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On the computation of the Fourier transform under the presence of nearby polar singularities

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Abstract. In this paper we present a procedure for the computation of integrals on the whole real line with nearby singularities. The method is based in collecting the possible singularities of the integrand within a weight function on the real line, passing to the unit circle by considering an associated weight function there and then making use of Szegő or interpolatory-type quadrature formulas. This algorithm is applied in order to provide a computational method for the Fourier transform of a function exhibiting polar singularities near the range of integration. Some error bounds for the estimations are presented and some numerical experiments carried out.

Contents

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1. Introduction

Given a pair of functions related by the expression

$$G(w) = \int_a^b g(t)\mathcal{K}(w,t)dt, \tag{1}$$

we say that $G(w)$ is the integral transform of $g(t)$ and $\mathcal{K}(w,t)$ its kernel. As it is known, the computation of the integral (1) is often a difficult task because of the possible infinite range of integration along with possible end-point singularities, making (1) sometimes an ill-posed problem. When $\mathcal{K}(w,t) = e^{2\pi iwt}$ the well known Fourier transform appears so that to the difficulty of infinite range we must add the "rapidly oscillatory integrand". Thus, under the assumption that g vanishes outside a finite interval, say $[0,T]$, when w is quite large the known special methods designed for rapidly oscillatory integrands could be used in order to efficiently compute (1). Indeed, by setting as the Fourier transform

$$G(w) = \int_{\mathbb{R}} g(t)e^{2\pi iwt}dt, \tag{2}$$

then we can write (see [6] for details)

$$G(w) = \int_0^T \left[\sum_{k\in\mathbb{Z}} g(t+kT)e^{2\pi iwkT}\right] e^{2\pi iwt}dt, \quad T > 0.$$

Thus, when concerned with the evaluation of the Fourier transform only for discrete values of r, namely $w = w_r = \frac{r}{T}$, $r = 0, \pm 1, \pm 2, \ldots$, it follows

$$G(w_r) = \int_0^T \left[\sum_{k\in\mathbb{Z}} g(t+kT)\right] e^{2\pi iw_r t}dt.$$

Now, setting $g_p(t) = \sum_{k\in\mathbb{Z}} g(t+kT)$, $-\infty < t < +\infty$ and $T > 0$, one can write

$$G(w_r) = \int_0^T g_p(t)e^{2\pi iw_r t}dt.$$

Observe that for any g, the function g_p is periodic of period T and coincides with g over $[0,T]$, if and only if, g vanishes outside this interval. This is the reason of the assumption $g(t) = 0$ for $t \neq [0,T]$. Although this condition is often satisfied in technological problems, in particular in communication theory, in general when g is nonzero over a larger interval than $[0,T]$ then an "overlap" or "aliasing error" occurs. This error is usually disregarded in most of the methods to compute the Fourier transform as it happens in the well known "discrete Fourier transform". In this respect, the second author et al. in [15] developed new methods to compute (2) under the same hypothesis on g but adding an extra difficulty, namely, the presence of singularities near the interval of integration.

Thus, setting $\tilde{g}(\theta) = f\left(\frac{T}{2\pi}\theta\right)$ and supposing that $\tilde{g}$ can be factorized as $\tilde{g}(\theta) = h(\theta)\sigma(\theta)$, with h smooth enough and σ possessing some possible singularities, then one has

$$G(w_r) = G(r) = \frac{T}{2\pi}\int_{-\pi}^{\pi} h(\theta)\sigma(\theta)e^{ir\theta}d\theta. \tag{3}$$

Hence, one sees that the initial problem reduces to the approximate calculation of weighted 2π-periodic integrals like (3) or more generally weighted integrals on the unit circle. In [15], the use of the so-called Szegő quadrature formulas or quadrature rules exactly integrating trigonometric polynomials up to the highest degree turned out to be crucial when computing (3).

In this paper, we will be concerned with the computation of the Fourier transform (2) for the discrete values $w = w_r = \frac{r}{2\pi}$, $r = 0, \pm 1, \pm 2, \ldots$, that is, the integral

$$G(w_r) = G(r) = \int_{-\infty}^{\infty} g(t)e^{2irt}dt,$$

by keeping the difficulty of the presence of polar singularities near the real line but now dropping off the assumption that g vanishes outside a finite interval. Instead, we will initially assume that g is a 2π-periodic function which will allow us to continue to using the Szegő quadrature formulas as done by the authors in the recent paper [9].

The rest of the paper has been organized as follows. In Section 2 we introduce and characterize quadrature formulas on the unit circle, of Szegő and interpolatory-type, along with certain error bounds requiring a low computational effort. In Section 3 we present a procedure for the computation of integrals on the whole real line with nearby singularities. This method consists in the introduction of the possible singularities of the integrand in a weight function defined on $\mathbb{R}$, to consider an associated weight function on the unit circle and then to make use of the rules presented in Section 2. In Section 4 we apply the results of Section 3 in order to provide a computational method for the Fourier transform of a function exhibiting polar singularities near the range of integration. Some numerical experiments are presented in a final section.

2. Quadrature formulas on the unit circle

In this section we will be mainly concerned with the approximate calculation of the weighted integral of a 2π-periodic function, i.e., with an integral like

$$I_\omega(f) = \int_{-\pi}^{\pi} f(\theta)\omega(\theta)d\theta,$$

ω being a weight function on $[-\pi, \pi]$, that is $\omega > 0$ a.e. on $[-\pi, \pi]$ and f 2π-periodic such that $f\omega$ is integrable on $[-\pi, \pi]$. By an n-point quadrature formula or quadrature rule for $I_\omega(f)$ or for ω we mean an expression like

$$I_n(f) = \sum_{j=0}^{n-1} \lambda_j f(\theta_j), \quad \theta_j \neq \theta_k \text{ if } j \neq k, \quad \{\theta_j\}_{j=0}^{n-1} \subset (-\pi, \pi], \tag{4}$$

so that the "nodes" $\{\theta_j\}_{j=0}^{n-1}$ and "weights" $\{\lambda_j\}_{j=0}^{n-1}$ in (4) are choosen by imposing that $I_\omega(T) = I_n(T)$ for any trigonometric polynomial $T(\theta) = \sum_{k=0}^{N} (a_k \cos k\theta + b_k \sin k\theta)$ with as high degree N as possible. In this respect, it is known that $N \leq n-1$ (see, e.g., [21, pp. 73–74]) and that the case $N = n-1$ gives rise to the quadrature formulas with the maximum trigonometric degree of precision. Such rules come characterized in terms of the so-called "bi-orthogonal systems of trigonometric polynomials" associated with ω (see [13] or [25] for further details). Alternatively, taking into account that any periodic function on $\mathbb{R}$ can be considered as a function defined on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and that a trigonometric polynomial T of degree k can be expressed in the form $T(\theta) = L(e^{i\theta})$ with

$$L \in \operatorname{span}\{z^j : -k \leq j \leq k\} =: \Lambda_{-k,k}$$

(Laurent polynomials), we can reformulate our problem as follows:

