walks is in the face whose fb-walk occurs twice at the other root, with the other end at either of the two occurrences of the other root on that fb-walk. For production $\overline{ss}$, all four ways of inserting edge $e$ place both edge-ends in the same face, so edge $e$ can be drawn in $S_i$. □

In the corollary, we write $G$ as the argument of the partials on the right side of the equation, rather than $(G, u, v)$, for the sake of brevity.

**Corollary 2.2.** *Let $(G, u, v)$ be a double-rooted graph with 2-valent co-roots. Then the genus distribution $\mathrm{gd}(G+e)$ is derivable by using the following equation:*

$$g_i(G+e) = 4dd_{i-1}(G) + 3dd'_{i-1}(G) + 2dd''_{i-1}(G) + 4ds_{i-1}(G) + 2ds'_{i-1}(G) + 4sd_{i-1}(G) + 2sd'_{i-1}(G) + 4ss_{i-1}(G) + dd'_i(G) + 2dd''_i(G) + 2ds'_i(G) + 2sd'_i(G) + 4\overline{ss}_i(G) \tag{2.10}$$

**Proof.** This is an immediate consequence of Theorem 2.1. □

**Remark.** The statement of Theorem 2.1 is sharpened within the proof of Theorem 3.1.

**2.4. Constructing $K_{3,3}$ from $K_4$.** As a preliminary to discussing how Equation (2.10) might be used in conjunction with methods of [Gro10a] to produce genus distributions for a recursively constructed family of graphs, we consider a simple application.

**Example 2.1.** Let $\ddot{K}_4$ be the graph obtained by inserting vertices $u$ and $v$ at the midpoints of two independent edges of the complete graph $K_4$, as in Figure 2.2, and by regarding them as roots. We observe in Figure 2.2 that the graph $\ddot{K}_4 + e$ is isomorphic to the complete bipartite graph $K_{3,3}$.

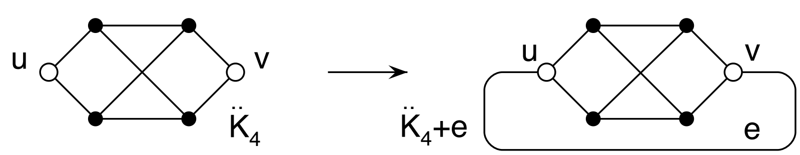


Figure 2.2: Adding an edge to $\ddot{K}_4$.

We use face-tracing (see [GT87]) to calculate the double-root partitioned genus distribution in Table 2.1 of the double-rooted graph $(\ddot{K}_4, u, v)$.

Table 2.1: Double-root partials of $\ddot{K}_4$.

| $i$ | $dd_i$ | $dd'_i$ | $dd''_i$ | $ds_i$ | $ds'_i$ | $sd_i$ | $sd'_i$ | $ss_i$ | $\overline{ss}_i$ | $g_i$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| 1 | 0 | 0 | 4 | 0 | 4 | 0 | 4 | 0 | 2 | 14 |

By Equation (2.10) we have, in agreement with [GF87] (and confirmable by face-tracing)

$$\begin{aligned} g_0(K_{3,3}) &= dd'_0(\ddot{K}_4) + 2dd''_0(\ddot{K}_4) + 2ds'_0(\ddot{K}_4) + 2sd'_0(\ddot{K}_4) + 4\overline{ss}_0(\ddot{K}_4) \\ &= 0+0+0+0+0 = 0. \end{aligned}$$

$$\begin{aligned} g_1(K_{3,3}) &= 4dd_0(\ddot{K}_4) + 3dd'_0(\ddot{K}_4) + 2dd''_0(\ddot{K}_4) + 4ds_0(\ddot{K}_4) \\ &\quad + 2ds'_0(\ddot{K}_4) + 4sd_0(\ddot{K}_4) + 2sd'_0(\ddot{K}_4) + 4ss_0(\ddot{K}_4) \\ &\quad + dd'_1(\ddot{K}_4) + 2dd''_1(\ddot{K}_4) + 2ds'_1(\ddot{K}_4) + 2sd'_1(\ddot{K}_4) + 4\overline{ss}_1(\ddot{K}_4) \\ &= 4\cdot 2 + 0+0+0+0+0+0+0 \\ &\quad + 0 + 2\cdot 4 + 2\cdot 4 + 2\cdot 4 + 4\cdot 2 = 40. \end{aligned}$$

