New York Journal of Mathematics

New York J. Math. **17a** (2011) 39–44.

Analyticity of a joint spectrum and a multivariable analytic Fredhom theorem

M. Stessin, R. Yang and K. Zhu

Abstract. For an n-tuple of compact operators $T = (T_1, \cdots, T_n)$ on a Hilbert space H we consider a notion of joint spectrum of T, denoted by $\Sigma(T)$, which consists of points $z = (z_1, \cdots, z_n)$ in $\mathbb{C}^n$ such that $I + z_1T_1 + \cdots + z_nT_n$ is not invertible, where I is the identity operator on H. Using the theory of determinants for certain Fredholm operators we show that $\Sigma(T)$ is always an analytic set of codimension 1 in $\mathbb{C}^n$. This result is in fact a special case of a multivariable version of the analytic Fredholm theorem.

Contents

1. Introduction

In this paper, H denotes a complex separable Hilbert space. $\mathcal{K}(H)$ and $\mathcal{S}_1$ denote the set of compact operators and trace class operators on H, respectively.

Given an n-tuple $T = (T_1, \cdots, T_n)$ of compact operators on H, we are interested in the set $\Sigma(T)$ consisting of $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$ such that $T(z)$ is not invertible, where

$$T(z) = I + z_1T_1 + \cdots + z_nT_n,$$

with I being the identity operator on H. One motivation of this paper is the study of the so-called projective spectrum for operator tuples that is defined and studied in [8].

Received August 13, 2010.

2000 *Mathematics Subject Classification.* Primery 30H05, 30E10, Secondary 30D55.

Key words and phrases. compact operators, Fredholm operators, analytic set, trace, determinant, joint spectrum.

ISSN 1076-9803/2011

Clearly, if A is any linear operator on H with $\|A\| < 1$, then the operator $I + A$ is invertible. Therefore, the joint spectrum $\Sigma(T)$ misses the entire open ball

$$|z| < 1/\sqrt{\|T_1\|^2 + \cdots + \|T_n\|^2}$$

in $\mathbb{C}^n$.

The main result of the paper is that $\Sigma(T)$ is always an analytic set of codimension 1 in $\mathbb{C}^n$. Recall that a set S in $\mathbb{C}^n$ is called an analytic set if for any $z \in S$ there exists some open neighborhood U of z in $\mathbb{C}^n$ such that $S \cap U$ is the intersection of zero sets of several holomorphic function in U. Furthermore, S is an analytic set of codimension 1 if $S \cap U$ above is the zero set of a single holomorphic function in U.

The case in which $n = 1$ is elementary. In fact, if $n = 1$ and T is a single operator, then it is clear that

$$\Sigma(T) = \left\{ -\frac{1}{z} : z \in \sigma(T) \setminus \{0\} \right\},$$

where $\sigma(T)$ is the classical spectrum of T, consisting of complex numbers z such that $zI - T$ is not invertible. If T is compact, then $\sigma(T)$ is a (finite or infinite) sequence with 0 as the only possible accumulation point, so $\Sigma(T)$ is a (finite or infinite) sequence in the punctured complex plane $\mathbb{C} - \{0\}$ without any finite accumulation point, which, according to the Weierstrass factorization theorem, is always the zero set of an entire function.

Our proof of the main result is more or less constructive and relies on the notion of determinant for a certain class of Fredholm operators. As a matter of fact, we are going to prove a multi-variable analytic Fredholm theorem, establish a trace-determinant formula, and derive the result mentioned above as a corollary.

2. Preliminaries on determinant and trace

If F is a trace class operator on H with eigenvalues $\lambda_1, \lambda_2, \ldots$, counting multiplicity, then the determinant of $I + F$ is defined by

$$\det(I + F) = \prod_{k=1}^{N(F)} (1 + \lambda_j),$$

where $N(F)$ is the rank of F. The following fact is a direct consequence of this definition.

