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The Riccati differential equation and a diffusion-type equation

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Abstract. We construct an explicit solution of the Cauchy initial value problem for certain diffusion-type equations with variable coefficients on the entire real line. The heat kernel is given in terms of elementary functions and certain integrals involving a characteristic function, which should be found as an analytic or numerical solution of a Riccati differential equation with time-dependent coefficients. Some special and limiting cases are outlined. Solution of the corresponding nonhomogeneous equation is also found.

Contents

1. Introduction

In this paper we discuss a method to construct the explicit solution (the time evolution operator is given explicitly as an integral operator) of the

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Cauchy initial value problem for the one-dimensional heat equation on the entire real line

$$\frac{\partial u}{\partial t} = Q\left(\frac{\partial}{\partial x},\ x,\ t\right)u, \tag{1.1}$$

where the right-hand side is a quadratic form $Q(p,x)$ of the coordinate x and the operator of differentiation $p=\partial/\partial x$ with time-dependent coefficients; see equation (2.1) in Section 2. The corresponding heat kernel or fundamental solution is given explicitly by the formulas (2.16)–(2.23), which has not been introduced in the literature before. The heat kernel is given in terms of elementary functions and certain integrals involving a characteristic function. This characteristic function is the solution of a second order differential equation with function coefficients (2.13) that has been obtained after certain substitutions from a Riccati differential equation (2.7). This Riccati differential equation was obtained in the process of splitting the diffusion equation in a system of ordinary differential equations (2.7)–(2.12). In Section 3 we solve this system of equations, deriving the heat kernel explicitly.

In Section 4 we prove the uniqueness of our solution for the Cauchy problem associated with (2.1) in the class of solutions satisfying the Tychonoff condition, see (4.1). We assume that the variable coefficients in (2.1) satisfy certain assumptions. The uniqueness is an immediate consequence of the maximum principle for parabolic equations on bounded domains and the extension method to unbounded domains introduced by M. Krzyzanski, see [Krz45], [Krz59] and [Frie64].

The connection between stochastic differential equations and fundamental solutions for certain parabolic equations (e.g., Feynman–Kac stochastic representation [Frie64], [Mil77], [Cra09], [Goa06]) has been applied in financial mathematics (e.g., probabilistic approach to pricing derivatives [Eth02]) and mathematical biology (e.g., Kolmogorov differential equations for epidemic, competition and predation processes [All07], [All03] and cable equations [JNT83]). In Sections 5 and 6 we give several examples of these types of equations included in our general equation (2.1), and finally in Section 7 the solution of the corresponding nonhomogeneous equation is obtained with the help of the Duhamel principle.

In [CLSS08], [CSS09], [CSS10], the case of a corresponding Schrödinger equation is investigated and classified in terms of elementary solutions of a characterization equation given by (2.13) below. These exactly solvable cases may be of interest in a general treatment of the nonlinear evolution equations; see [Can84], [Caz03], [CH98], [Tao06] and references therein. Moreover, these explicit solutions can also be useful when testing numerical methods of solving the semilinear heat equations with variable coefficients.

2. Solution of a Cauchy initial value problem: summary of results

The fundamental solution (or heat kernel) of the diffusion-type equation of the form

$$\frac{\partial u}{\partial t}=a(t)\frac{\partial^2 u}{\partial x^2}-\left(g(t)-c(t)x\right)\frac{\partial u}{\partial x}+\left(-b(t)x^2+f(t)x+d(t)\right)u, \tag{2.1}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, $f(t)$, and $g(t)$ are given real-valued functions of time t only, can be found by a familiar substitution

$$u=Ae^{S}=A(t)e^{S(x,y,t)} \tag{2.2}$$

with

$$A=A(t)=\frac{1}{\sqrt{2\pi\mu(t)}} \tag{2.3}$$

and

$$S=S\left(x,y,t\right)=\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2+\delta(t)x+\varepsilon\left(t\right)y+\kappa(t), \tag{2.4}$$

where $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$, $\varepsilon(t)$, and $\kappa(t)$ are differentiable real-valued functions of time t only. Indeed,

$$\frac{\partial S}{\partial t}=a\left(\frac{\partial S}{\partial x}\right)^2-bx^2+fx+\left(cx-g\right)\frac{\partial S}{\partial x} \tag{2.5}$$

provided

$$\frac{\mu'}{2\mu}=-a\frac{\partial^2 S}{\partial x^2}-d=-2\alpha(t)a(t)-d(t). \tag{2.6}$$

Equating the coefficients of all admissible powers of $x^m y^n$ with $0\le m+n\le 2$ gives the following system of ordinary differential equations

$$\frac{d\alpha}{dt}+b(t)-2c(t)\alpha-4a\left(t\right)\alpha^2=0, \tag{2.7}$$

$$\frac{d\beta}{dt}-\left(c(t)+4a(t)\alpha(t)\right)\beta=0, \tag{2.8}$$

$$\frac{d\gamma}{dt}-a(t)\beta^2(t)=0, \tag{2.9}$$

$$\frac{d\delta}{dt}-\left(c(t)+4a(t)\alpha(t)\right)\delta=f(t)-2\alpha(t)g(t), \tag{2.10}$$

$$\frac{d\varepsilon}{dt}+\left(g(t)-2a(t)\delta(t)\right)\beta(t)=0, \tag{2.11}$$

$$\frac{d\kappa}{dt}+g(t)\delta(t)-a\left(t\right)\delta^2(t)=0, \tag{2.12}$$

where (2.7) is the Riccati nonlinear differential equation; see, for example, [HS69], [Mol02], [Rai64], [RM08], [Wat44] and references therein.

