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A criterion for the existence of nonreal eigenvalues for a Dirac operator

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Abstract. The aim of this work is to explore the discrete spectrum generated by complex perturbations in $L^2(\mathbb{R}^3, \mathbb{C}^4)$ of the $3d$ Dirac operator

$$\alpha \cdot (-i\nabla - \mathbf{A}) + m\beta$$

with variable magnetic field. Here, $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and β are 4×4 Dirac matrices, and $m > 0$ is the mass of a particle. We give a simple criterion for the potentials to generate discrete spectrum near $\pm m$. In case of creation of nonreal eigenvalues, this criterion gives also their location.

Contents

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1. Introduction

In this paper, we consider a Dirac operator $D_m(b,V)$ defined as follows. Denoting $x=(x_1,x_2,x_3)$ the usual variables of $\mathbb{R}^3$, let

$$\mathbf{B}=(0,0,b) \tag{1.1}$$

be a nice scalar magnetic field with constant direction such that $b=b(x_1,x_2)$ is *an admissible magnetic field*. That is, there exists a constant $b_0>0$ satisfying

$$b(x_1,x_2)=b_0+\tilde{b}(x_1,x_2), \tag{1.2}$$

where $\tilde{b}$ is a function such that the Poisson equation $\Delta\tilde{\varphi}=\tilde{b}$ admits a solution $\tilde{\varphi}\in C^2(\mathbb{R}^2)$ verifying $\sup_{(x_1,x_2)\in\mathbb{R}^2}|D^\alpha\tilde{\varphi}(x_1,x_2)|<\infty$, $\alpha\in\mathbb{N}^2$, $|\alpha|\le 2$. Define on $\mathbb{R}^2$ the function φ_0 by $\varphi_0(x_1,x_2):=\frac{1}{4}b_0(x_1^2+x_2^2)$ and set

$$\varphi(x_1,x_2):=\varphi_0(x_1,x_2)+\tilde{\varphi}(x_1,x_2). \tag{1.3}$$

We obtain a magnetic potential $\mathbf{A}:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ generating the magnetic field $\mathbf{B}$ (i.e., $\mathbf{B}=\operatorname{curl}\mathbf{A}$) by setting

$$\begin{aligned} A_1(x_1,x_2,x_3)&=A_1(x_1,x_2)=-\partial_{x_2}\varphi(x_1,x_2),\\ A_2(x_1,x_2,x_3)&=A_2(x_1,x_2)=\partial_{x_1}\varphi(x_1,x_2),\\ A_3(x_1,x_2,x_3)&=0. \end{aligned} \tag{1.4}$$

Then, for a 4×4 complex matrix $V=\{V_{\ell k}(x)\}_{\ell,k=1}^4$, the Dirac operator $D_m(b,V)$ acting on $L^2(\mathbb{R}^3):=L^2(\mathbb{R}^3,\mathbb{C}^4)$ is defined by

$$D_m(b,V):=\alpha\cdot(-i\nabla-\mathbf{A})+m\beta+V, \tag{1.5}$$

where $m>0$ is the mass of a particle. Here, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and β are the Dirac matrices defined by the following relations:

$$\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk}\mathbf{1},\quad \alpha_j\beta+\beta\alpha_j=\mathbf{0},\quad \beta^2=\mathbf{1},\quad j,k\in\{1,2,3\}, \tag{1.6}$$

δ_{jk} being the Kronecker symbol defined by $\delta_{jk}=1$ if $j=k$ and $\delta_{jk}=0$ otherwise, (see, e.g., the book [Tha92, Appendix of Chapter 1] for other possible representations).

For $V=0$, it is known that the spectrum of $D_m(b,0)$ is $(-\infty,-m]\cup[m,+\infty)$ (see for instance [TidA11, Sam13]). Throughout this paper, we assume that V satisfies:

Assumption 1.1. $V_{\ell k}(x)\in\mathbb{C}$ for $1\le\ell,k\le 4$ with:

(1.7)
- $0\not\equiv V\in L^\infty(\mathbb{R}^3)$, $\quad|V_{\ell k}(x)|\lesssim F_\perp(x_1,x_2)\,G(x_3)$,
- $F_\perp\in\left(L^{\frac{q}{2}}\cap L^\infty\right)\left(\mathbb{R}^2,\mathbb{R}_+^*\right)$ for some $q\ge 4$,
- $0<G(x_3)\lesssim\langle x_3\rangle^{-\beta},\beta>3$, where $\langle y\rangle:=\sqrt{1+|y|^2}$ for $y\in\mathbb{R}^d$.

Remark 1.2. Assumption 1.1 is naturally satisfied by matrix-valued perturbations $V : \mathbb{R}^3 \to \mathbb{C}^4$ (not necessarily Hermitian) such that

$$|V_{\ell k}(x)| \lesssim \langle (x_1, x_2) \rangle^{-\beta_\perp} \langle x_3 \rangle^{-\beta}, \quad \beta_\perp > 0, \quad \beta > 3, \quad 1 \leq \ell, k \leq 4. \tag{1.8}$$

We also have the matrix-valued perturbations $V : \mathbb{R}^3 \to \mathbb{C}^4$ (not necessarily Hermitian) such that

$$|V_{\ell k}(x)| \lesssim \langle x \rangle^{-\gamma}, \quad \gamma > 3, \quad 1 \leq \ell, k \leq 4. \tag{1.9}$$

Indeed, it follows from (1.9) that (1.8) holds with any $\beta \in (3, \gamma)$ and $\beta_\perp = \gamma - \beta > 0$.

