Stable lengths on the pants graph are rational

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ABSTRACT. For the pants graph, there is little known about the behaviour of geodesics, as opposed to quasi-geodesics. Brock–Masur–Minsky showed that geodesics or geodesic segments connecting endpoints satisfying a bounded combinatorics condition, such as the stable/unstable laminations of a pseudo-Anosov, all have bounded combinatorics, *outside of annuli*. In this paper it is shown that there exist geodesics that also have bounded combinatorics within annuli. These geodesics are shown to have finiteness properties analogous to those of tight geodesics in the complex of curves, from which rationality of stable lengths of pseudo-Anosovs acting on the pants graph then follows from the arguments of Bowditch for the curve complex.

CONTENTS

1. Introduction

Suppose S is a closed, orientable, connected surface with genus at least 2, and let $C_P(S)$ be the pants graph of S, first defined by Hatcher–Thurston [11]. The vertices of $C_P(S)$ represent pants decompositions of S, and vertices are connected by an edge if it is possible to get from one vertex to the other via an elementary move, defined in Section 2.

In Theorem 12.3 of [4], Behrstock–Drutu–Mosher showed that when the genus of S is at least 3, Teichmüller space with the Weil–Petersson metric is

neither hyperbolic nor relatively hyperbolic relative to any nontrivial collection of subsets. Since Brock showed in [7], Theorem 1.1, that Teichmüller space of S with the Weil–Petersson metric is quasi-isometric to $C_P(S)$, the same is true for $C_P(S)$. As a result, geodesic stability is not true for geodesics in general, but only under certain assumptions. The K-bounded combinatorics condition of Brock–Masur–Minsky, [8], will be defined in Section 2. An example of objects satisfying the K-bounded combinatorics condition are the stable and unstable limit points, e and b respectively, of a pseudo-Anosov element g of the mapping class group $\mathrm{Mod}(S)$ of S. The K-bounded combinatorics condition ensures geodesic stability for geodesics with endpoints satisfying this condition. This fact was already known, although expressed in different language, as a consequence of Theorem 6.5 of [3].

The K-bounded combinatorics condition also helps to make sense of a notion of boundary points at infinity, as explained in Theorem 4.4 of [8]. Informally this is because a geodesic connecting two vertices satisfying the bounded combinatorics condition determines a direction in which $C_P(S)$ behaves as though it were negatively curved. Alternatively, a boundary at infinity for $C_P(S)$ can also be obtained from the construction in Example 2.15 of [10].

The *stable length* of g is defined to be

$$\lim_{n \to \infty} \frac{d(v, g^n v)}{n}$$

where $d(*, *)$ is the usual combinatorial distance on $C_P(S)$ obtained by assigning each edge length one, and v is any vertex of $C_P(S)$. Since the mapping class group acts by isometry, it is not hard to see that this quantity is locally constant, and hence independent of v, due to connectedness of $C_P(S)$. Connectedness of $C_P(S)$ is shown in the appendix of Hatcher–Thurston, [9].

A definition of Harvey's complex of curves $C(S)$ is given in Section 2. It was shown in [13], Proposition 7.6, that the stable length of a pseudo-Anosov acting on Harvey's complex of curves is nonzero. In Section 5 other results of Masur–Minsky will be summarised, that relate distances in $C(S)$ to distances in $C_P(S)$, from which it follows that the stable length of g acting on $C_P(S)$ must also be nonzero.

Benson Farb, [12], asked whether pseudo-Anosovs also have rational stable lengths on $C_P(S)$, and the main theorem of this paper answers that in the affirmative.

Theorem 1. *Let g be a pseudo-Anosov element of the mapping class group of S. Then the stable length of the action of g on $C_P(S)$ is rational.*

The Nielsen–Thurston classification of mapping classes states that every mapping class is either pseudo-Anosov, periodic or reducible. These categories are known to have many properties in common with hyperbolic, elliptic and parabolic isometries, respectively, of the hyperbolic plane. One

property of a hyperbolic isometry h acting on the complex plane $\mathbb{H}^2$ is that the hyperbolic isometry leaves invariant a geodesic connecting its limit points at infinity; the *axis* of h. That the stable length of a pseudo-Anosov acting on $C(S)$ is rational is a corollary of Proposition 7.6 of [14]. Rationality of the stable length of a pseudo-Anosov acting on $C(S)$ was then reproven in [6] by showing the existence of an axis for some finite power of the pseudo-Anosov. The stable length is then the translation length along the invariant geodesic divided by the power of the pseudo-Anosov that leaves the geodesic invariant. This is the approach taken in this paper for $C_P(S)$.

Let M be the mapping torus of the pseudo-Anosov g. In [17], Theorem 0.1, Thurston proved that the mapping torus of a pseudo-Anosov mapping class is hyperbolic, which is unique by the Mostov rigidity theorem. Let $\tilde{M} \equiv S \times \mathbb{R}$ be the infinite cyclic covering space corresponding to the fiber of the mapping torus. Fix an inclusion of S into $\tilde{M}$, and identify curves on S with curves in $\tilde{M}$. The *length* of a curve is then the length of its geodesic representative in $\tilde{M}$. The finiteness properties of tight geodesics used by Bowditch in [6] to prove rationality of stable length of a pseudo-Anosov acting on the complex of curves came from the fact that there are only finitely many orbits of short curves in $\tilde{M}$ under the action of $\langle g \rangle$.