We have

$$4a\alpha' + 4ab - 2c\left(4a\alpha\right) - \left(4a\alpha\right)^2 = 0, \qquad 4a\alpha = -2d - \frac{\mu'}{\mu}$$

from (2.7) and (2.6), and the substitution

$$4a\alpha' = -2d' - \frac{\mu''}{\mu} + \left(\frac{\mu'}{\mu}\right)^2 + \frac{a'}{a}\left(2d + \frac{\mu'}{\mu}\right)$$

results in the second order linear equation

$$\mu'' - \tau(t)\mu' - 4\sigma\left(t\right)\mu = 0 \tag{2.13}$$

with

$$\tau(t) = \frac{a'}{a} + 2c - 4d, \qquad \sigma\left(t\right) = ab + cd - d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right). \tag{2.14}$$

As we shall see later, equation (2.13) must be solved subject to the initial data

$$\mu\left(0\right) = 0, \qquad \mu'\left(0\right) = 2a\left(0\right) \neq 0, \qquad d(0) = 0 \tag{2.15}$$

in order to satisfy the initial condition for the corresponding Green function; see the asymptotic formula (2.24) below for a motivation. Then, the Riccati equation (2.7) can be solved by the back substitution (2.6).

We shall refer to Equation (2.13) as the *characteristic equation* and its solution $\mu(t)$, subject to (2.15), as the *characteristic function.* The special case (2.13) contains the generalized equation of hypergeometric type, whose solutions are studied in detail in [NU88]; see also [AAR99], [NSU91], [ST08], and [Wat44].

Thus, the Green function (fundamental solution or heat kernel) is explicitly given in terms of the characteristic function

$$u = K\left(x, y, t\right) = \frac{1}{\sqrt{2\pi\mu(t)}}\ e^{\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)}. \tag{2.16}$$

Here

$$\alpha(t) = -\frac{1}{4a(t)}\frac{\mu'(t)}{\mu(t)} - \frac{d(t)}{2a\left(t\right)}, \tag{2.17}$$

$$\beta(t) = \frac{1}{\mu(t)}\ \exp\left(\int_0^t \left(c\left(\tau\right) - 2d\left(\tau\right)\right)\ d\tau\right), \tag{2.18}$$

$$\begin{aligned}\gamma(t) = &-\frac{a(t)}{\mu(t)\mu'(t)}\ \exp\left(2\int_0^t \left(c\left(\tau\right) - 2d\left(\tau\right)\right)\ d\tau\right) \\ &- 4\int_0^t \frac{a\left(\tau\right)\sigma\left(\tau\right)}{\left(\mu'\left(\tau\right)\right)^2}\left(\exp\left(2\int_0^\tau \left(c\left(\lambda\right) - 2d\left(\lambda\right)\right)\ d\lambda\right)\right)\ d\tau,\end{aligned} \tag{2.19}$$

$$\delta(t)=\frac{1}{\mu(t)}\ \exp\left(\int_0^t\left(c\left(\tau\right)-2d\left(\tau\right)\right)\ d\tau\right)\int_0^t\exp\left(-\int_0^\tau\left(c\left(\lambda\right)-2d\left(\lambda\right)\right)\ d\lambda\right)\cdot\left(\left(f\left(\tau\right)+\frac{d\left(\tau\right)}{a\left(\tau\right)}g\left(\tau\right)\right)\mu\left(\tau\right)+\frac{g\left(\tau\right)}{2a\left(\tau\right)}\mu'\left(\tau\right)\right)\ d\tau, \tag{2.20}$$

$$\begin{aligned}\varepsilon(t)=&-\frac{2a(t)}{\mu'(t)}\delta(t)\ \exp\left(\int_0^t\left(c\left(\tau\right)-2d\left(\tau\right)\right)\ d\tau\right)\\&-8\int_0^t\frac{a\left(\tau\right)\sigma\left(\tau\right)}{\left(\mu'\left(\tau\right)\right)^2}\exp\left(\int_0^\tau\left(c\left(\lambda\right)-2d\left(\lambda\right)\right)\ d\lambda\right)\left(\mu\left(\tau\right)\delta\left(\tau\right)\right)\ d\tau\\&+2\int_0^t\frac{a\left(\tau\right)}{\mu'\left(\tau\right)}\exp\left(\int_0^\tau\left(c\left(\lambda\right)-2d\left(\lambda\right)\right)\ d\lambda\right)\left(f\left(\tau\right)+\frac{d\left(\tau\right)}{a\left(\tau\right)}g\left(\tau\right)\right)\ d\tau,\end{aligned} \tag{2.21}$$

$$\begin{aligned}\kappa(t)=&-\frac{a(t)\mu(t)}{\mu'(t)}\delta^2(t)-4\int_0^t\frac{a\left(\tau\right)\sigma\left(\tau\right)}{\left(\mu'\left(\tau\right)\right)^2}\left(\mu\left(\tau\right)\delta\left(\tau\right)\right)^2\ d\tau\\&+2\int_0^t\frac{a\left(\tau\right)}{\mu'\left(\tau\right)}\left(\mu\left(\tau\right)\delta\left(\tau\right)\right)\left(f\left(\tau\right)+\frac{d\left(\tau\right)}{a\left(\tau\right)}g\left(\tau\right)\right)\ d\tau\end{aligned} \tag{2.22}$$

with

$$\delta\left(0\right)=\frac{g\left(0\right)}{2a\left(0\right)},\qquad\varepsilon\left(0\right)=-\delta\left(0\right),\qquad\kappa\left(0\right)=0. \tag{2.23}$$

We have used integration by parts in order to resolve the singularities of the initial data; see Section 3 for more details. Then the corresponding asymptotic formula is

$$\begin{aligned}K\left(x,y,t\right)&=\frac{e^{S(x,y,t)}}{\sqrt{2\pi\mu\left(t\right)}}\\&\sim\frac{1}{\sqrt{4\pi a\left(0\right)t}}\exp\left(-\frac{\left(x-y\right)^2}{4a\left(0\right)t}\right)\exp\left(\frac{g\left(0\right)}{2a\left(0\right)}\left(x-y\right)\right)\end{aligned} \tag{2.24}$$

as $t\to 0^+$. Notice that the first term on the right hand side is a familiar heat kernel for the diffusion equation with constant coefficients (cf. Eq. (6.2) below).