Since we will deal with non-self-adjoint operators, it is useful to make precise the notion used of discrete and essential spectrum of an operator acting on a separable Hilbert space $\mathscr{H}$. Consider S a closed such operator. Let μ be an isolated point of $\mathrm{sp}(S)$, and $\mathscr{C}$ be a small positively oriented circle centred at μ, containing μ as the only point of $\mathrm{sp}(S)$. The point μ is said to be a discrete eigenvalue of S if it's algebraic multiplicity

$$\mathrm{mult}(\mu) := \mathrm{rank} \left(\frac{1}{2i\pi} \int_{\mathscr{C}} (S - z)^{-1} dz \right) \tag{1.10}$$

is finite. The discrete spectrum of S is then defined by

$$\mathrm{sp}_{\mathrm{disc}}(S) := \big\{ \mu \in \mathrm{sp}(S) : \mu \text{ is a discrete eigenvalue of } S \big\}. \tag{1.11}$$

Notice that the geometric multiplicity $\dim\big(\mathrm{Ker}(S - \mu)\big)$ of μ is such that $\dim\big(\mathrm{Ker}(S-\mu)\big) \leq \mathrm{mult}(\mu)$. Equality holds if S is self-adjoint. The essential spectrum of S is defined by

$$\mathrm{sp}_{\mathrm{ess}}(S) := \big\{ \mu \in \mathbb{C} : S - \mu \text{ is not a Fredholm operator} \big\}. \tag{1.12}$$

It's a closed subset of $\mathrm{sp}(S)$.

Under Assumption 1.1, we show (see Subsection 3.1) that V is relatively compact with respect to $D_m(b, 0)$. Therefore, according to the Weyl criterion on the invariance of the essential spectrum, we have

$$\begin{aligned} \mathrm{sp}_{\mathrm{ess}}\big(D_m(b, V)\big) = \mathrm{sp}_{\mathrm{ess}}(D_m(b, 0)) &= \mathrm{sp}(D_m(b, 0)) \\ &= (-\infty, -m] \cup [m, +\infty). \end{aligned} \tag{1.13}$$

However, V may generate complex eigenvalues (or discrete spectrum) that can only accumulate on $(-\infty, -m] \cup [m, +\infty)$ (see [GohGK90, Theorem 2.1, p. 373]). The situation near $\pm m$ is the most interesting since they play the role of spectral thresholds of this spectrum. For the quantum Hamiltonians, many studies on the distribution of the discrete spectrum near the essential spectrum have been done for self-adjoint perturbations, see for instance [Ivr98, Chap. 11-12], [PRV12, Sob86, Tam88, RoS09, Sam13, TidA11] and the references therein. Recently, there has been an increasing interest in the spectral theory of non-self-adjoint differential operators. We quote for instance the papers [Wan11, FLLS06, BO08, BGK09, DeHK09, DeHK13, Han13, GolK11, Sam14], see also the references therein. In most of these papers, (complex) eigenvalues estimates or Lieb–Thirring type inequalities are established. However, the problem of the existence and the localisation of the complex eigenvalues near the essential spectrum of the operators is not addressed. We can think that this is probably due to the technical difficulties caused by the non-self-adjoint aspect of the perturbation. By the same time, there are few results concerning non-self-adjoint Dirac operators, [Syr83, Syr87, CLT15, Dub14, Cue]. In this article, we will examine the problem of *the existence*, *the distribution* and *the localisation* of the nonreal eigenvalues of the Dirac operator $D_m(b, V)$ near $\pm m$. The case of the non-self-adjoint Laplacian $-\Delta + V(x)$ in $L^2(\mathbb{R}^n)$, $n \geq 2$, *near the origin*, is studied by Wang in [Wan11]. In particular, he proves that for slowly decaying potentials, 0 is the only possible accumulation point of de complex eigenvalues and if $V(x)$ decays more rapidly than $|x|^{-2}$, then there are no clusters of eigenvalues near the points of $[0, +\infty)$. Actually, in Assumption 1.1, the condition

$$0 < G(x_3) \lesssim \langle x_3 \rangle^{-\beta}, \quad \beta > 3, \quad x_3 \in \mathbb{R}, \tag{1.14}$$

is required in such a way we include perturbations decaying polynomially (as $|x_3| \longrightarrow +\infty$) along the direction of the magnetic field. In more restrictive setting, if we replace (1.14) by perturbations decaying exponentially along the direction of the magnetic field, i.e., satisfying

$$0 < G(x_3) \lesssim e^{-\beta \langle x_3 \rangle}, \quad \beta > 0, \quad x_3 \in \mathbb{R}, \tag{1.15}$$

then our third main result (Theorem 2.8) can be improved to get nonreal eigenvalues asymptotic behaviours near $\pm m$. However, this topic is beyond these notes in the sense that it requires the use of resonance approach, by defining in Riemann surfaces the resonances of the non-self-adjoint operator $D_m(b, V)$ near $\pm m$, and it will be considered elsewhere. Here, we extend and generalize to non-self-adjoint matrix case the methods of [Sam13, BBR07]. And, the problem studied is different. Moreover, due to the structure of the essential spectrum of the Dirac operator considered here (symmetric with respect to the origin), technical difficulties appear. In particular, these difficulties are underlying to the choice of the complex square root and the parametrization of the discrete eigenvalues in a neighbourhood of $\pm m$ (see (2.5), Remarks 2.1 and 4.1). To prove our main results, we reduce the study of the complex eigenvalues to the investigation of zeros of holomorphic functions. This allows us to essentially use complex analysis methods to solve our problem. Firstly, we obtain sharp upper bounds on the number of complex eigenvalues in small annulus near $\pm m$ (see Theorem 2.2). Secondly, under appropriate hypothesis, we prove the absence of nonreal eigenvalues in certain sectors adjoining $\pm m$ (see Theorem 2.5). By this way, we derive from Theorem 2.5 a relation between the properties of the perturbation V and the finiteness of the number of nonreal eigenvalues of $D_m(b, V)$ near $\pm m$ (see Corollary 2.6). Under additional conditions, we prove lower bounds implying the existence of nonreal eigenvalues near $\pm m$ (see Theorem 2.8). In more general setting, we conjecture a criterion of nonaccumulation of the discrete spectrum of $D_m(b,V)$ near $\pm m$ (see Conjecture 2.9). This conjecture is in the spirit of the Behrndt conjecture [Beh13, Open problem] on Sturm–Liouville operators. More precisely, he says the following: there exists nonreal eigenvalues of singular indefinite Sturm–Liouville operators accumulate to the real axis whenever the eigenvalues of the corresponding definite Sturm–Liouville operator accumulate to the bottom of the essential spectrum from below.