The major difficulty in working with curve complexes is that, apart from exceptional cases, they are not locally compact. In [14], Section 4, the notion of a tight geodesic was defined, in order to circumvent this obstacle. In [5], Section 1, a slightly modified definition of tightness is given, which is the one found in Section 2.

To prove Theorem 1, we need an analogue of the notion of tight geodesics for the pants graph in order to obtain finiteness properties similar to those used by Bowditch in [6]. Essentially, these are the geodesics that pass through subsurfaces in the most convex way possible, as explained in Subsection 3. The most obvious choice are geodesics with bounded combinatorics. A further consequence of Theorem 4.4 of [8] is that geodesics connecting b and e all satisfy the K-bounded combinatorics condition outside of annuli. All that remains to show is that there exist geodesics with bounded combinatorics also inside of annuli.

Outline of paper. Section 2 sets out the basic notation and definitions relating to curve complexes and bounded combinatorics. Section 3 is intended to reconcile the "bounded curve length" arguments of Bowditch with equivalent formulations using bounded combinatorics, in line with [8], and is not strictly necessary for the proof of Theorem 1. Bowditch's argument is outlined in Section 4. It will be convenient to make use of some well known results of Masur–Minsky, [14], which are summarised briefly in Section 5, and used to make rigorous the connections between combinatorics, distances and bounded curve lengths. Finally, the existence of K-bounded geodesics is proven in Section 7 using the short curve arguments from Section 6.

2. Standard definitions relating to curve complexes

A *curve* is an isotopy class of maps of S^1 into the manifold in question; here either the surface S or the 3-manifold $\tilde{M}$. The *intersection number* of two curves c_1 and c_2 on a surface is equal to the smallest possible number of crossings of two representatives of the isotopy classes. A curve will often be confused with the image of a particular representative of the isotopy class. When the (sub)surface has punctures or nonempty boundary, it will also be assumed that the curve is not homotopic onto the boundary or into a puncture. All curves are assumed to be simple, where intersection numbers of curves in $\tilde{M}$ are defined by projecting onto the image of a π_1 injective embedding of the surface S into $\tilde{M}$.

Pants graph. The *pants graph* $C_P(S)$ is the graph defined by Hatcher–Thurston in [11] with vertex set consisting of isotopy classes of pants decompositions of the surfaces S. Two vertices are connected by an edge if they represent pants decompositions that can be connected by a so-called *elementary move*. An elementary move takes a curve c_1 in a pants decomposition p_1 and replaces it with another curve c_2, such that $(p_1 \setminus c_1) \cup c_2$ is a new pants decomposition, and c_1 and c_2 intersect minimally in the component of $S \setminus (p_1 \setminus c_1)$ into which they can both be isotoped. To intersect minimally means that c_1 and c_2 have intersection number one inside a one holed torus or two inside a four-punctured sphere.

A geodesic in the pants graph will be said to *pass through a curve c*, or alternatively, the curve c will be said to be *on the geodesic* if there is a vertex of the geodesic representing a pants decomposition containing the curve c.

Curve complexes. A complex of curves will be defined for the surface S, and also, in order to define subsurface projections following Section 2 of Masur–Minsky,[14], curve complexes for subsurfaces of S. These complexes are all assumed to have the usual combinatorial distance, that assigns two vertices distance one iff they are connected by an edge. Suppose $S_{g,p}$ is an orientable surface with genus g and p boundary curves. Except for the annulus, the set of curves on the subsurface $S_{g,p}$ define the vertices of Harvey's complex of curves, $C(S_{g,p})$. Whenever $3g + p > 4$, two or more vertices of the curve complex $C(S_{g,p})$ span a simplex if the curves they represent can be realised disjointly. For all other subsurfaces of interest except the annulus, namely the four punctured sphere and the once punctured torus, $C(S_{g,p})$ has an edge connecting any pair of vertices representing curves that intersect minimally.

For an annulus $A \subset S$, the definition of $C(A)$ needs to be approached differently, and captures the concept of the amount of twisting around the core curve of the annulus. Let $\tilde{A}$ be the covering space of S to which A lifts homeomorphically. Since this annular covering is hyperbolic, there is a compactification $\hat{A}$ of $\tilde{A}$. A vertex in $C(A)$ is given by a lift of a curve to $\hat{A}$, modulo homotopies that fix the endpoints. Two vertices are connected by an edge if they represent lifts with disjoint interiors. It is not hard to see that this complex is quasi-isometric to $\mathbb{Z}$.

The notion of *subsurface projection* was defined by Masur–Minsky, Sections 2.3 and 2.4 of [14], as a means of breaking down curve complex problems into simpler pieces. Let Y be an incompressible, nonperipheral connected open subsurface of S. If c is a curve in S that intersects Y, the *subsurface projection of c to Y* is defined to be a union of curves in Y obtained by surgering each arc of $c \cap Y$ along the boundary of Y. Since there are some choices involved, in [14] it is shown that subsurface projections are coarsely well defined.