By the superposition principle, we obtain the solution of the Cauchy initial value problem

$$\frac{\partial u}{\partial t}=Qu,\qquad u\left(x,t\right)|_{t=0}=u_0\left(x\right) \tag{2.25}$$

on the infinite interval $-\infty<x<\infty$ with the general quadratic form $Q\left(p,x\right)$ in (2.1) as follows:

$$u\left(x,t\right)=\int_{-\infty}^{\infty}K\left(x,y,t\right)\ u_0\left(y\right)\ dy=Hu\left(x,0\right). \tag{2.26}$$

This yields a solution explicitly in terms of an integral operator H acting on the initial data provided that the integral converges and one can interchange differentiation and integration (for example if $\alpha,\, \gamma < 0$ and $\delta = \kappa = \varepsilon = 0$ in (2.16)). This integral is essentially the Laplace transform.

In a more general setting, solution of the initial value problem at time t_0

$$\frac{\partial u}{\partial t} = Qu, \qquad u\left(x, t\right)\big|_{t=t_0} = u\left(x, t_0\right) \tag{2.27}$$

on an infinite interval has the form

$$u\left(x, t\right) = \int_{-\infty}^{\infty} K\left(x, y, t, t_0\right)\ u_0\left(y, t_0\right)\ dy = H\left(t, t_0\right) u\left(x, t_0\right) \tag{2.28}$$

with the heat kernel given by

$$\begin{aligned} K\left(x, y, t, t_0\right) & \\ &= \frac{1}{\sqrt{2\pi\mu\left(t, t_0\right)}}\ e^{\alpha(t,t_0)x^2+\beta(t,t_0)xy+\gamma(t,t_0)y^2+\delta(t,t_0)x+\varepsilon(t,t_0)y+\kappa(t,t_0)}. \end{aligned} \tag{2.29}$$

The function $\mu(t) = \mu\left(t, t_0\right)$ is a solution of the characteristic equation (2.13) corresponding to the initial data

$$\mu\left(t_0, t_0\right) = 0, \qquad \mu'\left(t_0, t_0\right) = 2a\left(t_0\right) \neq 0, \qquad d(0) = 0. \tag{2.30}$$

If $\{\mu_1, \mu_2\}$ is a fundamental solution set of Equation (2.13), then

$$\mu\left(t, t_0\right) = \frac{2a\left(t_0\right)}{W\left(\mu_1, \mu_2\right)} \left(\mu_1\left(t_0\right)\mu_2\left(t\right) - \mu_1(t)\mu_2\left(t_0\right)\right) \tag{2.31}$$

and

$$\mu'\left(t, t_0\right) = \frac{2a\left(t_0\right)}{W\left(\mu_1, \mu_2\right)} \left(\mu_1\left(t_0\right)\mu_2'(t) - \mu_1'(t)\mu_2\left(t_0\right)\right), \tag{2.32}$$

where $W\left(\mu_1, \mu_2\right)$ is the value of the Wronskian at the point t_0.

Equations (2.17)–(2.22) are valid again but with the new characteristic function $\mu\left(t, t_0\right)$. The lower limits of integration should be replaced by t_0. Conditions (2.23) become

$$\delta\left(t_0, t_0\right) = -\varepsilon\left(t_0, t_0\right) = \frac{g\left(t_0\right)}{2a\left(t_0\right)}, \qquad \kappa\left(t_0, t_0\right) = 0 \tag{2.33}$$

and the asymptotic formula (2.24) should be modified as follows

$$\begin{aligned} K\left(x, y, t, t_0\right) &= \frac{e^{S(x,y,t,t_0)}}{\sqrt{2\pi\mu\left(t, t_0\right)}} \\ &\sim \frac{1}{\sqrt{4\pi a\left(t_0\right)\left(t - t_0\right)}} \exp\left(-\frac{\left(x-y\right)^2}{4a\left(t_0\right)\left(t - t_0\right)}\right) \\ &\quad \cdot \exp\left(\frac{g\left(t_0\right)}{2a\left(t_0\right)}\left(x - y\right)\right). \end{aligned} \tag{2.34}$$

We leave the details to the reader.

3. Derivation of the heat kernel

Here we obtain the above formulas (2.17)–(2.22) for the heat kernel. The first equation is a direct consequence of (2.6) and our Equation (2.8) takes the form

$$(\mu\beta)' = (c - 2d)(\mu\beta), \tag{3.1}$$

whose particular solution is (2.18).

From (2.9) and (2.18) one gets

$$\gamma(t) = \int \frac{a(t)}{\mu^2(t)} e^{2h(t)}\, dt, \qquad h(t) = \int_0^t (c(\tau) - 2d(\tau))\, d\tau \tag{3.2}$$

and integrating by parts

$$\gamma(t) = -\int \frac{ae^{2h}}{\mu'}\, d\left(\frac{1}{\mu}\right) = -\frac{ae^{2h}}{\mu\mu'} + \int \left(\frac{ae^{2h}}{\mu'}\right)' \frac{dt}{\mu}. \tag{3.3}$$

But the derivative of the auxiliary function

$$F(t) = \frac{a(t)}{\mu'(t)}\, e^{2h(t)} \tag{3.4}$$

is

$$F'(t) = \frac{(a' + 2h'a)\, e^{2h}\mu' - ae^{2h}\mu''}{(\mu')^2} = -\frac{4\sigma a\mu}{(\mu')^2} e^{2h} = -\frac{4\sigma\mu}{\mu'} F \tag{3.5}$$

in view of the characteristic equation (2.13)–(2.14). Substitution into (3.3) results in (2.19).