Lemma 1. *Suppose F is a trace class operator on H. Then the following are equivalent.*

(a) *$I + F$ is not invertible.*
(b) *-1 is an eigenvalue of F.*
(c) $\det(I + F) = 0$.

Proof. If $I+F$ is not invertible, then -1 belongs to the spectrum of F. But F is compact, so all nonzero points in its spectrum are eigenvalues. Thus -1 is an eigenvalue of F. This shows that (a) implies (b). That (b) implies (a) is obvious. It is also clear that (b) and (c) are equivalent. □

For two vectors x and y in H, the rank 1 operator $x \otimes y$ is defined by $x \otimes y(w) = \langle w,\ y\rangle x$. If F is a finite rank operator, say $\sum_{k=1}^{m} \phi_k \otimes f_k$, then $\det(I+F)$ is equal to the determinant of an associated $m \times m$ matrix ([4], Theorem 3.2), namely,

$$\det(I+F) = \det\left(\delta_{jk} + \langle \phi_j,\ f_k\rangle\right)_{j,k=1}^{m}.$$

3. A multivariable analytic Fredholm theorem

Now consider a holomorphic function f from a domain $\Omega \subset \mathbb{C}^n$ to $\mathcal{K}(H)$. Let

$$\Sigma(f) := \{z \in \Omega :\ I + f(z) \text{ is not invertible}\}.$$

In the case $n = 1$, the classical analytic Fredholm theorem says that $\Sigma(f)$ is either the entire Ω or a discrete subset of Ω ([7]). The purpose of this section is to establish a multivariable version of this result.

Lemma 2. *If f is holomorphic from a domain $\Omega \subset \mathbb{C}^n$ into $\mathcal{S}_1$, then the function $\det(I + f(z))$ is holomorphic on Ω. Consequently,*

$$\Sigma(f) = \{z \in \Omega :\ \det(I + f(z)) = 0\}$$

is the zero set of a global holomorphic function.

Proof. When $n = 1$, this is Theorem 8 on page 163 of [5]. When $n > 1$, we just consider the complex variables one at a time and use the $n = 1$ case. Since a complex-valued function defined on Ω is holomorphic whenever it is holomorphic in each variable separately, the desired result follows. □

The main result of this section is the following.

Theorem 3. *Suppose f is a holomorphic operator-valued function from a domain $\Omega \subset \mathbb{C}^n$ to $\mathcal{K}(H)$. Then $\Sigma(f)$ is either empty, or the entire Ω, or an analytic subset of Ω of codimension 1.*

Proof. Assume $\Sigma(f)$ is not empty or the entire Ω. In this case f is not a constant. Pick any point λ in $\Sigma(f)$. Since Ω is open and f is holomorphic, there is a small $r > 0$ such that $B(\lambda,\ r) \subset \Omega$ and $\|f(z) - f(\lambda)\| < 1/3$ for all $z \in B(\lambda,\ r)$. Since $f(\lambda)$ is compact, there is a finite rank operator K such that $\|f(\lambda) - K\| < 1/3$. It follows that

$$\|f(z) - K\| < 2/3, \qquad z \in B(\lambda,\ r). \tag{1}$$

Given any $z \in B(\lambda,\ r)$, the inequality in (1) implies that the operator $I + f(z) - K$ is invertible, so we can write

$$I + f(z) = [I + f(z) - K][I + (I + f(z) - K)^{-1}K].$$

Also, $I + f(z)$ is not invertible if and only if the operator

$$I + (I + f(z) - K)^{-1}K$$

is not invertible, and this is the case, according to Lemma 1, if and only if

$$\det[I + (I + f(z) - K)^{-1}K] = 0.$$

This shows that

$$\Sigma(f) \cap B(\lambda, r) = \{z \in B(\lambda, r) : \det[I + (I + f(z) - K)^{-1}K] = 0\}.$$