$$\begin{aligned} g_2(K_{3,3}) &= 4dd_1(\ddot{K}_4) + 3dd'_1(\ddot{K}_4) + 2dd''_1(\ddot{K}_4) + 4ds_1(\ddot{K}_4) \\ &\quad + 2ds'_1(\ddot{K}_4) + 4sd_1(\ddot{K}_4) + 2sd'_1(\ddot{K}_4) + 4ss_1(\ddot{K}_4) \\ &= 0+0+2\cdot 4 + 0 + 2\cdot 4 + 0 + 2\cdot 4 + 0 = 24. \end{aligned}$$

**Example 2.2.** Using the derivation of double-root partitioned genus distributions for the sequence of closed-end ladders in [PKG10] (a refinement of the original derivation in [FGS89]), Equation (2.10) yields the genus distributions of Ringel ladders, which were first calculated by [Tes00].

## 3. Deleting an edge

Deleting an edge $e$ of a graph $G$ that joins two vertices $u$ and $v$ inverts the operation of joining $u$ and $v$. The effect on the genus distribution of deleting an edge is easy to state. In an imbedding in which two distinct fb-walks are incident on the edge, the two faces are merged into one and the genus stays the same; if a single fb-walk is twice incident on the edge, then that face is split into two faces, and the genus drops by one. The catch is that several different imbeddings of $G$ may correspond to the same imbedding of $G - e$.

**3.1. Single-edge-root partitioned distributions.** While we used nine double-root partials of $G$ to construct a formula for the genus distribution of the result of adding an edge to a graph $G$, we can easily construct a formula for the genus distribution of the result $G - e$ of deleting an edge from $G$ with the aid of only two partials. Once again, the symbols $d$ and

$s$ are mnemonics for "different" and "same". This time we need only two *single-edge-root partitioned distributions*:

- $d_i(G, e)$ is the number of imbeddings of $G$ into the surface $S_i$ in which two different fb-walks are incident on edge $e$.
- $s_i(G, e)$ is the number of imbeddings of $G$ into the surface $S_i$ in which the same fb-walk is incident on both sides of $e$.

**Theorem 3.1.** *Let $(G, e)$ be a single-edge-rooted graph with two 3-valent endpoints. The following production rules describe the relationship between its partitioned genus distribution and the genus distribution of the graph $G - e$:*

$$d_i \longrightarrow \tfrac{1}{4}\, g_i \tag{3.1}$$

$$s_i \longrightarrow \tfrac{1}{4}\, g_{i-1}. \tag{3.2}$$

**Proof.** We observe that we could sharpen Theorem 2.1 by replacing each instance of $g_i$ on the right of the productions by $d_i$ and each instance of $g_{i+1}$ by $s_{i+1}$. Each imbedding of $G - e$ in $S_i$ corresponds to four imbeddings of $G$, and this theorem follows by reversing each of the productions of the shapened version of Theorem 2.1. □

**Corollary 3.2.** *Let $(G, e)$ be a single-edge-rooted graph with two 3-valent endpoints. Then the genus distribution $\mathrm{gd}(G - e)$ is derivable by using the following equation:*

$$g_i(G - e) = \tfrac{1}{4}\, d_i(G, e) + \tfrac{1}{4}\, s_{i+1}(G, e). \tag{3.3}$$

**Proof.** This is an immediate consequence of Theorem 3.1. □

## 4. Contracting an edge

In a graph $G$, let $e$ be an edge with endpoints $u$ and $v$. The *contraction of graph $G$ on (*or *along) edge $e$*, denoted $G/e$, is the graph obtained topologically by shrinking edge $e$ to a single vertex, so that vertices $u$ and $v$ are merged. The operation is called *contracting graph $G$ on edge $e$*.