Equation (2.10) can be rewritten as

$$\left(\mu e^{-h}\delta\right)' = \mu e^{-h}(f - 2\alpha g), \qquad h = \int_0^t (c - 2d)\, d\tau \tag{3.6}$$

and its direct integration gives (2.20).

We introduce another auxiliary function

$$G(t) = \mu(t)\delta(t)e^{-h(t)} \tag{3.7}$$

with the derivative given by (3.6). Then Equation (2.11) becomes

$$\frac{d\varepsilon}{dt} = -\frac{g}{\mu}e^h + \frac{2a\delta}{\mu}e^h$$

and

$$\varepsilon(t) = -\int \frac{g}{\mu}e^h\, dt + 2\int \frac{aG}{\mu^2}e^{2h}\, dt. \tag{3.8}$$

Integrating the second term by parts one gets

$$\begin{aligned} \int \frac{aG}{\mu^2}e^{2h}\, dt &= -\int \frac{aG}{\mu'}e^{2h}\, d\left(\frac{1}{\mu}\right) = -\int FG\, d\left(\frac{1}{\mu}\right) \\ &= -\frac{FG}{\mu} + \int \frac{(FG)'}{\mu}\, dt, \end{aligned} \tag{3.9}$$

where

$$(FG)' = F'G + FG' = -\frac{4a\sigma\mu}{(\mu')^2}(\mu\delta)\, e^h + \frac{a\mu}{\mu'}e^h f + \frac{d\mu}{\mu'}e^h g + \frac{1}{2}ge^h \tag{3.10}$$

in view of (3.5) and (3.6). Then substitution (3.10) into (3.9) allows us to cancel the divergent integrals. As a result one can resolve the singularity and simplify expression (3.8) to its final form (2.21).

Finally, by (2.12) and (3.7)

$$\kappa(t) = -\int g\delta \, dt + \int \frac{aG^2}{\mu^2}e^{2h}\, dt, \tag{3.11}$$

where the last integral can be transformed as follows

$$\int \frac{aG^2}{\mu^2}e^{2h}\, dt = -\int FG^2 \, d\left(\frac{1}{\mu}\right) = -\frac{FG^2}{\mu} + \int \frac{\left(FG^2\right)'}{\mu}\, dt \tag{3.12}$$

with

$$\left(FG^2\right)' = F'G^2 + 2FGG' = (FG)'\, G + FGG' = -\frac{4a\sigma\mu}{(\mu')^2}(\mu\delta)^2 + \frac{2a\mu}{\mu'}(\mu\delta)\, f + \frac{2d\mu}{\mu'}(\mu\delta)\, g + \mu g\delta. \tag{3.13}$$

Substitution (3.12)–(3.13) into (3.11) gives our final expression (2.22).

The details of derivation of the asymptotic formula (2.24) are left to the reader.

4. Uniqueness of the Cauchy problem

In this section we prove the uniqueness of the solution (2.26) of the Cauchy initial value problem (2.25) in the class of solutions satisfying the Tychonoff condition

$$|u(x,t)| \le B_2 \exp(B_1 x^2) \tag{4.1}$$

for some positive constants B_1 and B_2. The uniqueness is a direct consequence of using the maximum principle for parabolic equations on bounded domains and the extension method to unbounded domains introduced by M. Krzyzanski, see [Krz45], [Krz41]; we follow the presentation of [Frie64]. For the sake of clarity we outline the proof for our equation.

We also will use the following assumption on the coefficients:

Assumption A. $a(t) > 0, b(t), c(t), d(t), f(t)$ and $g(t)$ are continous functions in $[T_0, T_1]$.

We define the operator

$$Lu = a(t)\frac{\partial^2 u}{\partial x^2} + b_1(x,t)\, u + c_1(x,t)\, \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}, \tag{4.2}$$

where $b_1(x,t) = c(t)x - g(t)$, $c_1(x,t) = -b(t)x^2 + d(t) + f(t)x$.

We will need the following proposition:

Proposition 4.1. *If the initial data $u(x,0)$ satisfies (4.1) on $\mathbb{R}$, then (2.26) satisfies*

$$|u(x,t)| \leq M_2 \exp(M_1 x^2) \tag{4.3}$$

for some positive constants M_1, M_2 with $M_1 + \gamma(t) < 0$, $\mu(t) > 0$ for all t in $[T_0, T_1]$.

This proposition is a consequence of the integral

$$\int_{-\infty}^{\infty} e^{-ay^2+2by}\, dy = \sqrt{\frac{\pi}{a}}\, e^{b^2/a}, \qquad a > 0. \tag{4.4}$$

We will also use the following lemma:

Lemma 4.2. *Let S be the operator defined by*

$$Su = a(t)\frac{\partial^2 u}{\partial x^2} + b\,(x,t)\,u + c\,(x,t)\,\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}, \tag{4.5}$$

with $a(t) > 0$, $b\,(x,t)$ continous in $\mathbb{R}\times(T_0, T_1]$ and $c(x,t)$ bounded from above. If $Su \leq 0$ in $\mathbb{R} \times (T_0, T_1]$, $u(x,0) \geqslant 0$ in $\mathbb{R}$ and

$$\liminf_{|x|\to\infty}\ u(x,t) \geqslant 0 \tag{4.6}$$

uniformly with respect to t $\ (T_0 \leqslant t \leqslant T_1)$, then $u(x,t) \geqslant 0$ in $\mathbb{R} \times [T_0, T_1]$.