Given the integral on $\mathbb{T}$*,*

$$I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta,$$

find weights $\{\lambda_j\}_{j=0}^{n-1}$ *and distinct nodes* $\{z_j\}_{j=0}^{n-1} \subset \mathbb{T}$*,* $z_j = e^{i\theta_j}$ *for all* $j = 0, \ldots, n-1$*, such that*

$$I_\omega(L) = I_n(L), \quad \textit{for all } L \in \Lambda_{-(n-1),n-1}. \tag{5}$$

In general, given integers $r \leq s$ we set $\Lambda_{r,s} = \operatorname{span}\{z^j : r \leq j \leq s\}$ and $\Lambda = \bigcup_{r\geq 0} \Lambda_{-r,r}$ (space of Laurent polynomials). Observe that $\dim(\Lambda_{r,s}) = s - r + 1$ and that $\Lambda_{0,k} = \mathbb{P}_k$, the space of ordinary polynomials of degree at most k.

The space of all polynomials will be denoted by $\mathbb{P} = \bigcup_{k\geq 0} \mathbb{P}_k$ and given a polynomial $P_n(z)$ of degree exactly n, we define its reverse or reciprocal as $P_n^*(z) = z^n \overline{P_n(1/\bar{z})}$. Concerning the construction of the quadrature rule (5) one has the following (see, e.g., [8], [16] and [19]).

Theorem 2.1. *Set* $I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$ *and* $I_n(g) = \sum_{j=0}^{n-1} \lambda_j g(z_j)$*, with* $z_j \in \mathbb{T}$*,* $j = 1, \ldots, n$ *and let* $\{\varphi_k(z)\}_{k=0}^{\infty}$ *be the sequence of orthonormal (Szegő) polynomials for* ω*. Then,* $I_n(g) = I_\omega(g)$ *for all* $g \in \Lambda_{-(n-1),n-1}$*, if and only if:*

(1) $\{z_j\}_{j=0}^{n-1}$ *are the zeros of* $B_n(z, \tau_n) = \left[z\varphi_{n-1}(z) + \tau_n \varphi_{n-1}^*(z)\right]$*, for some* $\tau_n \in \mathbb{T}$*.*

(2) *The weights are given by* $\lambda_j = \left(\sum_{k=0}^{n-1} |\varphi_k(z_j)|^2\right)^{-1} > 0$ *for all* $j = 0, \ldots, n-1$*.*

$I_n(g)$ as given in Theorem 2.1 is called an n-point Szegő quadrature rule (see [19]) and represents the analog on the unit circle of the Gaussian formulas. On the other hand, if $\rho_n(z)$ denotes the monic Szegő polynomial for ω, i.e., $\rho_n(z) = \frac{\varphi_n(z)}{\gamma_n}$, where $\varphi_n(z) = \gamma_n z^n + \cdots$ $(\gamma_n > 0)$, one can also write (up to a multiplicative factor)

$$B_n(z, \tau_n) = z\rho_{n-1}(z) + \tau_n \rho^*_{n-1}(z). \tag{6}$$

Now, from the well known Szegő recurrence relations (see, e.g., [26, Theorem 11.4.2] or [24, Theorem 1.5.2])

$$\begin{pmatrix} \rho_{n+1}(z) \\ \rho^*_{n+1}(z) \end{pmatrix} = \begin{pmatrix} z & \delta_{n+1} \\ \overline{\delta_{n+1}} z & 1 \end{pmatrix} \begin{pmatrix} \rho_n(z) \\ \rho^*_n(z) \end{pmatrix}, \quad n = 0, 1, \ldots, \tag{7}$$

with $\rho_0(z) = \rho^*_0(z) \equiv 1$, $\delta_0 = 1$ and $\delta_n = \rho_n(0) \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for all $n \geq 1$ (Verblunsky parameters[1]), we see to generate an n-point Szegő formula, we take $\tau_n \in \mathbb{T}$ and consider the zeros of $B_n(z, \tau_n)$ given by (6) so that by (7) we essentially need the parameters $\delta_0 = 1, \delta_1, \ldots, \delta_{n-1}$ and τ_n. As for an efficient computation of Szegő formulas, see [2], [7] and [17, 18], among others.

Here it should be said that an explicit representation of the Szegő polynomials is seldom known so that they must be recursively computed from (7) (Levinson's algorithm, see [22]) and starting from the usual available information on the weight function ω, namely its trigonometric moments

$$\mu_k = \int_{-\pi}^{\pi} e^{-ik\theta} \omega(\theta) d\theta, \quad k = 0, \pm 1, \pm 2, \ldots.$$

Thus, in some cases the computational effort to calculate the Szegő formulas could be rather high or it could also happen that ω is a signed function or even more generally, a complex function so that the relation

$$\langle f, g \rangle_\omega = \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \omega(\theta) d\theta$$

is not an Hermitian inner product and orthogonality becomes meaningless. As an alternative, we could proceed as follows: given n distinct nodes $\{\tilde{z}_j\}_{j=0}^{n-1} \subset \mathbb{T}$, find weights $\{\tilde{\lambda}_j\}_{j=0}^{n-1}$ such that

$$\tilde{I}_n(g) = \sum_{j=0}^{n-1} \tilde{\lambda}_j g(\tilde{z}_j) = I_\omega(g), \tag{8}$$
$$\text{for all } g \in \Lambda_{-r,s} \quad \text{with} \quad \dim(\Lambda_{-r,s}) = r + s + 1.$$

In this case, the weights are uniquely determined by $\tilde{\lambda}_j = I_\omega(l_j(z))$, with $l_j \in \Lambda_{-r,s}$ such that $l_j(\tilde{z}_k) = \delta_{j,k}$, with $\delta_{j,k}$ the Kronecker delta symbol.