Since K is a trace class operator, the function

$$z \mapsto (I + f(z) - K)^{-1}K$$

is holomorphic from $B(\lambda,\ r)$ into $\mathcal{S}_1$. Combining this with Lemma 2, we conclude that the function

$$z \mapsto \det[I + (I + f(z) - K)^{-1}K]$$

is holomorphic on $B(\lambda,\ r)$. Thus $\Sigma(f) \cap B(\lambda,\ r)$ is the zero set of a holomorphic function in $B(\lambda,\ r)$. Clearly, $\lambda \in \Sigma(f) \cap B(\lambda,\ r)$. If $\Sigma(f) \cap B(\lambda,\ r) = B(\lambda,\ r)$, then $\Sigma(f) = \Omega$ by the connectness of Ω and this contradicts our assumption. Hence $\det[I + (I + f(z) - K)^{-1}K]$ is a nonconstant holomorphic function on $B(\lambda,\ r)$, and therefore $\Sigma(f) \cap B(\lambda,\ r)$ is an analytic subset of codimension 1. $\square$

Proposition 4. *Suppose f is a nonconstant holomorphic operator-valued function from a domain $\Omega \subset \mathbb{C}^n$ to $\mathcal{K}(H)$. If $\Sigma(f)$ is not empty or the entire Ω, then $\Sigma^c(f)$, the complement of $\Sigma(f)$ in Ω, is not simply connected.*

Proof. Pick a point $\lambda = (\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_n) \in \Sigma(f)$. Since by Theorem 3 $\Sigma(f)$ is an analytic set of codimension 1, we can pick an $r > 0$ and an $\eta \in \mathbb{C}^n$ with $\|\eta\| = 1$ such that $\{\lambda + w\eta :\ |w| < r\} \cap \Sigma(f)$ is a finite set. So for some smaller $0 < r' < r$, the set $D_\lambda := \{\lambda + w\eta :\ |w| \le r'\}$ contains only one point from $\Sigma(f)$, namely λ. Now consider the one variable function $g(w) = I + f(\lambda + w\eta)$, $|w| \le r'$. By construction $g(w)$ is Fredholm operator-valued and is invertible for all w with $|w| = r'$. By [6] (also see page 1 and 2 of [1]), the logarithmic integral

$$\frac{1}{2\pi i}\int_{\partial D_\lambda} [g(w)]^{-1} g'(w)\, dw \tag{2}$$

is a nonzero finite rank operator. Now consider the restriction of operator-valued 1-form

$$(I + f(z))^{-1} df(z) = (I + f(z))^{-1} \sum_{j=1}^{n} \frac{\partial f(z)}{\partial z_j} dz_j$$

to D_λ. Since $z = \lambda + w\eta$, $w = \langle z,\ \eta\rangle - \langle \lambda,\ \eta\rangle$, and it follows easily that $df(z) = g'(w)dw$ on D_λ. Hence

$$\frac{1}{2\pi i}\int_{\partial D_\lambda} (I + f(z))^{-1}\, df(z) = \frac{1}{2\pi i}\int_{\partial D_\lambda} [g(w)]^{-1} g'(w)\, dw \ne 0.$$

Since $(I+f(z))^{-1}df(z)$ is an analytic 1-form on $\Sigma^c(f)$, the fact above implies that ∂D_λ is not homotopic in $\Sigma^c(f)$ to a point. $\square$

We make some further remarks on the 1-form $(I+f(z))^{-1}df(z)$. Assume f is holomorphic from $\Omega \subset \mathbb{C}^n$ into $\mathcal{S}_1$. As in the proof of Proposition 4, the operator-valued 1-form

$$\omega_f(z) := (I+f(z))^{-1}\, df(z)$$

is well defined on $\Sigma^c(f) := \Omega \setminus \Sigma(f)$. Here again

$$d = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j.$$

Moreover, since $\frac{\partial f(z)}{\partial z_j}$ is trace class for all z, we see that $\omega_f(z)$ is in fact an $\mathcal{S}_1$-valued 1-form. Hence the trace $\operatorname{tr}[\omega_f(z)]$ is well defined and is an ordinary holomorphic 1-form on $\Sigma^c(f)$. The following formula is a multivariable version of its one variable case (cf. [5] Chapter IV formula (1.14)).