Now, we are ready to prove uniqueness, as follows:

Theorem 4.3. *If the initial data $u_0(x) \geq 0$ satisfies (4.1) on $\mathbb{R}$, there exists a unique solution to the Cauchy problem*

$$\begin{aligned} Lu &= 0, && (x,t) \text{ in } \mathbb{R} \times (T_0, T_1] \\ u(x,0) &= u_0(x), && x \text{ in } \mathbb{R} \end{aligned} \tag{4.7}$$

in the class of solutions satisfying (4.1).

We sketch the proof. Because $c_1(x,t)$ is not bounded we will apply the lemma to an alternative operator $\overline{L}$ and a function υ, see (4.18), (4.8) below.

First, we verify that we can apply the lemma to υ defined by

$$\upsilon = \frac{u}{F}. \tag{4.8}$$

For this we define the function

$$F(x,t) = \exp\left(\frac{kx^2}{1-\nu_1 t} + \nu_2 t\right),\ 0 \leq t \leq \frac{1}{2\nu_1} \tag{4.9}$$

where $k > M_1$ is fixed. Using Assumption A we can find M such that

$$|a(t)| \leq M, \qquad |b_1(x,t)| \leq M(x+1), \qquad |c_1(x,t)| \leq M(x^2+1). \tag{4.10}$$

Furthermore if $0 \leq t \leq 1/2\nu_1$ we can choose ν_1 and ν_2 such that

$$\frac{LF}{F} \leq 0. \tag{4.11}$$

To see this we observe that

$$\frac{\partial F}{\partial t}=\left(-\frac{kx^2(-\nu_1)}{(1-\nu_1 t)^2}+\nu_2\right)\exp\left(\frac{kx^2}{1-\nu_1 t}+\nu_2 t\right) \tag{4.12}$$

$$b_1(x,t)\frac{\partial F}{\partial x}=\frac{2k}{1-\nu_1 t}b_1(x,t)x\exp\left(\frac{kx^2}{1-\nu_1 t}+\nu_2 t\right) \tag{4.13}$$

$$c_1(x,t)F=c_1(x,t)\exp\left(\frac{kx^2}{1-\nu_1 t}+\nu_2 t\right) \tag{4.14}$$

$$a(t)\frac{\partial^2 F}{\partial x^2}=\frac{4k}{1-\nu_1 t}a(t)x^2\exp\left(\frac{kx^2}{1-\nu_1 t}+\nu_2 t\right)+\frac{2k}{1-\nu_1 t}a(t). \tag{4.15}$$

Therefore

$$\frac{LF}{F}=$$

$$\frac{4k}{1-\nu_1 t}a(t)x^2+\frac{2k}{1-\nu_1 t}a(t)+\frac{2k}{1-\nu_1 t}b_1(x,t)x+c_1(x,t)-\frac{\nu_1 kx^2}{(1-\nu_1 t)^2}-\nu_2$$

using (4.10), $0\le t\le 1/2\nu_1$ and $1\le 1/(1-\nu_1 t)\le 2$. Thus

$$\frac{LF}{F}\le(16k^2M+8kM+M-\nu_1 k)x^2+8kM+M-\nu_2. \tag{4.16}$$

So, we can choose ν_1 and ν_2 for (4.11) to follow.

Since $u(x,t)\ge -M_2\exp(M_1x^2)$ in $\mathbb{R}\times[T_0,T_1]$ and $0\le 1/2\le 1-\nu_1 t\le 1$, we have

$$\begin{aligned}\liminf_{|x|\to\infty}\ \upsilon(x,t)&=\liminf_{|x|\to\infty}\ \exp\left(-\frac{kx^2}{1-\nu_1 t}-\nu_2 t\right)u(x,t)\\&\ge\liminf_{|x|\to\infty}\ -M_2\exp\left(-\frac{kx^2}{1-\nu_1 t}\right)\exp(-\nu_2 t)\exp(M_1x^2)=0.\end{aligned}$$

The last equality follows from observing that $0\le 1/(1-\nu_1 t)\le 2$, $-k\ge -k/(1-\nu_1 t)$, so $M_1-k/(1-\nu_1 t)\le M_1-k\le 0$. We have proved that

$$\liminf_{|x|\to\infty}\ \upsilon(x,t)\ge 0 \tag{4.17}$$

uniformly with respect to t, $0\le t\le 1/2\nu_1$.

Second, it is easy to see that υ satisfies the equation

$$\overline{L}\upsilon=a(t)\frac{\partial^2\upsilon}{\partial x^2}+\overline{b}(x,t)\frac{\partial\upsilon}{\partial x}+\overline{c}(x,t)\upsilon-\frac{\partial\upsilon}{\partial t}=\overline{f} \tag{4.18}$$

where

$$\overline{f}=\frac{Lu}{F}\le 0;\qquad \overline{b_1}=b_1+2\frac{a(t)}{F}\frac{\partial F}{\partial x};\qquad \overline{c}=\frac{LF}{F}. \tag{4.19}$$

We observe that $\overline{L}\upsilon\le 0$ follows from the hypothesis that $Lu\le 0$. Furthermore by (4.16), $\overline{c}\le 0$. We can apply the lemma above and thus conclude that $v(x,t) \geq 0$ in $\mathbb{R} \times [0, 1/2\nu_1]$. The same is therefore true for $u(x,t)$. We can now proceed step by step to prove the positivity of $u(x,t)$ in $\mathbb{R} \times [T_0, T_1]$.

Now we are ready to prove uniqueness. Let's prove that if u satisfies (4.1) and $Lu = 0$; $\varphi = 0$, then $u \equiv 0$. Since $u(x,t) \geq -M_2 \exp(M_1 x^2)$, $Lu = 0$ in $\mathbb{R} \times (T_0, T_1)$ and $u(x,0) \geq 0$ implies $u(x,t) \geq 0$ in $\mathbb{R} \times [T_0, T_1]$. Similarly $-u(x,t) \geq -M_2 \exp(M_1 x^2)$, $L(-u) = 0$ in $\mathbb{R} \times (T_0, T_1)$ and $u(x,0) \geq 0$ implies $-u(x,t) \geq 0$ in $\mathbb{R} \times [T_0, T_1]$. Therefore we have $u(x,t) \equiv 0$.