The paper is organized as follows. We present our main results in Section 2. In Section 3, we estimate the Schatten–von Neumann norms (defined in Appendix A) of the (weighted) resolvent of $D_m(b,0)$. We also reduce the study of the discrete spectrum to that of zeros of holomorphic functions. In Section 4, we give a suitable decomposition of the (weighted) resolvent of $D_m(b,0)$. Section 5 is devoted to the proofs of the main results. Appendix A is a summary on basic properties of the Schatten–von Neumann classes. In Appendix B, we briefly recall the notion of the index of a finite meromorphic operator-valued function along a positive oriented contour.

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2. Formulation of the main results

In order to state our results, some additional notations are needed. Let $p = p(b)$ be the spectral projection of $L^2(\mathbb{R}^2)$ onto the (infinite-dimensional) kernel of

$$H_\perp^- := (-i\partial_{x_1} - A_1)^2 + (-i\partial_{x_1} - A_2)^2 - b, \tag{2.1}$$

(see [Rai10, Subsection 2.2]). For a complex 4×4 matrix $M = M(x)$, $x \in \mathbb{R}^3$, $|M|$ defines the multiplication operator in $L^2(\mathbb{R}^3)$ by the matrix $\sqrt{M^*M}$. Let $\mathbf{V}_{\pm m}$ be the multiplication operators by the functions

$$\begin{aligned} \mathbf{V}_m(x_1,x_2) &= \frac{1}{2}\int_{\mathbb{R}} v_{11}(x_1,x_2,x_3)dx_3, \\ \mathbf{V}_{-m}(x_1,x_2) &= \frac{1}{2}\int_{\mathbb{R}} v_{33}(x_1,x_2,x_3)dx_3, \end{aligned} \tag{2.2}$$

where $v_{\ell k}$, $1 \leq \ell, k \leq 4$, are the coefficients of the matrix $|V|$. Clearly, Assumption 1.1 implies that

$$0 \leq \mathbf{V}_{\pm m}(x_1,x_2) \lesssim \sqrt{F_\perp(x_1,x_2)}, \tag{2.3}$$

since $F_\perp$ is bounded. This together with [Rai10, Lemma 2.4] give that the self-adjoint Toeplitz operators $p\mathbf{V}_{\pm m}p$ are compacts. Defining

$$\mathbb{C}_\pm := \big\{ z \in \mathbb{C} : \pm \operatorname{Im}(z) > 0 \big\}, \tag{2.4}$$

we will adopt the following choice of the complex square root

$$\mathbb{C} \setminus (-\infty, 0] \xrightarrow{\sqrt{\cdot}} \mathbb{C}_+. \tag{2.5}$$

Let η be a fixed constant such that $0 < \eta < m$. For $\widetilde{m} \in \{\pm m\}$, we set

$$\mathcal{D}_{\widetilde{m}}^{\pm}(\eta) := \big\{ z \in \mathbb{C}_\pm : 0 < |z - \widetilde{m}| < \eta \big\}. \tag{2.6}$$

If $0 < \gamma < 1$ and $0 < \epsilon < \min\Big(\gamma, \frac{\eta(1-\gamma)}{2}\Big)$, we define the domains

$$\mathcal{D}_{\pm}^{*}(\epsilon) := \big\{ k \in \mathbb{C}_\pm : 0 < |k| < \epsilon : \operatorname{Re}(k) > 0 \big\}. \tag{2.7}$$

Note that $0 < \epsilon < \eta$. Actually, the singularities of the resolvent of $D_m(b, 0)$ at $\pm m$ are induced by those of the resolvent of the one-dimensionnal Laplacian $-\partial_{x_3}^2$ at zero (see (3.2)–(3.3)). Therefore, the complex eigenvalues z of $D_m(b, V)$ near $\pm m$ are naturally parametrized by

$$\begin{aligned} \mathbb{C} \setminus \operatorname{sp}\big(D_m(b, 0)\big) \ni z = z_{\pm m}(k) := \frac{\pm m(1 + k^2)}{1 - k^2} \\ \Leftrightarrow k^2 = \frac{z \mp m}{z \pm m} \in \mathbb{C} \setminus [0, +\infty). \end{aligned} \tag{2.8}$$

Remark 2.1.

(i) Observe that

$$\mathbb{C} \setminus \operatorname{sp}\big(D_m(b, 0)\big) \ni z \longmapsto \Psi_\pm(z) = \frac{z \mp m}{z \pm m} \in \mathbb{C} \setminus [0, +\infty) \tag{2.9}$$

are Möbius transformations with inverses $\Psi_\pm^{-1}(\lambda) = \frac{\pm m(1+\lambda)}{1-\lambda}$.