The *distance in the subsurface projection to Y* of the vertices v_1 and v_2 of $C(S)$, $d_Y(v_1, v_2)$, is zero when one or both of the vertices represent curves that can be isotoped out of Y. Otherwise, it is the distance in $C(Y)$ of the subsurface projections to Y of the vertices v_1 and v_2, which is shown to be coarsely well defined.

Bounded Combinatorics. Suppose v_1 and v_2 are vertices of $C(S)$. For any incompressible, connected open subsurface Y of S, $d_Y(v_1, v_2)$ is defined as above. The pair (v_1, v_2) is said to have *K-bounded combinatorics*, Section 1 of [8], if there is an upper bound K on the distance in the subsurface projection between v_1 and v_2 to all Y. If v_1 and v_2 are vertices of $C_P(S)$, the subsurface projections of the multicurves represented by the two vertices also have coarsely well defined subsurface projections to Y, consisting of a union of curves in Y. When the union of curves is not a multicurve, distance in the subsurface projection can be seen to be coarsely well defined by choosing some nontrivial multicurve contained in the union of curves. As for v_1 and v_2 in $C(S)$, the pair (v_1, v_2) is said to have *K-bounded combinatorics*, if there is an upper bound K on the distance in the subsurface projection between v_1 and v_2 to all Y.

A path in $C(S)$ or $C_P(S)$ will be said to be *K-bounded* if this is true for all pairs of vertices through which it passes.

Let $\mathcal{F}(b, e)$ be the graph consisting of the union of geodesics in $C_P(S)$ connecting b to e, and $\mathcal{F}_K(b, e)$ be the (possibly empty) subgraph consisting of the union of K-bounded geodesics. It follows from Theorem 4.4 of [8] that the geodesics in $\mathcal{F}_K(b, e)$ fellow travel.

3. A locally finite subgraph of $\mathcal{F}(b, e)$

The aim is to find a locally finite subgraph of $\mathcal{F}(b, e)$ closed under the action of g. By *locally finite* is meant that there are only finitely many geodesics in the subgraph connecting any two vertices.

Conjecture 5 of [2] states that all strata of $C_P(S)$, i.e., subcomplexes of $C_P(S)$ whose vertices correspond to pants decompositions containing a fixed multicurve, are convex. It was shown in the Appendix of [1] that this conjecture would imply the local finiteness we are looking for.

Apart from proving Conjecture 5 of [2], there are two strategies that might be employed to obtain a locally finite subgraph of $\mathcal{F}(b, e)$; namely using bounded combinatorics analogous to the approach taken in [8] or by using lengths of curves in mapping tori, following Bowditch, as explained in Section 4. The two approaches can be shown to give the same results; one direction of this will be proven in Lemma 4. Informally, the more unbounded the combinatorics become, the longer the associated curves in the mapping tori. In Section 6, bounded combinatorics also inside of annuli will be used to show the existence of geodesics in $\mathcal{F}(b, e)$ that only pass through curves that are not "too long". This subsection explains how to obtain a locally finite graph by bounding combinatorics. Proposition 2 is not essential for the proof of the theorem.

Let p_1 and p_2 be two pants decompositions. Since any two pants decompositions represent vertices in $C_P(S)$ at finite distance, p_1 and p_2 have K_1 bounded combinatorics, for some $K_1 > 0$. Any two vertices along any pant geodesics connecting p_1 to p_2 necessarily have bounded combinatorics, since the geodesics have finite length. A geodesic for which the supremum of K, taken over all pairs of vertices, is minimised, will be called a K-*minimising geodesic*.

The notion of K-minimising geodesics are intended to be a generalisation of Masur–Minsky's tight geodesics in $C(S)$. Informally speaking, a curve representing a vertex of a tight geodesic in $C(S)$ enters a subsurface of S only if it is forced to by a vertex on one side of it along the geodesic, and only as deeply as this vertex to one side of it. Distances in subsurface projections of the endpoints of a tight geodesic bound distances in subsurface projections of pairs of vertices along the geodesic. A key property of tight geodesics is that there are only finitely many tight geodesics between any two vertices of $C(S)$, [14], Theorem 6.14. The closest analogy of a tight geodesic in $C_P(S)$ is a hierarchy path, defined in Section 4 of [14]. However, it is not known whether or not hierarchy paths are geodesics. The condition that a geodesic in $C_P(S)$ is K-minimising is strong enough to obtain the local finiteness properties needed in this paper.