[1] There are at least four other terms: Szegő, reflection, Schur and Geronimus parameters, see [24, Chapter 1.5] .

$\tilde{I}_n(g)$ as given by (8) is called of interpolatory-type in $\Lambda_{-r,s}$ since $\tilde{I}_n(g) = I_\omega(L_n(g;\cdot))$ with $L_n(g,\cdot) \in \Lambda_{-r,s}$ such that $\tilde{L}_n(g,\tilde{z}_j) = g(\tilde{z}_j)$ for all $j = 1,\ldots,n$. Observe that now the weights $\tilde{\lambda}_j$ might not be positive (or, more generally, could be complex numbers) and this fact might become a drawback when dealing with the stability of $\tilde{I}_n(g)$.

As for an appropriate selection of the nodes on $\mathbb{T}$, we have the following (see [4] and also [5])

Theorem 2.2. *Fix a sequence $\{\tau_n\}_{n=1}^{\infty} \subset \mathbb{T}$, let $\tilde{z}_{j,n}$ be the n-th roots of τ_n and consider*

$$\tilde{I}_n(g) = \sum_{j=0}^{n-1} \tilde{\lambda}_{j,n} g(\tilde{z}_{j,n}) = I_\omega(g) \quad \text{for all } g \in \Lambda_{-r(n),s(n)},$$

where $\{r(n)\}_{n=1}^{\infty}$ and $\{s(n)\}_{n=1}^{\infty}$ are nondecreasing sequences of nonnegative integers such that $r(n)+s(n) = n-1$ for all $n \geq 1$ and $\lim_{n\to\infty} r(n) = \lim_{n\to\infty} s(n) = \infty$. Then, $\lim_{n\to\infty} \tilde{I}_n(g) = I_\omega(g)$ for any bounded g defined on $\mathbb{T}$ such that $g(e^{i\theta})\omega(\theta)$ is integrable. Furthermore, we also have

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \left|\tilde{\lambda}_{j,n}\right| g(z_{j,n}) = \int_{-\pi}^{\pi} g(e^{i\theta})|\omega(\theta)|d\theta.$$

From Theorem 2.2 it follows that

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \left|\tilde{\lambda}_{j,n}\right| = \int_{-\pi}^{\pi} |\omega(\theta)|d\theta$$

and consequently, the stability of the sequence $\{\tilde{I}_n(g)\}_{n=1}^{\infty}$ of quadrature rules is guaranteed.

Finally, for the error estimation of the above quadrature rules, when dealing with a Szegő formula we could make use of the upper bounds given in [20]. These bounds are quite sharp, but their calculation requires a high computational effort. Instead, we can take advantage of certain bounds obtained in [14] by the second author. We get:

Theorem 2.3. *Let $I_n(g) = \sum_{j=0}^{n-1} A_j g(x_j)$ be an n-point rule with distinct nodes $\{x_j\}_{j=0}^{n-1} \subset \mathbb{T}$ such that $I_n(g) = I_\omega(g) = \int_{-\pi}^{\pi} g(e^{i\theta})\omega(\theta)d\theta$ for any $g \in \Lambda_{-r,s}$, $(r,s \geq 0)$. Assume that $g(z)$ is analytic in a certain region G containing $\mathbb{T}$ and set $R_n(g) = I_\omega(g) - I_n(g)$. Then, there exist real numbers ϱ_1 and ϱ_2 with $0 < \varrho_1 < 1 < \varrho_2$ such that*

$$|R_n(g)| \leq (\mu_0 + \|I_n\|)\, \|g\|_{\Gamma_{\varrho_1}\cup\Gamma_{\varrho_2}} \left(\frac{\varrho_1^{r+1}}{1-\varrho_1^2} + \frac{\varrho_2^{1-s}}{1-\varrho_2^{-2}} \right). \tag{9}$$

Here, $\mu_0 = \int_{-\pi}^{\pi} \omega(\theta)d\theta$, $\|I_n\| = \sum_{j=1}^{n} |A_j|$, $\Gamma_\alpha = \{z \in \mathbb{C} : |z| = \alpha\}$ with $\alpha > 0$ and $\|g\|_B = \max\{|g(z)| : z \in B\}$, for $B \subset \mathbb{C}$.

Remark 2.4. Observe that Theorem 2.3 holds for both Szegő and interpolatory-type formulas. Thus, when applied to an n-point Szegő formula for ω, then $\|I_n\| = \mu_0$, so that (9) becomes (recall $r = s = n-1$)

$$|R_n(g)| \leq 2\mu_0 \|g\|_{\Gamma_{\varrho_1} \cup \Gamma_{\varrho_2}} \left(\frac{\varrho_1^n}{1-\varrho_1^2} + \frac{\varrho_2^{-n}}{1-\varrho_2^{-2}} \right).$$

3. Computation of integrals on the whole real line with nearby singularities

The main aim of this section is the approximate calculation of integrals of the form

$$\int_{-\infty}^{\infty} \frac{f(x)}{P(x)} dx,$$

with P a polynomial with real coefficients not vanishing on the real line. Thus, we can assume $P(x) > 0$ for any $x \in \mathbb{R}$. From the simple partial fraction decomposition of $1/P(x)$ and after making an elementary change of variable we can restrict ourselves to integrals like

$$\int_{\mathbb{R}} \frac{f(x)}{(x^2+\alpha^2)^p} dx, \tag{10}$$

with p a natural number and α a nonzero real number but sufficiently close to zero. Now, we will see how quadrature formulas on the unit circle, introduced in Section 2, can be used in order to compute (10). For this purpose suppose that f is a 2π-periodic function, so that setting $\sigma(x) = \sigma_p(x) = (x^2+\alpha^2)^{-p}$, then (10) becomes

$$I_{\sigma_p}(f) = \int_{\mathbb{R}} f(x)\sigma_p(x) dx. \tag{11}$$

Hence, we are now in a position to use the results of the authors concerning the application of Szegő polynomials to the computation of certain weighted integrals on the real line. Indeed (see [9]), we have:

Theorem 3.1. *Let σ be a bounded nonnegative function on $\mathbb{R}$ and f be a 2π-periodic function such that $f\sigma$ is integrable on $\mathbb{R}$. Assume there exist real numbers a and b with $-\infty < a < b < +\infty$ such that $\sigma(x)$ is monotonically nonincreasing on (b, ∞) and monotonically nondecreasing on $(-\infty, a)$. Then, the 2π-periodic function*

$$\omega(\theta) = \sum_{j \in \mathbb{Z}} \sigma(\theta + 2\pi j)$$

is a weight function on $[-\pi, \pi]$ and

$$\int_{\mathbb{R}} f(x)\sigma(x) dx = \int_{-\pi}^{\pi} f(\theta)\omega(\theta) d\theta.$$

As a straightforward consequence, we have:

Corollary 3.2. *Set $\sigma_p(x) = (x^2+\alpha^2)^{-p}$, for $\alpha \neq 0$, $p \geq 1$, and consider f a 2π-periodic function. Then,*

$$I_{\sigma_p}(f) = \int_{\mathbb{R}} f(x)\sigma_p(x)dx = \int_{-\pi}^{\pi} f(\theta)\omega_p(\theta)d\theta = I_{\omega_p}(f),$$

where

$$\omega_p(\theta) = \sum_{j\in\mathbb{Z}} \frac{1}{((\theta+2\pi j)^2+\alpha^2)^p}. \tag{12}$$

Thus, from Corollary 3.2 it clearly follows that the computation of the integral (11) on the whole real line reduces to the computation of a weighted integral on the unit circle. As already seen in Section 2, when using quadrature formulas on the unit circle (Szegő or interpolatory-type rules) to approximate the integral $I_{\omega_p}(f)$ in (11), then the trigonometric moments of $\omega_p(\theta)$ are the basic required information, i.e., the integrals

$$\mu_k^{(p)} = \int_{-\pi}^{\pi} e^{-ik\theta}\omega_p(\theta)d\theta, \quad k = 0, \pm1, \pm2, \ldots \tag{13}$$

need to be previously computed. In this respect we have:

Proposition 3.3. *The trigonometric moments $\mu_k^{(p)}$ given by (13) satisfy*

$$\mu_k^{(p)} = \begin{cases} \frac{1}{\alpha^2}\left[\frac{2p-3}{2p-2}\mu_k^{(p-1)} + \frac{k^2}{4(p-1)(p-2)}\mu_k^{(p-2)}\right] & p \geq 3, \\ \frac{(1+\alpha|k|)e^{-\alpha|k|}\pi}{2\alpha^3} & p = 2, \\ \frac{e^{-\alpha|k|}\pi}{\alpha} & p = 1. \end{cases} \tag{14}$$

Proof. Observe that $\mu_k^{(p)} = I_{\sigma_p}(e^{-ikx})$ and the relation

$$a^2 I_{\sigma_p}(e^{-ikx}) = I_{\sigma_{p-1}}(e^{-ikx}) - I_{\sigma_p}(x^2e^{-ikx}).$$

The proof follows by applying integration by parts twice in the expression $I_{\sigma_p}(x^2e^{-ikx})$. □

Remark 3.4. The case $p = 1$ was earlier studied by the authors in [9] by giving an explicit closed representation of the function ω_1 given by (12). Indeed, there it was proved that up to a positive multiplicative factor,

$$\omega_1(\theta) = \sum_{j\in\mathbb{Z}} \frac{1}{(\theta+2\pi j)^2+\alpha^2} = \frac{1}{|z-e^{-\alpha}|^2}, \quad z = e^{i\theta}. \tag{15}$$

Thus, setting $r = e^{-\alpha}$, it follows that $\omega_1(\theta) = \frac{1}{1-2r\cos\theta+r^2}$ (Poisson kernel).

As for the case $p = 2$ we can also give an explicit representation of ω_2 as shown in the following:

Theorem 3.5. *Under the above considerations it follows that*

$$\omega_2(\theta) = \frac{1}{2\alpha^2}\omega_1(\theta) - \frac{1}{16\alpha^2}\left(\csc^2\left(\frac{\alpha i-\theta}{2}\right) + \csc^2\left(\frac{\alpha i+\theta}{2}\right)\right).$$

Proof. Let us consider the so-called polygamma function

$$\psi_n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{n+1}}, \quad n \geq 1.$$

The following reflection formulas (see [1, p. 260]) are satisfied:

$$\psi_0(1-z) - \psi_0(z) = \pi \cot(\pi z), \quad \psi_1(1-z) + \psi_1(z) = \pi^2 \csc^2(\pi z). \tag{16}$$

Proceeding as in [9] for $p = 1$ and after some elementary but tedious calculations one can obtain the following relation:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} &\left(\alpha^2 + (\theta + 2\pi j)^2\right)^{-2} \\ = \tfrac{1}{16\pi^2} &\left\{ 2\pi i \left[\psi_0\left(-\tfrac{x+\alpha i}{2\pi}\right) - \psi_0\left(-\tfrac{x-\alpha i}{2\pi}\right) + \psi_0\left(-\tfrac{x-\alpha i}{2\pi}+1\right) - \psi_0\left(-\tfrac{x-\alpha i}{2\pi}-1\right) \right] \right. \\ &\left. - \left[\psi_1\left(-\tfrac{x-\alpha i}{2\pi}\right) + \psi_1\left(-\tfrac{x+\alpha i}{2\pi}\right) + \psi_1\left(-\tfrac{x-\alpha i}{2\pi}+1\right) + \psi_1\left(-\tfrac{x-\alpha i}{2\pi}-1\right) \right] \right\}.\end{aligned}$$

Thus, the proof follows from this last relation along with (16). □

In short, the steps to be given in order to approximately calculate the integral (10) making use of an n-point Szegő formula, can be summarized as follows:

(1) For a given $p \geq 1$, the corresponding trigonometric moments $\mu_k^{(p)}$, $k = 0, \pm 1, \pm 2, \ldots$ are computed by (14).
(2) By the recurrence relations (7) (Levinson's algorithm) the required Verblunsky parameters $\delta_1, \ldots, \delta_{n-1} \in \mathbb{D}$ are computed ($\delta_0 = 1$).
(3) Fixed $\tau_n \in \mathbb{T}$ along the above parameters, an n-point Szegő quadrature formula for $I_{\omega_p}(f) = \int_{-\pi}^{\pi} f(\theta)\omega_p(\theta)d\theta$ with $\omega_p(\theta)$ given by (12) can be efficiently computed.
(4) Set $I_n(f) = \sum_{j=0}^{n-1} \lambda_j f(z_j)$ for the above quadrature, so that $z_j = e^{i\theta_j}$ for $j = 0, \ldots, n-1$. Then,

$$I_{\sigma_p}(f) = \int_{\mathbb{R}} \frac{f(x)}{(\alpha^2 + x^2)^p} dx \approx \sum_{j=0}^{n-1} \lambda_j f(\theta_j) = I_n^{\sigma_p}(f). \tag{17}$$

As an illustration, several monic Szegő polynomials for ω_2 and $\alpha = 1$ have been computed along with the corresponding n-point Szegő formulas for different values of n and the parameter $\tau_n = 1$. The computations were implemented in MAPLE®[2] 9.5 with 16 digits and the corresponding Verblunsky coefficients and the nodes and weights for $n = 5, 6$ characterizing those rules are displayed on Tables 1 and 2, respectively.

Remark 3.6. Few weight functions on the unit circle give rise to explicit expressions of the corresponding Szegő polynomials. The above procedure shows how to compute the Szegő polynomials and to construct Szegő quadrature formulas for the weight function $\omega_p(\theta)$ defined in (12) which, as far as we know, has not been studied before in the literature if $p > 1$. Since we

[2] MAPLE is a registered trademark of Waterloo Maple, Inc.

n	δ_n
0	1
1	$-.735758882342885$
2	$.295067408390062$
3	$-.070167828110242$
4	$.016768660288210$
5	$-.004008490277504$
6	$.000958231141502$

TABLE 1. The Verblunsky parameters associated with $\omega_2(\theta)$ for $\alpha = 1$.

n	$Nodes$	$Weights$
5	-1	$.032800634680708$
	$.65541206018352 \pm .997849863613590i$	$.127576179753945$
	$.913443568148223 \pm .406965413528771i$	$.641421666303148$
6	$-.758428421357609 \pm .651756342260669i$	$.033983915212768$
	$.314685214430238 \pm .949196089234989i$	$.157719992791071$
	$.933150164882 \pm .359486814472310i$	$.593694255393610$

TABLE 2. Nodes and weights of the n-point Szegő quadrature formula for $\omega_2(\theta)$ with $n = 5$, 6 and $\alpha = \tau_n = 1$.

are dealing with a symmetric weight function on $[-\pi, \pi]$, Szegő polynomials have real coeffiencients, hence the Verblunsky parameters δ_n lie on $(-1, 1)$ for all $n \geq 1$, as displayed in Table 1, and the election of the parameter $\tau_n = 1$ implies that we have considered a symmetric rule, i.e., the nodes are either real ($\{\pm 1\}$) or appear in complex conjugate pairs. Moreover, as it is known, the weights associated with two complex conjugate nodes are equal. On the other hand, the symmetric character of $\omega_p(\theta)$ assures the existence of a weight function $\mu_p(x)$ on $[-1, 1]$ for which ω_p is the Joukowski transformation of μ_p. See [11] for more details. Studying the properties of the orthogonal polynomials for the weight function μ_p on $[-1, 1]$ could be an interesting task.

Otherwise, if the integral $I_{\omega_p}(f)$ is approximated by an n-point interpolatory-type formula in $\Lambda_{-r,s}$ with $r + s + 1 = n$ and taking as nodes the n-th roots of $\tau_n = e^{i\alpha_n}$, we only need to compute the trigonometric moments $\mu_0^{(p)}, \ldots, \mu_t^{(p)}$ with $t = \max\{r, s\}$. Setting $x_j = \sqrt[n]{\tau_n} = e^{\frac{i(\alpha_n + 2jn)}{n}}$, $j = 0, \ldots, n-1$, then the n-point interpolatory-type formula in $\Lambda_{-r,s}$ is given by $I_n(f) = \sum_{j=0}^{n-1} A_j f(x_j)$, where $A_j = \int_{-\pi}^{\pi} l_j(\theta)\omega_p(\theta)d\theta$, with $l_j \in \Lambda_{-r,s}$

such that $l_j(x_k) = \delta_{j,k}$. Thus, setting $z = e^{i\theta}$ it results in ([10, Section 2])

$$
\begin{aligned}
(18)\qquad A_j &= \frac{x_j^r}{nx_j^{n-1}} \int_{-\pi}^{\pi} \frac{z^n - x_j^n}{z^r(z - x_j)} \omega_p(\theta) d\theta \\
&= \frac{1}{nx_j^s} \int_{-\pi}^{\pi} \frac{1}{z^r} \left(z^{n-1} + x_j z^{n-2} + \cdots + x_j^{n-2} z + x_j^{n-1} \right) \omega_p(\theta) d\theta \\
&= \frac{1}{nz_j^s} \sum_{j=0}^{n-1} x_j^j \mu_{r+1+j-n}.
\end{aligned}
$$

Finally, we have obtained the approximation

$$
(19)\qquad I_{\sigma_p}(f) = \int_{\mathbb{R}} \frac{f(x)}{(\alpha^2 + x^2)^p} dx \approx \sum_{j=0}^{n-1} A_j f(\theta_j) = \tilde{I}_n^{\sigma_p}(f),
$$

with A_j given by (18) and $\theta_j = \frac{\alpha_n + 2j\pi}{n}$, $j = 0, \ldots, n-1$.

Remark 3.7. The weights A_j, $j = 0, \ldots, n-1$ given by (18) can be efficiently computed by means of the fast Fourier transform, as shown by the authors in [12].

We conclude this section with a result concerning convergence. Indeed, from the positivity of the weights in any Szegő formula and Theorem 2.2 we get:

Theorem 3.8. *Let f be a bounded 2π-periodic function such that*

$$f(x)(x^2 + \alpha^2)^{-p}$$

is integrable on $\mathbb{R}$ for $p = 1, 2, \ldots$ and $\alpha \neq 0$. Then

$$\lim_{n\to\infty} I_n^{\sigma_p}(f) = \lim_{n\to\infty} \tilde{I}_n^{\sigma_p}(f) = \int_{\mathbb{R}} f(x)\sigma_p(x) dx,$$

where $\sigma_p(x) = (x^2 + \alpha^2)^{-p}$ and $I_n^{\sigma_p}(f)$ and $\tilde{I}_n^{\sigma_p}(f)$ are given by (17) and (19), respectively.