Contraction of $G$ along $e$ is achieved combinatorially by deleting edge $e$ and then amalgamating the vertices $u$ and $v$ that were the endpoints of $e$. Accordingly, if $e$ is a cycle-edge, then we could apply the methods of [Gro10a] to the double-root partitioned genus distribution of $(G - e, u, v)$. Alternatively, if $e$ is a cut-edge, and if $G_u$ and $G_v$ are the components of $G - e$ that contain $u$ and $v$, respectively, then we could apply the methods of [GKP10] to the single-root partitioned genus distributions of $(G_u, u)$ and $(G_v, v)$. These partitioned genus distributions for $G - e$ are not inferrable from the single-edge-root partitioned genus distribution for $(G, e)$. Also, the Splitting Theorem of the next section can sometimes be used to determine the genus distributions of a contracted graph.

## 5. Splitting a vertex

Let $w$ be a vertex of a graph $G$, and let $U$ and $V$ be the cells of a bipartition of the neighbors of $w$ into nonempty parts. In the graph $G - w$, let every vertex of $U$ be joined to a new vertex $u$ and let every vertex of $V$ be joined to a new vertex $v$, and join the vertices $u$ and $v$. This operation is called *splitting graph $G$ at vertex $w$*, and the resulting graph is called a *split of the graph $G$ at the vertex $w$*. We may refer to $w$ as the *split vertex*.

**Proposition 5.1.** *Let $w$ be an $n$-valent vertex of a graph $G$, and let $r$ and $s$ be integers, each at least 2, with sum $r + s = n + 2$. Then the number of ways to split graph $G$ at vertex $w$ so that the endpoints of the new edge have degrees $r$ and $s$ is*

$$\begin{cases} \binom{n}{r-1} & \text{if } r \neq s \\ \frac{1}{2}\binom{n}{r-1} & \text{if } r = s. \end{cases}$$

**Proof.** This is elementary counting. □

**Example 5.1.** The 4-wheel $W_4$ has three splits at its hub-vertex, as shown in Figure 5.1. Two of the splits are isomorphic to $K_2 \times C_3$, and the other is isomorphic to $K_{3,3}$.

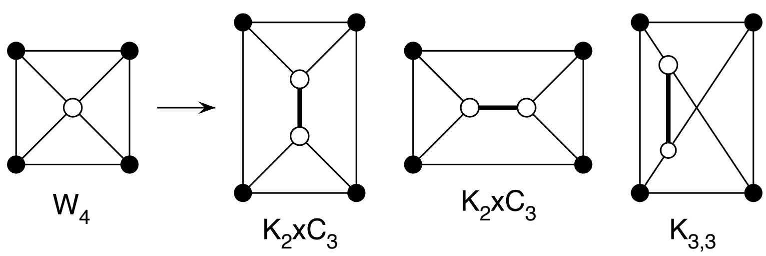


Figure 5.1: Splitting the 4-wheel $W_4$.

We observe that contracting any split of a graph along its new edge inverts the splitting. Conversely, if the sets of neighbors of the respective endpoints of an edge are used as the bipartition of the merged vertex, then splitting at the merged vertex inverts the contraction.

**5.1. A genus-distribution phenomenon.** In comparing the genus distribution of a graph to the genus distributions of its splits at a vertex of degree 4, we discover a rather interesting phenomenon: the genus distribution $\mathrm{gd}(G)$ of the unsplit graph is equal to exactly half the sum of the genus distributions of its three splits.

**Example 5.1, continued.** The genus distributions of $K_{3,3}$ and of $K_2 \times C_3$ are given by Examples 2 and 3 of [GF87]. The sum $\langle 4,\ 116,\ 72 \rangle$ of the genus distributions of these three splits is exactly double the genus distribution of $W_4$, which is $\langle 2,\ 58,\ 36\rangle$.

$$\begin{array}{rl} \mathrm{gd}(K_2\times C_3) = & \langle\, 2, \quad 38, \quad 24\,\rangle \\ \mathrm{gd}(K_2\times C_3) = & \langle\, 2, \quad 38, \quad 24\,\rangle \\ \mathrm{gd}(K_{3,3}) = & \langle\, 0, \quad 40, \quad 24\,\rangle \\ \hline \mathrm{sum} = & \langle\, 4, \quad 116, \quad 72\,\rangle \end{array}$$

In what follows, we prove that this phenomenon holds, in general.