(ii) For any $k \in \mathbb{C} \setminus \{\pm 1\}$, we have

$$z_{\pm m}(k) = \pm m \pm \frac{2mk^2}{1 - k^2} \quad \text{and} \quad \operatorname{Im}\big(z_{\pm m}(k)\big) = \pm \frac{2m \operatorname{Im}(k^2)}{|1 - k^2|^2}. \tag{2.10}$$

(iii) According to (2.10), $\pm \operatorname{Im}\big(z_m(k)\big) > 0$ if and only if $\pm \operatorname{Im}(k^2) > 0$. Then, it is easy to check that any $z_m(k) \in \mathbb{C}_\pm$ is respectively associated to a unique $k \in \mathbb{C}_\pm \cap \big\{k \in \mathbb{C} : \operatorname{Re}(k) > 0\big\}$. Moreover,

$$z_m(k) \in \mathcal{D}_m^{\pm}(\eta) \quad \text{whenever} \quad k \in \mathcal{D}_\pm^{*}(\epsilon). \tag{2.11}$$

(iv) Similarly, according to (2.10), we have $\pm \operatorname{Im}\big(z_{-m}(k)\big) > 0$ if and only if $\mp \operatorname{Im}(k^2) > 0$. Then, any $z_{-m}(k) \in \mathbb{C}_\pm$ is respectively associated to a unique $k \in \mathbb{C}_\mp \cap \big\{k \in \mathbb{C} : \operatorname{Re}(k) > 0\big\}$. Furthermore,

$$z_{-m}(k) \in \mathcal{D}_{-m}^{\pm}(\eta) \quad \text{whenever} \quad k \in \mathcal{D}_\mp^{*}(\epsilon). \tag{2.12}$$

In the sequel, to simplify the notations, we set

$$\begin{aligned} \operatorname{sp}_{\text{disc}}^{+}\big(D_m(b, V)\big) &:= \operatorname{sp}_{\text{disc}}\big(D_m(b, V)\big) \cap \mathcal{D}_{\pm m}^{+}(\eta), \\ \operatorname{sp}_{\text{disc}}^{-}\big(D_m(b, V)\big) &:= \operatorname{sp}_{\text{disc}}\big(D_m(b, V)\big) \cap \mathcal{D}_{\pm m}^{-}(\eta). \end{aligned} \tag{2.13}$$

We can now state our first main result.

Theorem 2.2 (Upper bound)**.** *Assume that Assumption 1.1 holds. Then, we have*

$$\sum_{\substack{z_{\pm m}(k)\in \mathrm{sp}^{+}_{\mathrm{disc}}\left(D_m(b,V)\right)\\ k\in\Delta_{\pm}}} \mathrm{mult}\big(z_{\pm m}(k)\big) + \sum_{\substack{z_{\pm m}(k)\in \mathrm{sp}^{-}_{\mathrm{disc}}\left(D_m(b,V)\right)\\ k\in\Delta_{\mp}}} \mathrm{mult}\big(z_{\pm m}(k)\big) = \mathcal{O}\Big(\operatorname{Tr}\mathbf{1}_{(r,\infty)}\big(p\mathbf{V}_{\pm m}p\big)|\ln r|\Big) + \mathcal{O}(1), \tag{2.14}$$

for some $r_0 > 0$ small enough and any $0 < r < r_0$, where $\mathrm{mult}\big(z_{\pm m}(k)\big)$ is defined by (1.10) and

$$\Delta_{\pm} := \big\{r < |k| < 2r : |\mathrm{Re}(k)| > \sqrt{\nu} : |\mathrm{Im}(k)| > \sqrt{\nu} : \nu > 0\big\} \cap \mathcal{D}^{*}_{\pm}(\epsilon).$$

In order to state the rest of the results, we put some restrictions on V.

Assumption 2.3. V satisfies Assumption 1.1 with

$$V = \Phi W,\ \Phi \in \mathbb{C}\setminus\mathbb{R}, \quad \text{and} \quad W = \big\{W_{\ell k}(x)\big\}_{\ell,k=1}^{4} \text{ is Hermitian.} \tag{2.15}$$

The potential W will be said to be of definite sign if $\pm W(x) \geq 0$ for any $x \in \mathbb{R}^3$. Let $J := \mathrm{sign}(W)$ denote the matrix sign of W. Without loss of generality, we will say that W is of definite sign $J = \pm$. For any $\delta > 0$, we set

$$\mathcal{C}_{\delta}(J) := \big\{k \in \mathbb{C} : -\delta J\,\mathrm{Im}(k) \leq |\,\mathrm{Re}(k)|\big\}, \quad J = \pm. \tag{2.16}$$

Remark 2.4. For $W \geq 0$ and $\pm\sin(\operatorname{Arg}\Phi) > 0$, the nonreal eigenvalues z of $D_m(b,V)$ verify $\pm\,\mathrm{Im}(z) > 0$. Then, according to Remark 2.1(iii)–(iv), they satisfy near $\pm m$:

(i) $z = z_{\pm m}(k) = \frac{\pm m(1+k^2)}{1-k^2} \in \mathcal{D}^{+}_{\pm m}(\eta)$, $k \in \mathcal{D}^{*}_{\pm}(\epsilon)$ if $\sin(\operatorname{Arg}\Phi) > 0$,

(ii) $z = z_{\pm m}(k) = \frac{\pm m(1+k^2)}{1-k^2} \in \mathcal{D}^{-}_{\pm m}(\eta)$, $k \in \mathcal{D}^{*}_{\mp}(\epsilon)$ if $\sin(\operatorname{Arg}\Phi) < 0$.

Theorem 2.5 (Absence of nonreal eigenvalues)**.** *Assume that V satisfies Assumptions 1.1 and 2.3 with $W \geq 0$. Then, for any $\delta > 0$ small enough, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, $D_m(b,\varepsilon V)$ has no nonreal eigenvalues in*

$$\left\{ z = z_{\pm m}(k) \in \begin{cases} \mathcal{D}^{+}_{\pm m}(\eta) : k \in \Phi\mathcal{C}_{\delta}(J)\cap\mathcal{D}^{*}_{\pm}(\epsilon) & \textit{for } \operatorname{Arg}\Phi \in (0,\pi), \\ \mathcal{D}^{-}_{\pm m}(\eta) : k \in -\Phi\mathcal{C}_{\delta}(J)\cap\mathcal{D}^{*}_{\mp}(\epsilon) & \textit{for } \operatorname{Arg}\Phi \in -(0,\pi) \end{cases} : 0 < |k| \ll 1 \right\}. \tag{2.17}$$