4. An application to the Fourier transform

Properties of the Fourier and Laplace transforms have been exhaustively studied during the last century because of their multiple applications to different fields in mathematics, physics and engineering. When dealing with the Laplace transform

$$F(p) = \int_0^{\infty} f(t) e^{-pt} dt$$

of a function f, a crucial problem is the recovery of $f(t)$ from $F(p)$, i.e., the calculation of the inverse transform. Now, we can convert the inversion formula

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(p) e^{pt} dt$$

to a Fourier transform as given by (2). Indeed, setting $p = c + i\sigma$ ($-\infty < \sigma < \infty$), we get

$$f(t) = \frac{e^{ct}}{2\pi} \int_{\mathbb{R}} F(c + i\sigma) e^{i\sigma t} d\sigma,$$

or equivalently

$$f(t) = e^{ct} \int_{\mathbb{R}} F(c + 2\pi i x) e^{2\pi i x t} dx,$$

yielding that the computation of the inverse Laplace reduces to the computation of the Fourier transform. On the other hand, the importance of the Fourier transform has given rise to a large number of extensions and generalizations as the so-called fractional Fourier transform that has become a powerful tool to solve problems, for instance in signal processes where the classical Fourier transform had not been able to provide a satisfactory answer. For details, see the basic text book [23] and also the survey [6] along with the references therein found. Essentially, the fractional Fourier transform is an integral transform whose kernel $\mathcal{K}_a(\xi, x)$ depends on a parameter $a \in \mathbb{R}$ so that when $a = 1$ the classical Fourier transform is recovered. Furthermore, in [23] it is shown that the fractional Fourier transform of a function f can be expressed in terms of the Fourier transform of a new function $\tilde{f}$ associated with f but much more difficult to handle. Thus, the methods designed for the computation of the Fourier transform could be used both for the fractional Fourier and the inverse Laplace transform. In this respect, the aim of this section is to apply the results of Section 3 in order to provide a computational method for the Fourier transform of a function exhibiting polar singularities near the range of integration, i.e., the real axis. More precisely, we will be concerned with the approximate calculation of the Fourier transform

$$G(w) = \int_{\mathbb{R}} g(x) e^{2\pi i x w} dx,$$

for the discrete values of $w_k = \frac{k}{2n}$, $k = 0, \pm 1, \pm 2, \ldots$ and $g(x) = \frac{f(x)}{P(x)}$ with f sufficiently smooth and P a real polynomial such that $P(x) \neq 0$ for any $x \in \mathbb{R}$.

As pointed out in Section 3, we can restrict without loss of generality to the computation of

$$G(w_k) = \int_{\mathbb{R}} \frac{f(x)}{(x^2 + \alpha^2)^p} e^{ikx} dx, \quad k = 0, \pm 1, \pm 2, \ldots$$

with p a fixed natural number. For this purpose we will first assume that f is 2π-periodic. Hence, by Corollary 3.2,

$$(20) \quad \begin{aligned} G(k) &= G(w_k) \\ &= \int_{-\pi}^{\pi} f(\theta) e^{ik\theta} \omega_p(\theta) d\theta \quad \text{where} \quad \omega_p(\theta) = \sum_{j \in \mathbb{Z}} \frac{1}{[(\theta + 2\pi j) + \alpha^2]^p}. \end{aligned}$$

Here, ω_p is a weight function whose trigonometric moments can be calculated by Theorem 3.3. Clearly, under these conditions the value of the Fourier transform $G(k)$ could be approximated by means of an N-point quadrature rule on the unit circle, either Szegő or interpolatory-type on $\Lambda_{-r,s}$ such that $r+s=N-1$. Thus, let

$$I_n(g)=\sum_{j=0}^{n-1}\lambda_j g(z_j)$$

be a Szegő rule for ω_p so that when applied to the integral (20) it results in

$$G(k)\approx\sum_{j=0}^{N-1}\lambda_j f(\theta_j)e^{ik\theta_j}=G_N(k),\quad z_j=e^{i\theta_j},\ \theta_j\in[-\pi,\pi),\ \lambda_j>0, \tag{21}$$

for all $j=0,\dots,N-1$ and $\theta_i\neq\theta_j$ if $j\neq k$.

If we assume that f is real, one has $\overline{G_N(k)}=\sum_{j=0}^{N-1}\lambda_j f(\theta_j)z_j^{-k}=G_N(-k)$. Hence, in this case we can restrict ourselves to nonnegative values of k. If we take $k=0,\dots,N-1$ then (21) transforms the sequence $f(j)=f(\theta_j)$ for $j=0,\dots,N-1$ into $G_N(0),\dots,G_N(N-1)$ (compare with the "discrete Fourier transform"). So, we see that (21) can be rewritten in a matrix notation as $G_N=D_Nf_N$, where

$$G_N=(G_N(0),\dots,G_N(N-1))^T,\quad f_N=(f(0),\dots,f(N-1))^T \text{ and}$$

$$D_N=\begin{pmatrix}\lambda_0 & \lambda_1 & \cdots & \lambda_{N-1}\\ \lambda_0 z_0 & \lambda_1 z_1 & \cdots & \lambda_{N-1}z_{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_0 z_0^{N-1} & \lambda_1 z_1^{N-1} & \cdots & \lambda_{N-1}z_{n-1}^{N-1}\end{pmatrix}.$$

Clearly, since $\lambda_j>0$ and $z_j\neq z_k$ if $j\neq k$,

$$\det(D_N)=\lambda_0\cdots\lambda_{N-1}\begin{vmatrix}1 & 1 & \cdots & 1\\ z_0 & z_1 & \cdots & z_{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ z_0^{N-1} & z_1^{N-1} & \cdots & z_{n-1}^{N-1}\end{vmatrix}\neq 0.$$

Thus we can write $f_N=D_N^{-1}G_N$, so that the inverse Fourier transform can be also recovered (although in this work we are not concerned with this problem).