**5.2. Relative genus distributions.** Let $G$ be a graph, let $U$ be a subset of its vertex set, and let $\rho_U$ be an assignment of rotations to every vertex of $U$. Let $g_i^{\rho_U}(G)$ denote the number of imbeddings of the graph $G$ in the surface $S_i$ such that the rotation at every vertex $u \in U$ is $\rho_U(u)$. The sequence

$$\mathrm{gd}^{\rho_U}(G) = g_0^{\rho_U}(G),\ g_1^{\rho_U}(G),\ g_2^{\rho_U}(G),\ \ldots$$

is called the *relative genus distribution* of $G$ with respect to $\rho_U$.

**Notation.** We denote the set of all rotation assignments for $U$ by $R_U$.

**Proposition 5.2.** *Let $G$ be a graph and let $U$ be a subset of its vertex set. Then for all $i \geq 0$,*

$$g_i(G) = \sum_{\rho\in R_U} g_i^{\rho}(G).$$

**Proof.** The full set of imbeddings of graph $G$ can be partitioned according to assignments of rotations on the set $U$. For any given assignment $\rho$ of rotations on the set $U$, the relative genus distribution $\mathrm{gd}^{\rho}(G)$ is a genus distribution for the imbeddings in the cell of that partition, corresponding to the assignment $\rho$. □

**Corollary 5.3.** *Let $G$ be a graph, let $U$ be a subset of its vertex set. Then*

$$\mathrm{gd}(G) = \sum_{\rho\in R_U} \mathrm{gd}^{\rho}(G).$$

**Proof.** This follows immediately from Proposition 5.2. □

Although relative genus distributions are introduced here primarily for their conceptual use in deriving a genus distribution for a whole graph, it is helpful to consider a small illustration of Proposition 5.2 with concrete numbers.

**Example 5.2.** In the graph $K_2 \times C_3$, let the set $U$ comprise two adjacent vertices $u$ and $v$ on different 3-cycles. Since $u$ and $v$ are both 3-valent, there are four possible assignments of rotations to $U$. Figure 5.2 illustrates these four assignments.

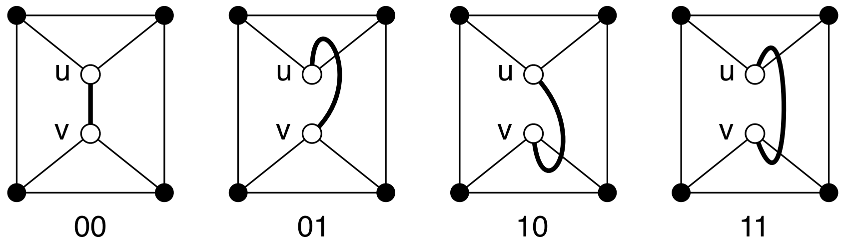


Figure 5.2: Rotations on a subset of vertices of $K_2 \times C_3$.

Of course, there are 16 possible combinations of rotations on the other four vertices, for each of the assignments to the vertices of $U$. The figure uses the same assignment of rotations to the other four vertices with all four of the assignments to $u$ and $v$. Table 5.1 gives the relative genus distributions for the four assignments of rotations to $U$ and their sum, which is the genus distribution of $K_2 \times C_3$.