Finally, if we assume that u_1 and u_2 are solutions of (4.7) and if we define $w = u_1 - u_2$, then

$$\begin{aligned} Lw &= Lu_1 - Lu_2 = 0 \\ w(x,0) &= u_1(x,0) - u_2(x,0) = \varphi(x) - \varphi(x) = 0. \end{aligned} \tag{4.20}$$

Then by the argument above $w = 0$ and so $u_1 = u_2$.

As a consequence our solution (2.26) is a unique solution under the conditions of the theorem above.

5. Special initial data

In the case $u(x,0) = u_0 = \text{constant}$, our solution (2.26) takes the form

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} K(x,y,t)\ u_0\ dy \\ &= u_0 \frac{e^{\alpha(t)x^2+\delta(t)x+\kappa(t)}}{\sqrt{2\pi\mu(t)}} \int_{-\infty}^{\infty} e^{(\beta(t)x+\varepsilon(t))y+\gamma(t)y^2}\ dy \\ &= \frac{u_0}{\sqrt{-2\mu\gamma}} \exp\left(\frac{\left(4\alpha\gamma-\beta^2\right)x^2 + 2\left(2\gamma\delta-\beta\varepsilon\right)x + 4\gamma\kappa - \varepsilon^2}{4\gamma} \right), \end{aligned} \tag{5.1}$$

provided $\gamma(t) < 0$ with the help of an elementary integral (4.4).

The details of taking the limit $t \to 0^+$ in (5.1) are left to the reader.

When $u(x,0) = \delta(x - x_0)$, where $\delta(x)$ is the Dirac delta function, one gets formally

$$u(x,t) = \int_{-\infty}^{\infty} K(x,y,t)\ \delta(y - x_0)\ dy = K(x, x_0, t). \tag{5.2}$$

Thus, in general, the heat kernel (2.16) provides an evolution of this initial data, concentrated originally at a point x_0, into the entire space for a suitable time interval $t > 0$. $K(x, x_0, t)$ may be thought as the temperature at (x,t) caused by an initial burst of heat at $(x_0, 0)$

6. Some examples

Now let us consider several elementary solutions of the characteristic equation (2.13); more complicated cases may include special functions, like Bessel, hypergeometric or elliptic functions [AAR99], [NU88], [Rai60], and

[Wat44]. We encourage the reader to verify our formula with the examples presented in this section; for each case under consideration one must first solve the characteristic equation (2.13) subject to (2.14)–(2.15) and then one must find the expressions (2.17)–(2.23) to obtain explicitly the FS (2.16). Among important elementary cases of our general expressions for the Green function (2.16)–(2.22) are the following:

Example 1. For the traditional diffusion equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}, \qquad a = \text{constant} > 0 \tag{6.1}$$

the heat kernel is

$$K(x, y, t) = \frac{1}{\sqrt{4\pi a t}} \exp\left(-\frac{(x-y)^2}{4at}\right), \qquad t > 0. \tag{6.2}$$

Equation (5.1) gives the steady solution $u_0 = \text{constant}$ for all times $t \geq 0$. See [Can84] and references therein for a detailed investigation of the classical one-dimensional heat equation.

Example 2. In mathematical models of the nerve cell, certain dendritic branches can also be treated as equivalent cylinders in their transient response. A dendritic branch is typically modeled using the cylindrical cable equation. Understanding of the cable equation can bring insights on the dynamics of structural deformation to surrounding neurons that could affect voltage propagation. In fact the cable equation can be used to model the effects of an aneurysm on transmission of electrical signals in a dendrite. If we consider the cable equation on an infinite cylinder:

$$\tau\frac{\partial u}{\partial t} = \lambda^2\frac{\partial^2 u}{\partial x^2} + u, \tag{6.3}$$

then the fundamental solution is given by

$$K(x, y, t) = \frac{\sqrt{\tau}e^{t/\tau}}{\sqrt{4\pi\lambda^2 t}} \exp\left(\frac{-\tau(x-y)^2}{4\lambda^2 t}\right), \qquad t > 0. \tag{6.4}$$

Example 3. We consider the Fokker–Planck equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + u. \tag{6.5}$$

The fundamental solution is given by

$$K(x, y, t) = \frac{1}{\sqrt{2\pi(1-e^{-2t})}} \exp\left(\frac{-\left(x - e^{-t}y\right)^2}{2(1-e^{-2t})}\right). \tag{6.6}$$

Example 4. Now let's consider the equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + (g - kx)\frac{\partial u}{\partial x}, \tag{6.7}$$

with $g \geq 0$, $k > 0$. The case $g = 0$ corresponds to the heat equation with linear drift [Mil77] and in stochastic differential equations this equation corresponds to the Kolmogorov forward equation associated to the regular Ornstein–Uhlenbeck process [Cra09]. The characteristic equation associated to (6.7) is

$$\mu'' + 2k\mu' = 0 \tag{6.8}$$

with solution $\mu(t) = ak^{-1}\left(1 - e^{-2kt}\right) = 2ak^{-1}e^{-kt}\sinh kt$ and the corresponding fundamental solution is given by

$$K(x,y,t) = \frac{\sqrt{k}e^{\frac{kt}{2}}}{\sqrt{4\pi a\sinh(kt)}} \exp\left(-\frac{e^{-kt}\left(g\left(e^{kt}+1\right) + k\left(x - e^{kt}y\right)\right)^2}{4ak\sinh(kt)}\right),$$

so our solution matches the one found in [Cra09] using Lie symmetries of parabolic PDEs.