Proposition 5. *If f is holomorphic from $\Omega \subset \mathbb{C}^n$ into $\mathcal{S}_1$, then*

$$\operatorname{tr}[\omega_f(z)] = d \log \det(I+f(z))$$

on $\Sigma^c(f)$.

In this case, the integral (2) in the proof of Proposition 4 transforms to an elementary logarithmic integral by the trace. More specifically, in this case, λ is a zero of the function $\det(I+f(z))$, and by Proposition 5 above and the residue theorem,

$$\begin{aligned}\frac{1}{2\pi i}\operatorname{tr}\int_{\partial D_\lambda} g^{-1}(w)\, dg(w) &= \frac{1}{2\pi i}\int_{\partial D_\lambda} \operatorname{tr}[(I+f(\lambda+w\eta))^{-1} df(\lambda+w\eta)]\\ &= \frac{1}{2\pi i}\int_{\partial D_\lambda} d\log\det(I+f(\lambda+w\eta))\\ &= m(\lambda),\end{aligned}$$

where $m(\lambda)$ is the order of zero of $\det(I+f(\lambda+w\eta))$ at $w=0$.

References

[1] Bart, H.; Ehrhardt, T.; Silbermann, B. Trace conditions for regular spectral behavior of vector-valued analytic functions. *Linear Algebra and its Applications* **430** (2009), 1945–1965. MR2503944 (2010f:30098), Zbl 1188.47014.

[2] Chirka, E. M. Complex analytic sets. Translated from the Russian by R. A. M. Hoksbergen. Mathematics and Its Applications: Soviet Series, 46. *Kluwer Academic Publishers, Dordrecht*, 1989. xx+372 pp. ISBN 0-7923-0234-6. MR1111477 (92b:32016), Zbl 0683.32002.

[3] Gohberg, I. C.; Goldberg, S.; Kaashoek, M. Classes of linear operators. Volume 1. Operator Theory: Advances and Applications, 49. *Birkhäuser Verlag, Basel*, 1990. xiv+468 pp. ISBN: 3-7643-2531-3. MR1130394 (93d:47002), Zbl 0745.47002.

[4] Gohberg, I. C.; Goldberg, S. Krupnik, N. Traces and determinants of linear operators. Operator Theory: Advances and Applications, 116. *Birkhäuser Verlag, Basel*, 2000. x+258 pp. ISBN: 3-7643-6177-8. MR1744872 (2001b:47035), Zbl 0946.47013.

[5] Gohberg, I. C; Krein, M. G. Introduction to the theory of linear nonselfadjoint operators. Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, 18. *American Mathematical Society, Providence, R.I.*, 1969. xv+378 pp. MR0246142 (39 #7447), Zbl 0181.13504.

[6] Markus, A. S.; Sigal, E. I. The multiplicity of a characteristic value of an analytic operator-valued function. *Mat. Issled.* **5** (1970), no. 3(17), 129–147. (Russian). MR0312306 (47 #868), Zbl 0234.47013.

[7] Reed, M.; Simon, B. Methods of modern mathematical physics. I: Functional analysis. Rev. and enl. ed. *Academic Press, A Subsidiary of Harcourt Brace Jovanovich, Publishers, New York*, 1980, xv+400 pp. MR0493419 (58 #12429a), Zbl 0459.46001.

[8] Yang, R. Projective spectrum in Banach algebras. *J. of Topology and Analysis* **1** (2009) 289–306. MR2574027 (2010m:47008), Zbl 1197.47015.

Department of Mathematics and Statistics, SUNY at Albany, Albany, NY 12222
stessin@math.albany.edu

Department of Mathematics and Statistics, SUNY at Albany, Albany, NY 12222
ryang@math.albany.edu

Department of Mathematics and Statistics, SUNY at Albany, Albany, NY 12222
kzhu@math.albany.edu

This paper is available via http://nyjm.albany.edu/j/2011/17a-3.html.