Table 5.1: Relative genus with respect to four rotations.

| $\rho$ | $g_0$ | $g_1$ | $g_2$ |
|---|---|---|---|
| 00 | 1 | 9 | 6 |
| 01 | 0 | 10 | 6 |
| 10 | 0 | 10 | 6 |
| 11 | 1 | 9 | 6 |
| $\mathrm{gd}(K_2 \times C_3)$ | 2 | 38 | 24 |

**5.3. The splitting theorem.** We now prove our main result about splitting and contracting.

**Theorem 5.4** (Splitting Theorem)**.** *Let $G$ be a graph and $w$ a 4-valent vertex of $G$. Let $H_1$, $H_2$, and $H_3$ be the three graphs into which $G$ can be split at $w$, so that the two new vertices of each split are 3-valent. Then*

$$2\mathrm{gd}(G) = \mathrm{gd}(H_1) + \mathrm{gd}(H_2) + \mathrm{gd}(H_3). \tag{5.1}$$

**Proof.** In the graph $H_i$, for $i = 1, 2, 3$, we let $u_i$ and $v_i$ be the vertices into which vertex $w$ of graph $G$ splits, and we let $U_i = \{u_i, v_i\}$. In the graph $G$, we let $U = \{w\}$. According to Corollary 5.3,

$$\mathrm{gd}(H_i) = \sum_{\rho \in R_{U_i}} \mathrm{gd}^{\rho}(H_i) \quad \text{for } i = 1, 2, 3 \tag{5.2}$$

and

$$\mathrm{gd}(G) = \sum_{\rho \in R_U} \mathrm{gd}^{\rho}(G). \tag{5.3}$$

We consider the imbeddings of $H_1$, $H_2$, and $H_3$ to be mutually disjoint. We make two assertions about the operation of contracting the edge $u_i v_i$.

(1) It induces a 2-to-1 correspondence from the union of the sets of imbeddings of these three graphs onto the set of imbeddings of the graph $G$.
(2) It preserves the genus of the surface.

In regard to assertion (1), we consider an arbitrary imbedding $\iota : G \to S$, in which we suppose that the rotation at vertex $w$ is *abcd*. We observe that each of the three graphs $H_1$, $H_2$, and $H_3$ has exactly four imbeddings that coincide with the imbedding $\iota$ on all the vertices of $V_G - \{w\}$, so there are 12 imbeddings altogether in the union of the imbeddings of these three graphs. Of course, there are exactly six imbeddings of $G$ (including $\iota$) that have the same rotations as $\iota$ on the vertices of $V_G - \{w\}$.

Figure 5.3 shows the 2-to-1 correspondence. In each column, contraction of the imbeddings in rows 1 and 2 along the edge $u_i v_i$ maps those imbeddings to the imbedding in row 3, in that same column.

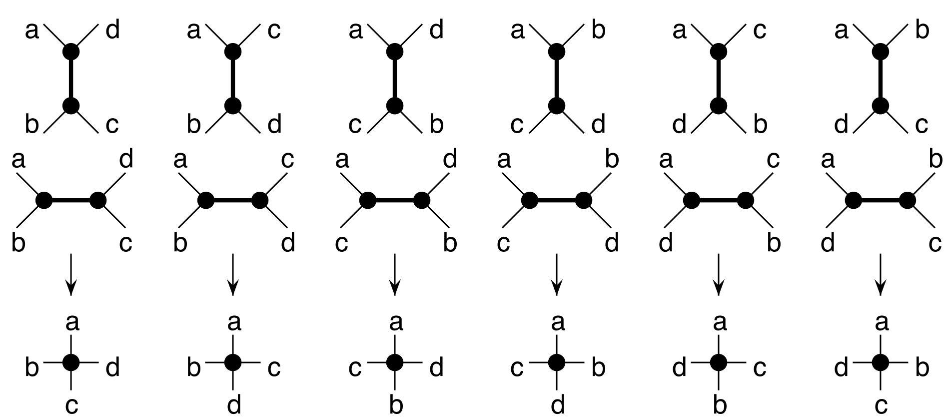

Figure 5.3: The 2-to-1 correspondence to rotations at a split vertex.

Regarding assertion (2), we observe that a contraction operation decreases the numbers of vertices and edges, each by 1, and preserves the number of faces. This implies that the genus of the imbedding surface is unchanged. With this in mind, Equation (5.1) follows from Equations (5.2) and (5.3). □

Example 5.1 could serve as an example of *direct application* of the Splitting Theorem. That is, we could use values for the three terms on the right of Equation (5.1) to calculate $\mathrm{gd}(G)$.