Example 5. The diffusion-type equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + fxu, \tag{6.9}$$

where $a > 0$ and f are constants (see [Fey05], [Fey205], [Fey49a], [Fey49b], [FH65], [CLSS08] and references therein regarding to similar cases of the Schrödinger equation), has the the characteristic function of the form $\mu = 2at$. The heat kernel is

$$K(x,y,t) = \frac{1}{\sqrt{4\pi at}} \exp\left(-\frac{(x-y)^2}{4at}\right) \exp\left(\frac{f}{2}(x+y)\,t + \frac{af^2}{12}t^3\right) \tag{6.10}$$

provided $t > 0$. Evolution of the uniform initial data $u(x,0) = u_0 =$ constant is given by

$$u(x,t) = u_0 e^{fxt + af^2t^3/3}. \tag{6.11}$$

Example 6. The initial value problem for the following diffusion-type equation with variable coefficients

$$\frac{\partial u}{\partial t} = a\left(\frac{\partial^2 u}{\partial x^2} - x^2 u\right) + \omega\left(\cosh\left((2a-1)\,t\right)\ xu + \sinh\left((2a-1)\,t\right)\ \frac{\partial u}{\partial x}\right), \tag{6.12}$$

where $a > 0$ and ω are two constants, was solved in [LS07] by using the eigenfunction expansion method and a connection with the representations of the Heisenberg–Weyl group $N(3)$. Here we apply a different approach. The solution of the characteristic equation

$$\mu'' - 4a^2\mu = 0 \tag{6.13}$$

is $\mu=\sinh(2at)$ and the corresponding heat kernel is given by

$$\begin{aligned}K(x,y,t)&\\ &=\frac{1}{\sqrt{2\pi\sinh(2at)}}\ \exp\left(-\frac{\left(x^2+y^2\right)\cosh(2at)-2xy}{2\sinh(2at)}\right)\\ &\quad\cdot\exp\left(2\omega\frac{x\sinh(t/2)+y\sinh((2a-1/2)t)}{\sinh(2at)}\sinh\left(\frac{t}{2}\right)\right)\\ &\quad\cdot\exp\left(-2\omega^2\frac{\cosh(2at)}{\sinh(2at)}\sinh^4\left(\frac{t}{2}\right)\right)\\ &\quad\cdot\exp\left(\frac{\omega^2}{2}\left(t-2\sinh t+\frac{1}{2}\sinh(2t)\right)\right),\qquad t>0.\end{aligned}\tag{6.14}$$

Indeed, by (2.17)–(2.19)

$$\alpha=\gamma=-\frac{\cosh(2at)}{2\sinh(2at)},\qquad \beta=\frac{1}{\sinh(2at)}.\tag{6.15}$$

In this case

$$\begin{aligned}f\mu+\frac{g}{2a}\mu'&=\omega\left(\cosh((2a-1)t)\sinh(2at)-\sinh((2a-1)t)\cosh(2at)\right)\\ &=\omega\sinh t\end{aligned}$$

and Equation (2.20) gives

$$\delta=\omega\frac{\cosh t-1}{\sinh(2at)}=2\omega\frac{\sinh^2(t/2)}{\sinh(2at)}.\tag{6.16}$$

By (2.21)

$$\begin{aligned}\varepsilon=\omega\frac{1-\cosh t}{\sinh(2at)\cosh(2at)}&+2a\omega\int_0^t\frac{1-\cosh\tau}{\cosh^2(2a\tau)}\ d\tau\\ &+\omega\int_0^t\frac{\cosh((2a-1)\tau)}{\cosh(2a\tau)}\ d\tau,\end{aligned}\tag{6.17}$$

where the integration by parts gives

$$2a\int_0^t\frac{1-\cosh\tau}{\cosh^2(2a\tau)}\ d\tau=(1-\cosh t)\frac{\sinh(2at)}{\cosh(2at)}+\int_0^t\frac{\sinh(2a\tau)}{\cosh(2a\tau)}\sinh\tau\ d\tau.$$

Thus

$$\varepsilon=\omega(1-\cosh t)\frac{\cosh(2at)}{\sinh(2at)}+\omega\int_0^t\frac{\sinh(2a\tau)\sinh\tau+\cosh((2a-1)\tau)}{\cosh(2a\tau)}\ d\tau$$

and an elementary identity

$$\sinh(2at)\sinh t+\cosh((2a-1)t)=\cosh(2at)\cosh t\tag{6.18}$$

leads to an integral evaluation. Two other identities

$$\cosh(2at)\cosh t-\sinh(2at)\sinh t=\cosh((2a-1)t),\tag{6.19}$$

$$\cosh(2at)-\cosh((2a-1)t)=2\sinh(t/2)\sinh((2a-1/2)t)\tag{6.20}$$

result in

$$\varepsilon = \omega \frac{\cosh(2at) - \cosh((2a-1)t)}{\sinh(2at)} = 2\omega \frac{\sinh(t/2)\sinh((2a-1/2)t)}{\sinh(2at)}. \tag{6.21}$$

In a similar fashion,

$$\kappa = -2\omega^2 \sinh^4(t/2) \frac{\cosh(2at)}{\sinh(2at)} + \frac{1}{2}\omega^2 \left(t - 2\sinh t + \frac{1}{2}\sinh(2t)\right), \tag{6.22}$$

and Equation (6.14) is derived. In the limit $\omega \to 0$ this kernel also gives a familiar expression in statistical mechanics for the density matrix for a system consisting of a simple harmonic oscillator [FH65].