For Ω a small pointed neighbourhood of $\widetilde{m} \in \{\pm m\}$, let us introduce the counting function of complex eigenvalues of the operator $D_m(b,V)$ lying in

Ω, taking into account the multiplicity:

$$(2.18)\quad \begin{aligned}&\mathcal{N}_{\widetilde{m}}\big(\big(D_m(b,V)\big),\Omega\big)\\ &\qquad := \#\big\{z = z_{\widetilde{m}}(k) \in \mathrm{sp}_{\mathrm{disc}}\big(D_m(b,V)\big) \cap \mathbb{C}_\pm \cap \Omega : 0 < |k| \ll 1\big\}.\end{aligned}$$

As an immediate consequence of Theorem 2.5, we have the following:

Corollary 2.6 (Nonaccumulation of nonreal eigenvalues)**.** *Let the assumptions of Theorem 2.5 hold. Then, for any $0 < \varepsilon \leq \varepsilon_0$ and any domain Ω as above, we have*

$$(2.19)\qquad \begin{cases}\mathcal{N}_m\big(\big(D_m(b,\varepsilon V)\big),\Omega\big) < \infty & \textit{for } \operatorname{Arg}\Phi \in \pm\big(0,\frac{\pi}{2}\big),\\ \mathcal{N}_{-m}\big(\big(D_m(b,\varepsilon V)\big),\Omega\big) < \infty & \textit{for } \operatorname{Arg}\Phi \in \pm\big(\frac{\pi}{2},\pi\big).\end{cases}$$

Indeed, near m, for $\operatorname{Arg}\Phi \in \pm\big(0,\frac{\pi}{2}\big)$ and δ small enough, we have respectively $\pm\Phi\mathcal{C}_\delta(J) \cap \mathcal{D}^*_\pm(\epsilon) = \mathcal{D}^*_\pm(\epsilon)$. Near $-m$, for $\operatorname{Arg}\Phi \in \pm\big(\frac{\pi}{2},\pi\big)$ and δ small enough, we have respectively $\pm\Phi\mathcal{C}_\delta(J) \cap \mathcal{D}^*_\mp(\epsilon) = \mathcal{D}^*_\mp(\epsilon)$. Therefore, Corollary 2.6 follows according to (2.11) and (2.12).

Similarly to (2.2), let $\mathbf{W}_{\pm m}$ define the multiplication operators by the functions $\mathbf{W}_{\pm m} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with respect to the matrix $|W|$. Hence, let us consider the following:

Assumption 2.7. The functions $\mathbf{W}_{\pm m}$ satisfy

$$0 < \mathbf{W}_{\pm m}(x_1,x_2) \leq e^{-C\langle(x_1,x_2)\rangle^2}$$

for some positive constant C.

For $r_0 > 0$, $\delta > 0$ two fixed constants, and $r > 0$ which tends to zero, we define

$$(2.20)\qquad \Gamma^\delta(r,r_0) := \big\{x + iy \in \mathbb{C} : r < x < r_0, -\delta x < y < \delta x\big\}.$$

Theorem 2.8 (Lower bounds)**.** *Assume that V satisfies Assumptions 1.1, 2.3 and 2.7 with $W \geq 0$. Then, for any $\delta > 0$ small enough, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, there is an accumulation of nonreal eigenvalues $z_{\pm m}(k)$ of $D_m(b,\varepsilon V)$ near $\pm m$ in a sector around the semi-axis*[1]

$$(2.21)\qquad \begin{cases} z = \pm m \pm e^{i(2\operatorname{Arg}\Phi - \pi)}]0,+\infty) & \textit{for } \operatorname{Arg}\Phi \in \big(\frac{\pi}{2}\big)_\pm + \big(0,\frac{\pi}{2}\big)\,,\\ z = \pm m \pm e^{i(2\operatorname{Arg}\Phi + \pi)}]0,+\infty) & \textit{for } \operatorname{Arg}\Phi \in -\big(\frac{\pi}{2}\big)_\pm - \big(0,\frac{\pi}{2}\big)\,.\end{cases}$$

More precisely, for

$$(2.22)\qquad \operatorname{Arg}\Phi \in \Big(\frac{\pi}{2}\Big)_\pm + \Big(0,\frac{\pi}{2}\Big)\,,$$

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[1] For $r \in \mathbb{R}$, we set $r_\pm := \max(0,\pm r)$.

there exists a decreasing sequence of positive numbers $(r_\ell^{\pm m})$, $r_\ell^{\pm m} \searrow 0$, *such that*

$$\sum_{\substack{z_{\pm m}(k)\,\in\,\mathrm{sp}^+_{\mathrm{disc}}\left(D_m(b,\varepsilon V)\right)\\ k\,\in\,-iJ\Phi\Gamma^\delta(r^{\pm m}_{\ell+1},r^{\pm m}_\ell)\cap\mathcal{D}^*_\pm(\epsilon)}} \mathrm{mult}\big(z_{\pm m}(k)\big) \geq \mathrm{Tr}\,\mathbf{1}_{(r^{\pm m}_{\ell+1},r^{\pm m}_\ell)}\big(p\mathbf{W}_{\pm m}p\big). \tag{2.23}$$

For

$$\mathrm{Arg}\,\Phi \in -\left(\frac{\pi}{2}\right)_\pm - \left(0,\frac{\pi}{2}\right), \tag{2.24}$$

(2.23) *holds again with* $\mathrm{sp}^+_{\mathrm{disc}}\big(D_m(b,\varepsilon V)\big)$ *replaced by* $\mathrm{sp}^-_{\mathrm{disc}}\big(D_m(b,\varepsilon V)\big)$, k *by* $-k$, *and* $\mathcal{D}^*_\pm(\epsilon)$ *by* $\mathcal{D}^*_\mp(\epsilon)$.