Proposition 2. *Let $d(v_1, v_2)$ denote the distance in $C_P(S)$ between two vertices v_1 and v_2. Suppose also that v_1 and v_2 do not represent pants decompositions with curves in common. The number of K-minimising geodesics*

in $C_P(S)$ connecting v_1 and v_2 is bounded from above by

$$\left(-\frac{3K}{2}\chi(S)\right)^{d(v_1,v_2)}.$$

Proof. Starting at the vertex v_1, an elementary move might possibly be performed on any one of $-\frac{3}{2}\chi(S)$ of the curves in the pants decomposition of S. Suppose the elementary move is performed on the curve c. Since v_2 represents a pants decomposition containing a curve that intersects c, applying the K-bounded combinatorics condition to the annular subsurface with core curve c restrains the number of twists the elementary move can perform within the one holed torus or four punctured sphere containing c. This gives at most $-\frac{3K}{2}\chi(S)$ edges emerging from v_1 that a K-bounded geodesic connecting v_1 to v_2 might take. From each of the endpoints of these finite number of edges, again there are only finitely many edges that a K-bounded geodesic might take, etc. The bound follows.   □

Remark. Since the action of the mapping class group on $C_P(S)$ preserves distance in subsurface projections, any mapping class maps K-minimising geodesic segments to K-minimising geodesic segments. If two pants decompositions have a curve c in common, it is not clear whether K-minimising geodesics, (or geodesics in general) connecting the corresponding vertices in $C_P(S)$ only pass through vertices representing pants decompositions containing c; this amounts to proving Conjecture 5 of [2]. If not, the assumption in Proposition 2 is necessary, because there is a mapping class consisting of a Dehn-twist around c that fixes v_1 and v_2, but has an infinite orbit of K-minimising geodesics.

Suppose now that p_1 and p_2 are no longer pants decompositions, but a pair of laminations satisfying the K-bounded combinatorics condition, such as the limit points of a pseudo-Anosov. It is not yet clear that K-minimising geodesics connecting these two boundary points exist.

4. Short curves and invariant quasi-geodesics

Recall that Brock showed that the pants graph is quasi-isometric to Teichmüller space of S, $\mathcal{T}(S)$, with the Weil–Petersson metric, [7], Theorem 1.1. The intuition behind this is that a vertex v in $C_P(S)$ determines a neighbourhood $\mathcal{N}(v)$ in $\mathcal{T}(S)$, consisting of the set of points in $\mathcal{T}(S)$ corresponding to metrics that make the curves in v "short". There is some freedom in choosing what is meant by "short"; from now on it will be fixed to mean shorter than twice Bers' constant L. By the *length* of a curve is meant here, the length of the unique geodesic in the free homotopy class in a given metric. That sufficiently short curves determine a pants decomposition is a consequence of Margulis' lemma. The proof in [7] makes use of the fact that the neighbourhood $\mathcal{N}(v)$ has bounded diameter in the Weil–Petersson metric. The way the position of the short curves change when traversing a path in $\mathcal{T}(S)$ is modelled by a path in $C_P(S)$.

Recall that $\tilde{M}$ is the infinite cyclic covering space of the mapping torus with monodromy g. As the covering space of a compact manifold without boundary, $\tilde{M}$ has injectivity radius bounded from below, so it follows from Theorem A of [15] that $\tilde{M}$ determines a quasi-geodesic (with respect to the Teichmüller metric) in Teichmüller space. By construction, this quasi-geodesic is in the ϵ-thick part of Teichmüller space. In addition, since g is pseudo-Anosov, the image in $C_P(S)$ of this quasi-geodesic under the quasi-isometry from Theorem 1.1 of [7] satisfies K-bounded combinatorics, also in annuli. It follows from Theorem 1.1 of [16] that a quasi-geodesic with respect to the Weil–Petersson metric is obtained. Composing the quasi-geodesic in Teichmüller space with the quasi-isometry from Teichmüller space to $C_P(S)$, a family, $\mathcal{Q}_{2L}(b,e)$, of quasi-geodesics in $C_P(S)$ connecting b to e, is obtained.

By construction, the elements of $\mathcal{Q}_{2L}(b,e)$ only pass through short curves in $\tilde{M}$. The property of short curves in $\tilde{M}$ of interest for this proof is that, by Margulis' lemma, there are only finitely many of them in M, so the vertices of quasi-geodesics in $\mathcal{Q}_{2L}(b,e)$ are contained in finitely many orbits of g. This will be shown to be the source of the periodicity used to prove the theorem.

Since g acts by isometry, both on $C_P(S)$ and on $\tilde{M}$, $\mathcal{Q}_{2L}(b,e)$ is closed under the action of g. It then follows from exactly the same argument given in [6], that there is a quasi-geodesic Q in $\mathcal{Q}_{2L}(b,e)$ invariant under g^m for some m.

Clearly, the existence of Q does not prove the theorem, since Q is only a quasi-geodesic. In Section 6, the proximity of geodesics to Q will be used to show the existence of geodesics in $\mathcal{F}_K$ for large enough K.

For the sake of completeness, Bowditch's argument is sketched below.

Bowditch argument. Let $\mathcal{G}(b,e)$ be the graph consisting of the union of tight, directed geodesics in $C(S)$ connecting b to e, and let E be the set of directed edges of $\mathcal{G}(b,e)$. The mapping class group maps tight geodesics to tight geodesics, so $\mathcal{G}(b,e)$ is closed under the action of g. The set $E/\langle g \rangle$ is shown by Bowditch in Section 7 of [6] to be finite using an argument of Minsky's that puts vertices on a geodesic in $\mathcal{G}(b,e)$ in correspondence with short curves in $\tilde{M}$. The finite number of g orbits of short curves in $\tilde{M}$ gives a bound on the number of elements of the set $E/\langle g \rangle$.