Example 7. The case $a = 1/2$ corresponds to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} - x^2 u\right) + \omega\, xu \tag{6.23}$$

and the heat kernel (6.14) is simplified to the form

$$K(x,y,t) = \frac{e^{\omega^2 t/2}}{\sqrt{2\pi \sinh t}} \exp\left(-\frac{\left((x-\omega)^2 + (y-\omega)^2\right)\cosh t - 2(x-\omega)(y-\omega)}{2\sinh t}\right),$$

when $t > 0$. A similar diffusion-type equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2} + x^2 u\right) + \omega\, xu \tag{6.24}$$

can be solved with the aid of the kernel

$$K(x,y,t) = \frac{e^{-\omega^2 t/2}}{\sqrt{2\pi \sin t}} \exp\left(-\frac{\left((x+\omega)^2 + (y+\omega)^2\right)\cos t - 2(x+\omega)(y+\omega)}{2\sin t}\right)$$

provided $0 < t < \pi/2$. We leave the details to the reader.

Example 8. Following the case of exactly solvable time-dependent Schrödinger equation found in [MCS07], we consider the diffusion-type equation of the form

$$\frac{\partial u}{\partial t} = \cosh^2 t\, \frac{\partial^2 u}{\partial x^2} + \sinh^2 t\, x^2 u + \frac{1}{2}\sinh 2t \left(2x\frac{\partial u}{\partial x} + u\right). \tag{6.25}$$

The corresponding characteristic equation

$$\mu'' - 2\tanh t\, \mu' + 2\mu = 0 \tag{6.26}$$

has two linearly independent solutions

$$\mu_1 = \cos t \sinh t + \sin t \cosh t, \tag{6.27}$$

$$\mu_2 = \sin t \sinh t - \cos t \cosh t \tag{6.28}$$

with the Wronskian $W(\mu_1, \mu_2) = 2\cosh^2 t$, and the first one satisfies the initial conditions (2.15). The heat kernel is

$$\begin{aligned} K(x,y,t) \\ &= \frac{1}{\sqrt{2\pi(\cos t \sinh t + \sin t \cosh t)}} \\ &\quad \cdot \exp\left(\frac{(y^2 - x^2)\sin t \sinh t + 2xy - (x^2 + y^2)\cos t \cosh t}{2(\cos t \sinh t + \sin t \cosh t)}\right) \end{aligned} \tag{6.29}$$

provided $0 < t < T_1 \approx 0.9375520344$, where T_1 is the first positive root of the transcendental equation $\tanh t = \cot t$. Then $\gamma(t) < 0$ and the integral (2.26) converges for suitable initial data.

Example 9. A similar diffusion-type equation

$$\frac{\partial u}{\partial t} = \cos^2 t \, \frac{\partial^2 u}{\partial x^2} + \sin^2 t \; x^2 u - \frac{1}{2}\sin 2t \left(2x\frac{\partial u}{\partial x} + u\right) \tag{6.30}$$

has the characteristic equation of the form

$$\mu'' + 2\tan t \; \mu' - 2\mu = 0 \tag{6.31}$$

with the same solution (6.27). It appeared in [MCS07] and [CLSS08] for a special case of the Schrödinger equation. The corresponding heat kernel has the same form (6.29) but with x and y interchanged:

$$\begin{aligned} K(x,y,t) &= \frac{1}{\sqrt{2\pi(\cos t \sinh t + \sin t \cosh t)}} \\ &\quad \cdot \exp\left(\frac{(x^2 - y^2)\sin t \sinh t + 2xy - (x^2 + y^2)\cos t \cosh t}{2(\cos t \sinh t + \sin t \cosh t)}\right) \end{aligned}$$

provided $0 < t < T_2 \approx 2.347045566$, where T_2 is the first positive root of the transcendental equation $\tanh t = -\cot t$. We leave the details for the reader.

7. Solution of the nonhomogeneous equation

A diffusion-type equation of the form

$$\left(\frac{\partial}{\partial t} - Q(t)\right) u = F, \tag{7.1}$$

where Q stands for the second order linear differential operator in the right-hand side of Equation (2.1) and $F = F(t,x,u)$, can be rewritten formally as an integral equation (the Duhamel principle; see [Caz03], [CH98], [LSU68], [Lev07], [SS08], [Tao06] and references therein)

$$u(x,t) = H(t,0)\, u(x,0) + \int_0^t H(t,s)\, F(s,x,u)\ ds. \tag{7.2}$$

Operator $H(t,s)$ is given by (2.28). When F does not depend on u, one gets a solution of the nonhomogeneous equation (7.1).

Indeed, a formal differentiation gives

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} H(t,0)\, u(x,0) + \frac{\partial}{\partial t}\int_0^t H(t,s)\, F(s,x,u)\ ds, \tag{7.3}$$

where

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^t H(t,s)\ \ F(s,x,u)\ ds \\ = H(t,t)\ \ F(t,x,u) + \int_0^t \frac{\partial}{\partial t} H(t,s)\, F(s,x,u)\ ds\end{aligned} \tag{7.4}$$

and we assume that $H(t,t)$ is the identity operator. Also

$$Q(t)u = Q(t)H(t,0)\, u(x,0) + \int_0^t Q(t)H(t,s)\, F(s,x,u)\ ds \tag{7.5}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial t} - Q(t)\right) u = \left(\frac{\partial}{\partial t} - Q(t)\right) H(t,0)\, u(x,0) + F \\ + \int_0^t \left(\frac{\partial}{\partial t} - Q(t)\right) H(t,s)\, F(s,x,u)\ ds,\end{aligned} \tag{7.6}$$

where

$$\left(\frac{\partial}{\partial t} - Q(t)\right) H(t,s) = 0, \qquad 0 \le s < t \tag{7.7}$$

by construction of the operator $H(t,s)$ in (2.28). This completes our formal proof. A rigorous proof will be given elsewhere.

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