**5.4. Indirect application of the splitting theorem.** Combining some preexisting results with the Splitting Theorem, we can calculate genus distributions for various graphs and infinite families *indirectly*. That is, we calculate the genus distribution of one of the splits by using the values of $\mathrm{gd}(G)$ and the genus distributions of one or two of the other splits. To illustrate, we begin with the graph $H$ depicted in Figure 5.4.

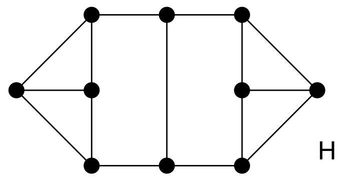


Figure 5.4: To calculate: the genus distribution of this graph.

We seek a graph $G$ with known genus distribution and a 4-valent vertex $w$ such that

- $H$ can be obtained by splitting $G$ at $w$;
- each of the other two splits either has known genus distribution or is isomorphic to $H$.

To construct such a graph $G$, we insert a midpoint on one edge of $K_4$ and thereby obtain a graph we call $\dot{K}_4$. We amalgamate two copies of $\dot{K}_4$ at their 2-valent vertices and take the resulting graph as $G$. Figure 5.5 shows the graph $G$ and the three splits at its 4-valent vertex. The split labeled $\dot{K}_4|\dot{K}_4$ is called a *bar-amalgamation* of two copies of $\dot{K}_4$, and the other two are copies of $H$.

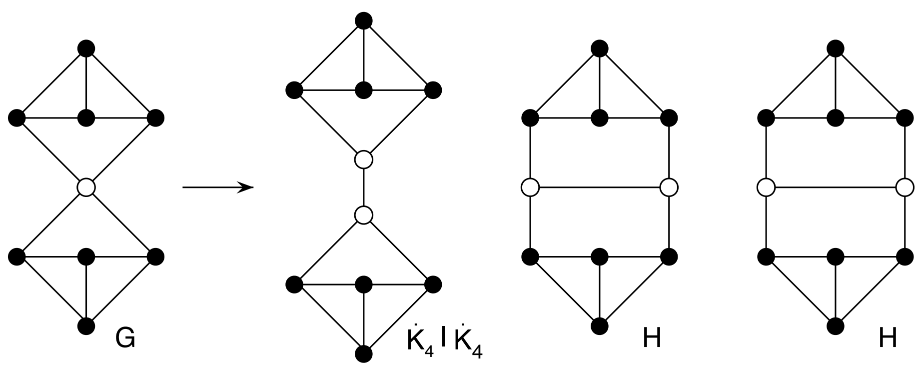


Figure 5.5: Splitting the graph $G$.

By face-tracing, we calculate

$$d_0(\dot{K}_4) = 2 \quad d_1(\dot{K}_4) = 8 \quad s_1(\dot{K}_4) = 6$$

The two preexisting results we need are as follows:

**Theorem 5.5.** *Let $(A, t)$ and $(B, u)$ be single-rooted graphs with 2-valent roots, and let $(C, w) = (A, t) * (B, u)$. Then*

(5.4)

$$g_k(C) = 4\sum_{i=0}^{k} d_i(A)d_{k-i}(B) + 2\sum_{i=0}^{k-1} d_i(A)d_{k-i-1}(B) + 6\sum_{i=0}^{k} d_i(A)s_{k-i}(B) + 6\sum_{i=0}^{k} s_i(A)d_{k-i}(B) + 6\sum_{i=0}^{k} s_i(A)s_{k-i}(B).$$

**Proof.** This is Corollary 2.3 of [GKP10]. □

**Theorem 5.6.** *Let $(A, u)$ and $(B, v)$ be rooted graphs. The genus distribution of the bar-amalgamation $(A, u)|(B, v)$ is obtained by multiplying the convolution of the genus distributions of $A$ and $B$ by the product of the degrees of vertices $u$ and $v$ in the graphs $A$ and $B$, respectively.*