A graphic illustration of Theorems 2.5 and 2.8 near m with $V = \Phi W$, $W \geq 0$, is given in Figure 2.

$$\mathrm{Arg}\,\Phi \in (\tfrac{\pi}{2}, \pi),\ W \geq 0$$
$$V = \Phi W$$

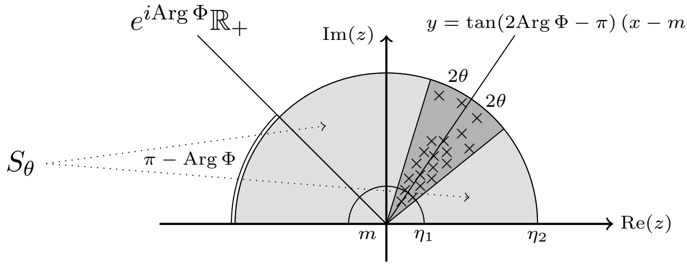

FIGURE 2.1. Localisation of the nonreal eigenvalues near m with $0 < \eta_1 < \eta_2 < \eta$ small enough: For θ small enough and $0 < \varepsilon \leq \varepsilon_0$, $D_m(b,\varepsilon V) := D_m(b,0) + \varepsilon V$ has no eigenvalues in S_θ (Theorem 2.5). They are concentrated around the semi-axis $z = m + e^{i(2\mathrm{Arg}\,\Phi - \pi)}]0,+\infty)$ (Theorem 2.8).

Here, the accumulation of the nonreal eigenvalues of $D_m(b,\varepsilon V)$ near $\pm m$ holds for any $0 < \varepsilon \leq \varepsilon_0$. We expect this to be a general phenomenon in the sense of the following conjecture:

Conjecture 2.9. *Let* $V = \Phi W$ *satisfy Assumption* 1.1 *with*

$$\mathrm{Arg}\,\Phi \in \mathbb{C} \setminus \mathbb{R}e^{ik\frac{\pi}{2}},$$

$k \in \mathbb{Z}$, *and* W *Hermitian of definite sign. Then, for any domain* Ω *as in* (2.18), *we have*

$$\mathcal{N}_{\pm m}\big(D_m(b,V),\Omega\big) < \infty \tag{2.25}$$

if and only if $\pm\mathrm{Re}(V) > 0$.

$\operatorname{Arg}\Phi$ $V = \Phi W$	$\left(-\pi, -\frac{\pi}{2}\right)$	$\left(-\frac{\pi}{2}, 0\right)$	$\left(0, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \pi\right)$	
$W \geq 0$ $W \in \mathfrak{B}_h(\mathbb{C}^4)$	accumulation near m around the semi-axis $m + e^{i(2\operatorname{Arg}\Phi+\pi)}]0, +\infty)$ $\Re(V) \leq 0$	non-accumulation near m $\Re(V) \geq 0$	non-accumulation near m $\Re(V) \geq 0$	accumulation near m around the semi-axis $m + e^{i(2\operatorname{Arg}\Phi-\pi)}]0, +\infty)$ $\Re(V) \leq 0$	Threshold $+m$
	non-accumulation near $-m$ $\Re(V) \leq 0$	accumulation near $-m$ around the semi-axis $-m - e^{i(2\operatorname{Arg}\Phi+\pi)}]0, +\infty)$ $\Re(V) \geq 0$	accumulation near $-m$ around the semi-axis $-m - e^{i(2\operatorname{Arg}\Phi-\pi)}]0, +\infty)$ $\Re(V) \geq 0$	non-accumulation near $-m$ $\Re(V) \leq 0$	Threshold $-m$
Location of the discrete eigenvalues	Lower half-plane		Upper half-plane		
	Discrete eigenvalues near $\pm m$ of $D_m(b, \varepsilon V)$ for $0 < \varepsilon \leq \varepsilon_0$				

FIGURE 2.2. Summary of results.

3. Characterisation of the discrete eigenvalues

From now on, for $\widetilde{m} \in \{\pm m\}$, $\mathcal{D}_{\widetilde{m}}^{\pm}(\eta)$ and $\mathcal{D}_{\pm}^{*}(\epsilon)$ are the domains given by (2.6) and (2.7) respectively.

3.1. Local properties of the (weighted) free resolvent.

In this subsection, we show in particular that under Assumption 1.1, V is relatively compact with respect to $D_m(b, 0)$.

Let $P := p \otimes 1$ define the orthogonal projection onto $\operatorname{Ker} H_{\perp}^{-} \otimes L^2(\mathbb{R})$, where $H_{\perp}^{-}$ is the two-dimensional magnetic Schrdinger operator defined by (2.1). Denote $\mathbf{P}$ the orthogonal projection onto the union of the eigenspaces of $D_m(b, 0)$ corresponding to $\pm m$. Then, we have

$$\mathbf{P} = \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} := \mathrm{I} - \mathbf{P} = \begin{pmatrix} I-P & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I-P & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \tag{3.1}$$

(see [Sam13, Section 3]). Moreover, if $z \in \mathbb{C} \setminus (-\infty, -m] \cup [m, +\infty)$, then

$$\big(D_m(b, 0) - z\big)^{-1} = \big(D_m(b, 0) - z\big)^{-1}\mathbf{P} + \big(D_m(b, 0) - z\big)^{-1}\mathbf{Q} \tag{3.2}$$

with

(3.3)
$$\big(D_m(b, 0) - z\big)^{-1}\mathbf{P} = \Big[p \otimes \mathscr{R}(z^2 - m^2)\Big] \begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \Big[p \otimes (-i\partial_{x_3})\mathscr{R}(z^2 - m^2)\Big] \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here, the resolvent $\mathscr{R}(z) := \left(-\partial_{x_3}^2 - z\right)^{-1}$, $z \in \mathbb{C} \setminus [0, +\infty)$, acts in $L^2(\mathbb{R})$. It admits the integral kernel

$$I_z(x_3, x_3') := -\frac{e^{i\sqrt{z}|x_3 - x_3'|}}{2i\sqrt{z}}, \tag{3.4}$$

with $\operatorname{Im}\left(\sqrt{z}\right) > 0$. In what follows below, the definition of the Schatten–von Neumann class ideals $\mathbf{S}_q$ is recalled in Appendix A.