It may not be the case that all geodesics contained in $\mathcal{G}(b,e)$ are tight, so let $\mathcal{L}(b,e)$ be the set of all geodesics contained in $\mathcal{G}(b,e)$.

Remark. The geodesics in $\mathcal{L}(b,e)$ are all constructed by connecting edges of $E/\langle g \rangle$. There is no assumption being made that all paths constructed in this way are geodesics.

Short curve arguments gave us finiteness of the set of edges of $E/\langle g \rangle$. It is not yet clear that a geodesic invariant under some power of g exists. The following argument of Delzant is used to show that there exist geodesics passing through the finite set of edges with cyclic repetitions. Assign each

of the finite elements of $E/\langle g \rangle$ a number. A geodesic γ in $\mathcal{L}(b,e)$ will be said to be *lexicographically least* for all vertices v, w of γ if the sequence of labels of directed edges in the segment of γ connecting v to w is lexicographically least amongst all geodesic segments in $\mathcal{G}(b,e)$ connecting v to w. Let $\mathcal{L}_L(b,e)$ be the subgraph of lexicographically least geodesics in $\mathcal{G}(b,e)$. It is shown that:

- $\mathcal{L}_L(b,e)$ is nonempty.
- $\mathcal{L}_L(b,e)$ is closed under the action of g.
- $\mathcal{L}_L(b,e)$ contains finitely many elements.

Since there is a finite, nonempty, set of geodesics connecting b to e, closed under the action of g, it follows that some finite power m of g has an axis.

5. Masur–Minsky background and K-bounded combinatorics

Hierarchy paths are defined in Section 4 of [14]. A hierarchy is a combinatorial object, informally described in [14] as a thickening of a geodesic in $C(S)$. The construction in Section 4 of [14] results in a path in the marking graph. The *marking graph* is a graph defined similarly to the pants graph, but with the curves in the pants decompositions provided with transversals. There is an action of the mapping class group on the marking graph, and the transversals ensure that vertex stabiliser subgroups of the mapping class group are trivial. A simpler construction of hierarchy paths in the pants graph is all that will be needed here. A discussion of how the results in [14] for the marking graph can be applied to the pants graph is given in Section 8 of [14].

Let ϕ be an element of the mapping class group, and v be a vertex of $C_P(S)$ representing the pants decomposition by the curves

$$(p_1, p_2, \ldots, p_{-\frac{3}{2}\chi(S)}).$$

The hierarchy path is constructed around a tight geodesic α in $C(S)$ passing from p_1 to $\phi(p_1)$. Suppose α passes through the vertices

$$p_1, \alpha_1, \alpha_2, \ldots, \alpha_m, \phi(p_1).$$

In the subsurface projection to $S \setminus \alpha_i$, the curves α_{i-1} and α_{i+1} are a certain distance apart. For every i, construct a tight geodesic β^i in the subsurface projection to $S \setminus \alpha_i$. Suppose the geodesic β^i passes through the vertices $\alpha_{i-1}, \beta^i_1, \beta^i_2, \ldots, \alpha_{i+1}$. For the vertex p_1, construct a tight geodesic β^0 in the subsurface projection to $S \setminus \alpha_1$ between p_2 and α_1, and for the vertex $\phi(p_1)$ construct a tight geodesic in the subsurface projection to $S \setminus \phi(p_1)$ between α_m and $\phi(p_2)$. These tight geodesics are then used to obtain an ordered set of pairs

$$P := \{(p_1, p_2), (p_1, \beta^0_1), (p_1, \beta^0_2), \ldots, (p_1, \alpha_1), (\beta^1_1, \alpha_1), (\beta^1_2, \alpha_1),$$
$$\ldots (\alpha_3, \alpha_2), \ldots (\phi(p_1), \phi(p_2))\}.$$

The ordered set P is the basis around which a curve in the pants graph is built up, by iterating the construction in the previous paragraph. More precisely, we next construct a tight geodesic γ^{i-1} in the subsurface projection to $S \setminus P_i$, where P_i is the ith element of P. The geodesic γ^i connects the subsurface projection of P_{i-1} to the subsurface $S \setminus P_i$ to the subsurface projection of P_{i+1} to $S \setminus P_i$. Recall the convention that the subsurface projection of a curve homotopic to a boundary curve of the subsurface is empty. So the subsurface projection of P_{i-1} and P_{i+1} to $S \setminus P_i$ each consist of a single curve, and it is possible to construct a geodesic in the subsurface projection connecting these two curves. Then construct an ordered set of triples as before. If S has genus 2, this ordered set determines a path in $C_P(S)$, if not, keep iterating until a path in $C_P(S)$ between p_1 and $\phi(p_1)$ is obtained. This path is called a hierarchy path, and it follows from Theorem 6.12 of [14], together with the arguments in Section 8 of [14], that it is a quasi-geodesic with uniform bounds on the constants.