**Proof.** This is Theorem 5 of [GF87]. □

Using Theorem 5.5 we can calculate $\mathrm{gd}(G)$, for $G = \dot{K}_4 * \dot{K}_4$:

$$\begin{aligned}
g_0(G) &= 4d_0(\dot{K}_4)d_0(\dot{K}_4) = 4 \cdot 2 \cdot 2 = 16\\
g_1(G) &= 4(d_0(\dot{K}_4)d_1(\dot{K}_4) + d_1(\dot{K}_4)d_0(\dot{K}_4)) + 2d_0(\dot{K}_4)d_0(\dot{K}_4)\\
&\quad + 6d_0(\dot{K}_4)s_1(\dot{K}_4) + 6s_1(\dot{K}_4)d_0(\dot{K}_4)\\
&= 4(2 \cdot 8 + 8 \cdot 2) + 2 \cdot 2 \cdot 2 + 6 \cdot 2 \cdot 6 + 6 \cdot 6 \cdot 2 = 280\\
g_2(G) &= 4d_1(\dot{K}_4)d_1(\dot{K}_4) + 2(d_0(\dot{K}_4)d_1(\dot{K}_4) + d_1(\dot{K}_4)d_0(\dot{K}_4))\\
&\quad + 6d_1(\dot{K}_4)d_1(\dot{K}_4) + 6s_1(\dot{K}_4)d_1(\dot{K}_4)\\
&= 4 \cdot 8 \cdot 8 + 2 \cdot (2 \cdot 8 + b \cdot 2) + 6 \cdot 8 \cdot 6 + 6 \cdot 6 \cdot 8 = 1112\\
g_3(G) &= 2d_1(\dot{K}_4)d_1(\dot{K}_4) = 128.
\end{aligned}$$

Applying Theorem 5.6 to the task of calculating the genus distribution of $(\dot{K}_4, u)|(\dot{K}_4, u)$, we now multiply the product of the degrees of the roots by the convolution of the genus distribution for $\dot{K}_4$ with itself.

$$\mathrm{gd}(\dot{K}_4|\dot{K}_4) = 4(4,\ 56,\ 196) = (16,\ 224,\ 784). \tag{5.5}$$

To calculate $\mathrm{gd}(H)$, we solve this linear equation

$$\begin{aligned}
\mathrm{gd}(H) &= \frac{1}{2}\left[2\mathrm{gd}(G) - \mathrm{gd}(\dot{K}_4|\dot{K}_4)\right]\\
&= \frac{1}{2}\left[(32,\ 560,\ 2224,\ 256) - (16,\ 224,\ 784,\ 0)\right]\\
&= \frac{1}{2}\left[(16,\ 336,\ 1440,\ 256)\right]\\
&= (8,\ 168,\ 720,\ 128).
\end{aligned}$$

**5.5. Genus distribution for an infinite sequence.** We can generalize the graph $H$ of Figure 5.5 to an infinite sequence of graphs, in which the number of "horizontal rungs" is arbitrarily large. Each such graph can be obtained from a double-vertex-rooted closed-end ladder (see [PKG10] for a recursion for the genus distributions of all such graphs) with the appropriate number of rungs by vertex-amalgamation with $\dot{K}_4$ at one root of the ladder, then a split of the resulting 4-valent vertex, followed by vertex-amalgamation of another copy of $\dot{K}_4$ to the remaining root of the ladder, and then another split.

## 6. Conclusions

The methods presented in this paper enable us to calculate previously unknown genus distributions for various graphs, including the following:

- the genus distribution of the result of joining the roots of a double-rooted graph $(G, u, v)$ with 2-valent co-roots whose double-root partitioned genus distributions is known;
- the genus distribution of the result of deleting the root edge $e$, where edge $e$ has 3-valent endpoints, from an edge-rooted graph $(G, e)$ whose edge-root partitioned genus distribution is known;
- the genus distributions of graphs obtained by splitting a 4-valent vertex or by contracting an edge whose endpoints are 3-valent;
- the genus distributions of infinite families of graphs, which are obtained by combining the results here with preexisting results.

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