Suppose now that (20) is approximated by an N-point interpolatory-type formula in $\Lambda_{r,s}$, with $r+s=N-1$ and taking as nodes the N-th roots of a complex number $\tau_N\in\mathbb{T}$, say $\tau_N=1$. For this formula, set

$$\tilde{I}_N(f)=\sum_{j=0}^{N-1}A_jf(x_j),$$

with $x_j = \sqrt[N]{1}$ for $j = 0, \ldots, N-1$ and $\{A_j\}_{j=0}^{N-1}$ given by (18). Thus, when applied to (20) one obtains

$$G(k) \approx \tilde{G}_N(k) = \sum_{j=0}^{N-1} A_j f(\theta_j) e^{ik\theta_j} \quad \text{where} \quad \theta_j = \frac{2\pi j}{N},\ j = 0, \ldots, N-1. \tag{22}$$

Here again, it should be taken into account that (22) may be considered as an entity in its own right. Indeed, by setting $w = e^{\frac{2\pi i}{N}}$, one can write

$$G(k) \approx \tilde{G}_N(k) = \sum_{j=0}^{N-1} A_j f(j) w^{jk} \quad \text{with} \quad f(j) = f\left(\frac{2\pi j}{N}\right). \tag{23}$$

Actually, (23) can be considered as a basic linear transformation mapping the sequence $f(0), \ldots, f(N-1)$ onto the sequence $\tilde{G}_N(0), \ldots, \tilde{G}_N(N-1)$. Again, (23) can be written in the matrix form $\tilde{G}_N = \tilde{D}_N f_N$, where

$$\tilde{G}_N = \left(\tilde{G}_N(0), \ldots, \tilde{G}_N(N-1)\right)^T,\ f_N = (f(0), \ldots, f(N-1))^T \text{ and}$$

$$\tilde{D}_N = \begin{pmatrix} A_0 & A_1 & \cdots & A_{N-1} \\ A_0 & A_1 w & \cdots & A_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_0 & A_1 w^{N-1} & \cdots & A_{N-1} w^{(N-1)(N-1)} \end{pmatrix}.$$

Clearly,

$$\det\left(\tilde{D}_N\right) = A_0 \cdots A_{N-1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & \cdots & w^{(N-1)(N-1)} \end{vmatrix},$$

which is nonzero if and only if $A_j \neq 0$, for all $j = 0, \ldots, N-1$. In this case, one can write $f_N = \tilde{D}_N^{-1} \tilde{G}_N$. Now, making use of the well-known identity,

$$\sum_{r=0}^{N-1} w^{r(j-k)} = \frac{1 - w^{N(j-k)}}{1 - w^{j-k}} = \begin{cases} N & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

we have for k fixed that

$$\sum_{r=0}^{N-1} \tilde{G}(r) w^{-kr} = \sum_{r=0}^{N-1} \sum_{j=0}^{N-1} A_j f(j) w^{rj} w^{-kr} = \sum_{r=0}^{N-1} \sum_{j=0}^{N-1} A_j f(j) w^{(j-k)r}$$
$$= \sum_{j=0}^{N-1} A_j f(j) \sum_{r=0}^{N-1} w^{(j-k)r} = A_k f(k) N.$$

Hence, provided that $A_k \neq 0$ for $k = 0, \ldots, N-1$,

$$f(k) = \frac{1}{A_k N} \sum_{r=0}^{N-1} \tilde{G}_N(r) w^{-kr}$$

gives an approximation to the inverse transformation.

As N tends to infinity, by Theorem 3.8, one readily has for the estimations $G_N(k)$ and $\tilde{G}_N(k)$ given by (21) and (22) the following:

Corollary 4.1. *Let f be a bounded 2π-periodic function such that*

$$f(x)(x^2+\alpha^2)^{-p}$$

is integrable on $\mathbb{R}$ for $\alpha \neq 0$ and p a fixed natural number. Then for any integer k,

$$\lim_{N\to\infty} G_n(k) = \lim_{N\to\infty} \tilde{G}_n(k) = \int_{\mathbb{R}} \frac{f(x)}{(x^2+\alpha^2)^p} e^{ikx} dx.$$

Finally, we could make use of Theorem 2.3 in order to give an error bound of the estimations $G_N(k)$ and $\tilde{G}_N(k)$ given above. Indeed, take into account that any 2π-periodic function f on $\mathbb{R}$ can be viewed as a function on $\mathbb{T}$, so that one can write $f(\theta) = \tilde{f}(e^{i\theta})$. Suppose $\tilde{f}(z)$ is an analytic function on a certain region containing $\mathbb{T}$, and let $G_N(k)$ and $\tilde{G}_N(k)$ be the estimations of

$$G\left(\frac{k}{2\pi}\right) = \int_{\mathbb{R}} \frac{f(x)e^{ikx}}{(x^2+\alpha^2)^p} dx,$$

given by (21) and (22), respectively for any integer k. Then, the following holds:

Corollary 4.2. *Under the above assumptions, there exist real numbers ϱ_1 and ϱ_2 with $0 < \varrho_1 < 1 < \varrho_2$ such that for any integer k,*

$$\left| G\left(\frac{k}{2\pi}\right) - \tilde{G}_N(k) \right| \leq \varrho^{\max} \left(\mu_0^{(p)} + \|\tilde{I}_N\| \right) \left(\frac{\varrho_1^{r+1}}{1-\varrho_1^2} + \frac{\varrho_2^{1-s}}{1-\varrho_2^{-2}} \right), \quad r+s = N-1, \tag{24}$$

$$\left| G\left(\frac{k}{2\pi}\right) - G_N(k) \right| \leq 2\mu_0^{(p)} \varrho^{\max} \left(\frac{\varrho_1^N}{1-\varrho_1^2} + \frac{\varrho_2^{-N}}{1-\varrho_2^{-2}} \right). \tag{25}$$

Here,

$$\varrho^{\max} = \max\left\{ \varrho_1^k \|\tilde{f}\|_{\Gamma_{\varrho_1}}, \varrho_2^k \|\tilde{f}\|_{\Gamma_{\varrho_2}} \right\},$$

$$\mu_0^{(p)} = \int_{-\pi}^{\pi} \omega_p(\theta) d\theta = \int_{\mathbb{R}} \frac{dx}{(x^2+\alpha^2)^p},$$

$$\|\tilde{I}_N\| = \sum_{j=0}^{N-1} |A_j|,$$

$$\Gamma_a = \{z \in \mathbb{C} : |z| = a\},$$

$$\|\tilde{f}\|_B = \max\{|\tilde{f}(z)| : z \in B\},$$

for $a > 0$ and $B \subset \mathbb{C}$.