Lemma 3. *Suppose the endpoints v and $\phi(v)$ of a hierarchy path in $C_P(S)$ have K-bounded combinatorics. Let $l(\alpha)$ be the length of the geodesic α in $C(S)$ around which the hierarchy path is constructed. Then*

$$d(v, \phi(v)) \leq l(\alpha)K^{-\frac{3}{2}\chi(S)-1}$$

where $d(*, *)$ denotes distance in $C_P(S)$.

Proof. When the endpoints of a hierarchy path have K-bounded combinatorics, each iteration of the construction outlined above increases the number of elements in the ordered sets by at most a factor of K. $\qquad\square$

Recall that Q is a quasi-geodesic connecting b to e that only passes through short curves, and whose existence was proven in Section 4. We would like to define a quantity $D(\gamma)$, similar to the Hausdorff distance in $C_P(S)$ between Q and the geodesic γ, with the only difference being that the usual distance between two points in $C_P(S)$ is replaced by the length of the shortest hierarchy path between them. Since hierarchy paths are uniform quasi-geodesics, and it follows from Theorem 4.4 of [8] that γ fellow travels Q, $D(\gamma)$ must be finite.

Defining twists. Using the quasi-isometry between the pants graph and Teichmüller space with the Weil–Petersson metric, it is possible to define an approximate notion of the number of Dehn twists performed by an elementary move corresponding to an edge. Alternatively, Theorem 4.4 of [8] states that a geodesic γ in $\mathcal{F}(b, e)$ is within a bounded distance of Q. Since the short curves in $\tilde{M}$ have bounded combinatorics, lengths of curves in $\tilde{M}$ could be used. This is the approach taken here. Let $s_{L,c}$ be the set of all curves in $\tilde{M}$ of length less than twice Bers' constant that intersect c. A curve will be said to be *twisted around c at least n times* if it has distance

at least n from any curve in $s_{L,c}$ in the subsurface projection to the annulus with core curve c.

Suppose κ is a geodesic in $\mathcal{F}_{K^*}(b, e)$; the existence of which will be proven in Section 6, with $2\mathcal{T}$ being the bound on the number of times a curve through which κ passes can be twisted around a simple curve. The next lemma shows that a bound on $\mathcal{T}$ gives rise to the length bounds required for a proof of Theorem 1 analogous to Bowditch's argument from [6].

Lemma 4 (Bounded combinatorics implies bounded length). *There is a bound on the length of curves through which κ passes, depending on Bers' constant L, K^*, $D(\kappa)$ and the upper bound l_c on the width of the collar of a curve on the Teichmüller quasi-geodesic corresponding to $\tilde{M}$.*

Proof. Recall that Q passes through curves whose length are all bounded from above by $2L$. Since any vertex of κ can be connected to a vertex of Q by a hierarchy path δ of length at most $D(\kappa)$, an upper bound on the length of curves will be found by iterating: for the first vertex on δ, a bound of $b_1 := 4L + 4\mathcal{T}L + 2l_c = 4L(\mathcal{T} + 1) + 2l_c$ is obtained. This is found by assuming a worst case scenario, as illustrated in Figure 1.
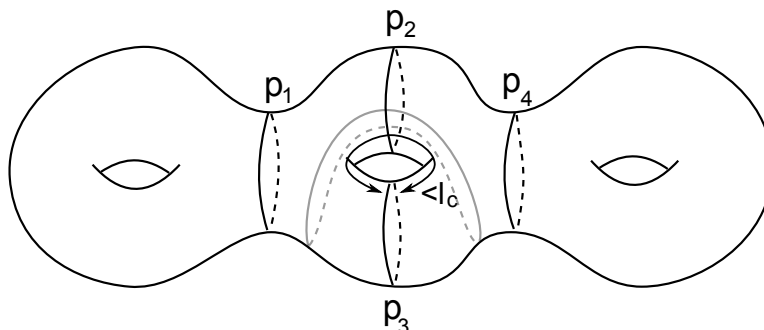

FIGURE 1. Illustration of the worst case scenario, from which the expression for b_1 is derived. The curve drawn in grey has length less than $2l_c$ plus twice the length of the curve p_3. Since p_3 has length less than $2L$, the grey curve has length less than $4L + 2l_c$. The grey curve could also be twisted at most $2\mathcal{T}$ times around the curve p_2.

The same argument with b_{n-1} in place of $2L$ gives

$$b_n = 2b_{n-1} + 4b_{n-1}\mathcal{T} + 2l_c. \qquad \square$$

Basic problem. Although the elements of $\mathcal{F}(b, e)$ fellow travel Q, since this is not the marking graph, what might happen is that every geodesic in $\mathcal{F}(b, e)$ passes through curves whose length approaches infinity. In this case, the hierarchy paths connecting points of Q to points of $\mathcal{F}(b, e)$ necessarily perform arbitrarily large numbers of twists, precluding the possibility of bounding either the combinatorics or the lengths of curves through which

the geodesic passes. Since g acts by isometry both on $\tilde{M}$ and $C_P(S)$, if g is to have an axis, it is necessary to rule out the possibility that all geodesics in $\mathcal{F}(b, e)$ pass through arbitrarily long curves or have unbounded combinatorics.