Lemma 3.1. *Let $U \in L^q(\mathbb{R}^2)$, $q \in [2, +\infty)$ and $\tau > \frac{1}{2}$. Then, the operator-valued function*

$$\mathbb{C} \setminus \operatorname{sp}\big(D_m(b,0)\big) \ni z \longmapsto U \langle x_3 \rangle^{-\tau} \big(D_m(b,0) - z\big)^{-1} \mathbf{P}$$

is holomorphic with values in $\mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$. Moreover, we have

$$\left\| U \langle x_3 \rangle^{-\tau} \big(D_m(b,0) - z\big)^{-1} \mathbf{P} \right\|_{\mathbf{S}_q}^q \leq C \|U\|_{L^q}^q M(z,m)^q, \tag{3.5}$$

where

$$\begin{aligned} M(z,m) := \|\langle x_3 \rangle^{-\tau}\|_{L^q} \big(|z+m| + |z-m|\big) \sup\nolimits_{s \in [0,+\infty)} \left| \frac{s+1}{s - z^2 + m^2} \right| \\ + \frac{\|\langle x_3 \rangle^{-\tau}\|_{L^2}}{\big(\operatorname{Im} \sqrt{z^2 - m^2}\big)^{\frac{1}{2}}}, \end{aligned} \tag{3.6}$$

$C = C(q,b)$ being a constant depending on q and b.

Proof. The holomorphicity on $\mathbb{C} \setminus \operatorname{sp}\big(D_m(b,0)\big)$ is evident. Let us prove the bound (3.5). Constants are generic (i.e., changing from a relation to another). Set

$$L_1(z) := \Big[p \otimes \mathscr{R}(z^2 - m^2)\Big] \begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

and

$$L_2(z) := \Big[p \otimes (-i\partial_{x_3}) \mathscr{R}(z^2 - m^2)\Big] \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.8}$$

Then, from (3.3), we get

$$U \langle x_3 \rangle^{-\tau} \big(D_m(b,0) - z\big)^{-1} \mathbf{P} = U \langle x_3 \rangle^{-\tau} L_1(z) + U \langle x_3 \rangle^{-\tau} L_2(z). \tag{3.9}$$

First, we estimate the $\mathbf{S}_q$-norm of the first term of the RHS of (3.9). Thanks to (3.7), we have

$$U \langle x_3 \rangle^{-\tau} L_1(z) = \Big[U p \otimes \langle x_3 \rangle^{-\tau} \mathscr{R}(z^2 - m^2)\Big] \begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

By an easy adaptation of [Rai10, Proof of Lemma 2.4], it can be similarly proved that the operator Up satisfies $Up \in \mathbf{S}_q\big(L^2(\mathbb{R}^2)\big)$ with

$$\|Up\|_{\mathbf{S}_q}^q \leq \frac{b_0}{2\pi} \mathrm{e}^{2 \operatorname{osc} \tilde{\varphi}} \|U\|_{L^q}^q, \tag{3.11}$$
$$\operatorname{osc} \tilde{\varphi} := \sup_{(x_1,x_2)\in\mathbb{R}^2} \tilde{\varphi}(x_1,x_2) - \inf_{(x_1,x_2)\in\mathbb{R}^2} \tilde{\varphi}(x_1,x_2).$$

On the other hand, we have

$$\begin{aligned} \big\|\langle x_3\rangle^{-\tau} \mathscr{R}(z^2-m^2)\big\|_{\mathbf{S}_q}^q \leq \Big\| \langle x_3\rangle^{-\tau} \left(-\partial_{x_3}^2+1\right)^{-1}\Big\|_{\mathbf{S}_q}^q \\ \times \big\|\left(-\partial_{x_3}^2+1\right) \mathscr{R}(z^2-m^2)\big\|^q . \end{aligned} \tag{3.12}$$

By the Spectral mapping theorem, we have

$$\big\|\left(-\partial_{x_3}^2+1\right) \mathscr{R}(z^2-m^2)\big\|^q \leq \sup\nolimits_{s\in[0,+\infty)}^q \left| \frac{s+1}{s-z^2+m^2}\right|, \tag{3.13}$$

and by the standard criterion [Sim79, Theorem 4.1], we have

$$\big\|\langle x_3\rangle^{-\tau} \left(-\partial_{x_3}^2+1\right)\big\|_{\mathbf{S}_q}^q \leq C \|\langle x_3\rangle^{-\tau}\|_{L^q}^q \left\| \Big(|\cdot|^2+1\Big)^{-1} \right\|_{L^q}^q . \tag{3.14}$$

Combining (3.10), (3.11), (3.12), (3.13) with (3.14), we get

(3.15)

$$\begin{aligned} \big\|U\langle x_3\rangle^{-\tau} L_1(z)\big\|_{\mathbf{S}_q}^q \leq C(q,b) \|U\|_{L^q}^q \|\langle x_3\rangle^{-\tau}\|_{L^q}^q \\ \times \big(|z+m|+|z-m|\big)^q \sup\nolimits_{s\in[0,+\infty)}^q \left| \frac{s+1}{s-z^2+m^2}\right| . \end{aligned}$$

Now, we estimate the $\mathbf{S}_q$-norm of the second term of the RHS of (3.9). Thanks to (3.8), we have

$$U\langle x_3\rangle^{-\tau} L_2(z) = \Big[Up \otimes \langle x_3\rangle^{-\tau}(-i\partial_{x_3})\mathscr{R}(z^2-m^2)\Big] \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix} . \tag{3.16}$$