Remark 4.3. Observe that both upper bounds (24) and (25) essentially depend on the domain of analiticy of $\tilde{f}$ and the domain of validity of the used quadrature rules.

5. Numerical examples

In this section we will numerically illustrate the effectiveness of the procedure given in Section 4 to compute the Fourier transform of a function of the form $f(x)/P(x)$, P being a polynomial with real coefficients not vanishing on the real line. For our purposes, we will take $f(x) = f_j(x)$, $j = 1, 2$ with

$$f_1(x) = \cos^7 x, \quad f_2(x) = \frac{\sin^2 x}{\cos x + 2} \quad \text{and} \quad P(x) = (x^2 + \alpha^2)^2. \tag{26}$$

As estimations we will use quadrature formulas exact in $\Lambda_{-5,5}$. More precisely, a 6-point Szegő quadrature formula with $\tau_6 = 1$ and an interpolatory-type rule with 11 nodes, which are the 11-th roots of unity will be considered. Both approximations are compared with the results given by using the standard method of the discrete Fourier transform (DFT). As explained in Section 4, the Fourier transform

$$G(w) = \int_{\mathbb{R}} \frac{f_j(x)}{(x^2 + \alpha^2)^2} e^{2\pi i w x} dx, \quad j = 1, 2$$

will be computed for the discrete values $w = w_k = \frac{k}{2\pi}$, $k = 0, \ldots, 5$. The computations were implemented in MAPLE® 9.5 with 16 digits and the relative errors are displayed in Tables 3 and 4.

From these results we observe that the relative errors obtained by using the interpolatory-type rules are very competitive in comparision with those obtained with the Szegő quadrature formula and that both rules clearly improve the results obtained by means of the DFT, especially when the poles are close to the range of integration, i.e., when α is close enough to zero. We emphasize here that despite that the interpolatory-type rule uses eleven nodes, these are the 11-th roots of unity and hence the computational effort for those rules is extremely low in comparision with the computational cost required for the construction of Szegő quadrature formulas.

Now, we will try to check the "goodness" of the upper error bounds given in Corollary 4.2. For this purpose, we will take

$$f(x) = f_3(x) = \frac{1}{(\cos x + M)}, \quad |M| > 1 \tag{27}$$

in (10) and again $p = 2$. In this case one can write

$$f(x) = \tilde{f}(e^{ix}) \quad \text{with} \quad \tilde{f}(z) = \frac{2z}{z^2 + 2Mz + 1},$$

α	k	Interpolatory-type	Szegő	DFT
1	0	.0003327230568994	.026901785277821	.501255163022280
1	3	.011034089407123	.366423427377691	.544879627889607
1	4	.04257230233186	.771276938936903	.605806335161480
.5	1	.002192800039399	.017545727235752	.571118102659482
.5	2	.009640505456589	.040265984545946	.593137505053635
.5	5	.185108978090672	.508365589525011	.881875249382593
.1	0	.000544431992941	.0001361725826865	.966599365120779
.1	3	.007729061901771	.002089448008441	.977513935435605
.1	4	.018247711351388	.004653758825189	.985752349128655

TABLE 3. A comparision of the relative errors in a 6-point Szegő quadrature formula, a 11-point interpolatory-type rule with nodes the 11-th roots of 1 (both exact in $\Lambda_{-5,5}$) and the discrete Fourier transform in the estimation of $G(w_k)$ for $f_1(x)$ given by (26).

α	k	Interpolatory-type	Szegő	DFT
1	0	.000011127049588	.002531907754003	.511355152673849
1	1	.000006217099662	.014146717614614	.487521854761410
1	2	.000258217645989	.058756209807850	.518710579190526
1	3	.001107751936501	.252063738497163	.501965828743609

TABLE 4. A comparision of the relative errors in a 6-point Szegő quadrature formula, a 11-point interpolatory-type rule with nodes the 11-th roots of unity (both exact in $\Lambda_{-5,5}$) and the discrete Fourier transform in the estimation of $G(w_k)$ for $f_2(x)$ given by (26).

(observe that $\tilde{f}(z) = \tilde{f}(1/z)$) exhibiting two real poles given by $y_1 = -M+\sqrt{M^2-1}$ and $y_2 = -M-\sqrt{M^2-1} = y_1^{-1}$. Thus, in Corollary 4.2 one can take $\varrho_1 < \min\{|y_1|, |y_1|^{-1}\}$ and $\varrho_2 = \varrho_1^{-1}$. Considering as above both Szegő and interpolatory-type rules with the same domain of validity $\Lambda_{-5,5}$, we observed numerically that the corresponding weights of the interpolatory-type quadrature formulas were positive and hence, the upper error bounds are equal for both procedures. The absolute errors of the numerical experiments implemented in MAPLE® 9.5 with 10 digits are displayed in Table 5.

We can observe the "goodness" of these upper error bounds from the results presented in Table 5. Moreover, the more the modulus of the function $\|\tilde{f}\|_{\Gamma_{\varrho_1 \cup \varrho_2}}$ in (24)–(25) increases, the greater the bounds are.

Finally, it should be recalled also that we have assumed from the beginning the hypothesis that f is a periodic function. If we drop this assumption,

M	k	error bound
5	0	$.2558252315E-02$
5	2	$.2998338219E-02$
5	4	$.4104725024E-01$
10	1	$.2876999356E-06$
10	3	$.1038596767E-03$
10	4	$.1973333858E-02$
20	0	$.3391315496E-07$
20	2	$.1076039338E-06$
20	3	$.6360698256E-06$

TABLE 5. Upper error bounds in Corollary 4.2 for both Szegő and interpolatory-type quadrature formulas with the same domain of validity $\Lambda_{-5,5}$ in the estimation of (10) with $f(x) = f_3(x)$ given by (27) and $p = 2$.

we could also pass to the unit circle by using a Cayley transformation. For further details, see [3] where a more general set of orthogonal functions than polynomials, that is orthogonal rational functions, are studied and where such transformation allows one to recover properties for orthogonal rational functions from the real line to the unit circle, and vice versa. An analog of the method presented in this paper when the function f is not necessarily periodic and by passing from the whole real line to the unit circle by a Cayley transformation will be studied in a forthcoming paper.

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