6. Short curves in the pants complex

We finally have all the ingredients to start the proof of Theorem 1.

Proof. Suppose γ is a geodesic in $\mathcal{F}(b, e)$ passing through the curve c_n, where c_n is long because it has been twisted around a curve c at least n times. It follows from Lemma 4 that if c_n does not have a large subsurface projection to an annulus, its length is bounded.

Let v be a vertex of γ that represents a pants decomposition containing the curve c_n. Cut the geodesic γ at the vertex v to obtain two rays; one connecting v to e, call it r_1, and the other connecting b to v, r_2.

Lemma 5. *If n is larger than some constant $\mathcal{T}(K, D(\gamma))$, both r_1 and r_2 have to pass through c.*

Proof. By assumption, every hierarchy path connecting v to the nearest point(s) on Q performs a large number of twists around the curve c. Clearly, this can not be the case for every vertex of γ, because the limits of Q and γ are the same, and therefore can not differ when projected to a subsurface Y consisting of an annulus with core curve c. It follows that γ necessarily passes through curves that are not twisted a large number of times around c and have arbitrarily large intersection number with c.

Fact: any curve that intersects c_n minimally within a four punctured sphere or a once punctured torus either does not pass through the annulus with core curve c, or is twisted almost as many times as c_n; at most two times more or fewer. Similarly, if two curves are disjoint and both pass through Y, the number of twists around c can only differ by at most one.

If r_1 does not pass through c, how does r_1 get from c_n to any curve on r_1 that is not twisted a large number of times around c and intersects c arbitrarily often?

Suppose c intersects more than one curve in the pants decomposition corresponding to v. By the previous fact, in order to reach a pants decomposition that does not have a large subsurface projection to the annulus with core curve c, an elementary move at the vertex v can not decrease the number of twists by more than one. An elementary move might increase/decrease by one the number of curves in the pants decomposition passing through the annulus with core curve c.

If c only intersects one curve, i.e., c_n, in the pants decomposition corresponding to v, then c and c_n fill the subsurface in which elementary moves involving c_n can occur. So an elementary move can undo the twists of each arc of a curve passing through Y at most two at a time, or increase/decrease the intersection number of the pants decomposition with c, for example, via

an elementary move that twists c around c_n. However, it can not decrease the intersection number of the pants decomposition with c to 0 without passing through c, because c is the only nontrivial, nonperipheral curve in the 1-holed torus or 4-holed sphere in question disjoint from c.

It will now be argued that, for n sufficiently large, an element of $\mathcal{F}(b,e)$ can not afford to undo the twists one or two at a time, because then it would be possible to find a quasi-geodesic segment connecting two points on the geodesic that is shorter than the geodesic segment with the same endpoints.

Let r_n^c be a connected subsegment of r_1 containing vertices that are all twisted around c more than n times. By assumption, there is a vertex v_δ on a hierarchy path δ connecting Q to γ, where v_δ represents a pants decomposition containing the curve c and is one endpoint of an edge representing the elementary move that introduces the large number of twists around c. This is shown in part (a) of Figure 2. Let Δ be the set of all such v_δs. A vertex v_γ on r_n^c is a distance less than $D(\gamma)$ from Δ.
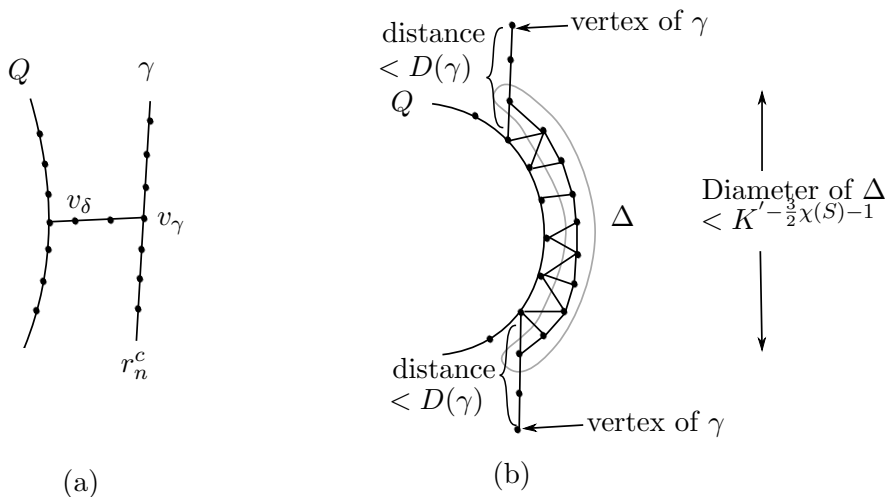
FIGURE 2. How to bound the length of r_n^c.

Since any vertex in Δ is within $D(\gamma)$ of γ, and γ has K-bounded combinatorics, it follows that there exists a K' depending on K and $D(\gamma)$ such that any two vertices in Δ have K'-bounded combinatorics. By Lemma 3, Δ therefore has diameter bounded from above by

$$K'^{-\frac{3}{2}\chi(S)-1}.$$

Since r_n^c also stays within distance $D(\gamma)$ of Δ, the length of r_n^c is bounded by

$$K'^{-\frac{3}{2}\chi(S)-1} + 2D(\gamma).$$

This is illustrated in part (b) of Figure 2.

The bound on the length of r_n^c gives a bound of $2K^{-\frac{3}{2}\chi(S)-1} + 4D(\gamma)$ on the number $\mathcal{T}$ of twists around c that can be untwisted one or two at a time by elementary moves along γ. It follows that if n is larger than this bound, r_1 must pass through c in order to be able to undo the large number of twists around c. A symmetric argument shows that r_2 also passes through the curve c.   □

7. Existence of K-minimising geodesics.

The Lemma 5 will now be used to show the existence of geodesics in $\mathcal{F}(b,e)$ with bounded combinatorics, also in annuli, from which Theorem 1 then follows.

Lemma 6. *Starting with a geodesic in $\mathcal{F}(b,e)$, it is possible to untwist all the large subsurface projections to annuli to obtain a geodesic in $\mathcal{F}_{K^*}(b,e)$, for K^* less than the maximum of $2\mathcal{T}$, K.*

Proof. Suppose $2\mathcal{T} < n$, and let v_1 and v_2 be two vertices containing c on the boundary of a geodesic subsegment I_c of γ containing v. Suppose also that v_1 and v_2 are as close as possible to v, so that all vertices in the interior of I_c arel twisted many times around c. The curves in the pants decompositions represented by v_1 and v_2 are all either disjoint from c or c itself, so both v_1 and v_2 are fixed by the mapping class T_c^{-n} that performs n twists backwards along c. Since the mapping class group acts on $C_P(S)$ by isometry, T_c^{-n} takes I_c to a geodesic segment with the same endpoints. Let γ' be the geodesic constructed from γ by replacing I_c with its image under T_c^{-n}.

The proof of Lemma 5 shows that, in the interior of the interval I_c, the variation in the number of times any vertex can be twisted around c is bounded from above by $\mathcal{T}$. Therefore, T_c^{-n} takes I_c to a geodesic segment on which no vertex is twisted around c more than $\mathcal{T}$ times.

Either the image v' of v under T_c^{-n} is not twisted many times around any curve, in which case v' consists only of curves with length less than the bound derived in Lemma 4, or v' is twisted more than $2\mathcal{T}$ times around some other curve, call it d. In the second case, we have seen how to modify γ' in such a way as to undo the large number of twists around d. If we keep untwisting like this, after finitely many iterations, we obtain a geodesic γ_v, for which the image of the vertex v can not be untwisted any further. This is because at each step, at least one curve in the image of v is shortened by an amount that is uniformly bounded from below, as will now be shown.

Length reduction from untwisting. The infinite cyclic covering $\tilde{M}$ represents a path through Teichmüller space; choose the point t at which the length of the curve c_n in the pants decomposition representing v (notation taken from Lemma 5) is as short as possible. The point t in Teichmüller

space defines a pants decomposition of short curves $(p_1, p_2, \ldots, p_{-\frac{3}{2}\chi(S)})$. If none of the p_i happen to be the curve c, then c_n has distance at least n in the subsurface projection to the annulus with core curve c from at least one of the p_i, so undoing $n > 2\mathcal{T}$ twists around c will decrease the length of c_n by an amount bounded from below by approximately $2\mathcal{T}s$, where $s > 0$ is the length of the shortest curve in the mapping torus M. If one of the p_i is c, then take $t^{'}$ to be the closest point from t along the path in Teichmüller space represented by $\tilde{M}$ for which the corresponding pants decomposition does not contain c. Since Teichmüller space is quasi-isometric to the pants graph, the distance between t and $t^{'}$ is uniformly bounded from above by a constant depending on K. So an estimation of the length reduction in the metric determined by $t^{'}$ gives an estimation for the length reduction in the metric given by t, which gives a lower bound for the length reduction in $\tilde{M}$, because the length of c_n in $\tilde{M}$ is realised in the metric on S coming from the point t in Teichmüller space.

Now that we know how to modify γ to obtain at least one vertex with bounded combinatorics inside of annuli, we need to check that it is possible to do this consistently for all vertices. Choose a vertex w on γ_v that is twisted many times around some curve f. It can be assumed without loss of generality that w exists, because otherwise we have a geodesic with bounded combinatorics also inside of annuli, as desired. It is not possible that the interior of such an interval contains the image of v, because the image of v could not be twisted around f more than $\mathcal{T} - 1$ times. $\square$

It follows from Lemmas 4 and 6 that the family of geodesics $\mathcal{F}_{K^*}(b, e)$ only passes through curves of bounded length. Therefore, exactly the same arguments as in Section 4 show the existence of an axis γ_a invariant under some power m of g.

The stable length of g is then equal to the rational number

$$\frac{d(p, g^m p)}{m}$$

for any vertex p on γ_a. $\square$

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