According to (3.4), the operator $\langle x_3\rangle^{-\tau}(-i\partial_{x_3})\mathscr{R}(z^2-m^2)$ admits the integral kernel

$$- \langle x_3\rangle^{-\tau} \frac{x_3-x_3'}{2|x_3-x_3'|} e^{i\sqrt{z^2-m^2}|x_3-x_3'|}. \tag{3.17}$$

An estimate of the $L^2(\mathbb{R}^2)$-norm of (3.17) shows that

$$\langle x_3\rangle^{-\tau}(-i\partial_{x_3})\mathscr{R}(z^2-m^2) \in \mathbf{S}_2\big(L^2(\mathbb{R})\big)$$

with

$$\big\|\langle x_3\rangle^{-\tau}(-i\partial_{x_3})\mathscr{R}(z^2-m^2)\big\|_{\mathbf{S}_2}^2 \leq \frac{C\|\langle x_3\rangle^{-\tau}\|_{L^2}^2}{\operatorname{Im}\sqrt{z^2-m^2}}. \tag{3.18}$$

By combining (3.16), (3.11) with (3.18), we get

$$\left\| U \langle x_3 \rangle^{-\tau} L_2(z) \right\|_{\mathbf{S}_q} \leq C(q,b)^{\frac{1}{q}} \frac{\|U\|_{L^q} \| \langle x_3 \rangle^{-\tau} \|_{L^2}}{\left(\operatorname{Im} \sqrt{z^2 - m^2} \right)^{\frac{1}{2}}}. \tag{3.19}$$

Then, (3.5) follows immediately from (3.9), (3.15) and (3.19), which gives the proof. ☐

For simplicity of notation in the sequel, we set

$$H^{\pm} := (-i\nabla - \mathbf{A})^2 \pm b = H_{\perp}^{\pm} \otimes 1 + 1 \otimes (-\partial_{x_3}^2), \tag{3.20}$$

where $H_{\perp}^{-}$ is the operator defined by (2.1), $H_{\perp}^{+}$ being the corresponding operator with $-b$ replaced by b. We recall from [Rai10, Subsection 2.2] that we have

$$\dim \operatorname{Ker} H_{\perp}^{-} = \infty, \quad \dim \operatorname{Ker} H_{\perp}^{+} = 0, \quad \text{and} \quad \sigma(H_{\perp}^{\pm}) \subset \{0\} \cup [\zeta, +\infty), \tag{3.21}$$

with

$$\zeta := 2b_0 \mathrm{e}^{-2\operatorname{osc}\tilde{\varphi}} > 0, \tag{3.22}$$

$\operatorname{osc}\tilde{\varphi}$ being defined by (3.11). Since the spectrum of the one-dimensional Laplacian $-\partial_{x_3}^2$ coincides with $[0,+\infty)$, we deduce from (3.20) and (3.21) that, on one hand, the spectrum of the operator H^+ belongs to $[\zeta,+\infty)$ (notice that in the constant magnetic field case $b = b_0$, we have $\zeta = 2b_0$, the first Landau level of H^+). On the other hand, that the spectrum of the operator H^- coincides with $[0,+\infty)$.

Lemma 3.2. *Let $g \in L^q(\mathbb{R}^3)$, $q \in [4,+\infty)$. Then, the operator-valued function*

$$\mathbb{C} \setminus \left\{ \left(-\infty, -\sqrt{m^2+\zeta} \right] \cup \left[\sqrt{m^2+\zeta}, +\infty \right) \right\} \ni z \longmapsto g\big(D_m(b,0) - z\big)^{-1} \mathbf{Q} \tag{3.23}$$

is holomorphic with values in $\mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$. Moreover, we have

$$\left\| g\big(D_m(b,0) - z\big)^{-1} \mathbf{Q} \right\|_{\mathbf{S}_q}^q \leq C \|g\|_{L^q}^q \widetilde{M}(z,m)^q, \tag{3.24}$$

where

$$\widetilde{M}(z,m) := \sup_{s \in [\zeta,+\infty)} \left| \frac{s+1}{s+m^2} \right|^{\frac{1}{2}} + \big(|z| + |z|^2\big) \sup_{s \in [\zeta,+\infty)} \left| \frac{s+1}{s+m^2-z^2} \right|, \tag{3.25}$$

$C = C(q)$ being a constant depending on q.

Proof. For $z \in \rho\big(D_m(b,0)\big)$ $\big($the resolvent set of $D_m(b,0)\big)$, we have

$$\big(D_m(b,0) - z\big)^{-1} = D_m(b,0)^{-1} + z\big(1 + zD_m(b,0)^{-1}\big)\big(D_m(b,0)^2 - z^2\big)^{-1}. \tag{3.26}$$

By setting

$$L_3(z) := z\big(1 + zD_m(b,0)^{-1}\big)\big(D_m(b,0)^2 - z^2\big)^{-1}, \tag{3.27}$$

we get from (3.26)

$$g\big(D_m(b,0)-z\big)^{-1}\mathbf{Q} = gD_m(b,0)^{-1}\mathbf{Q} + gL_3(z)\mathbf{Q}. \tag{3.28}$$

It can be proved that

$$\begin{aligned}&\big(D_m(b,0)^2 - z^2\big)^{-1}\mathbf{Q}\\ &= \begin{pmatrix} \big(H^-+m^2-z^2\big)^{-1}(I-P) & 0 & 0 & 0 \\ 0 & \big(H^++m^2-z^2\big)^{-1} & 0 & 0 \\ 0 & 0 & \big(H^-+m^2-z^2\big)^{-1}(I-P) & 0 \\ 0 & 0 & 0 & \big(H^++m^2-z^2\big)^{-1} \end{pmatrix},\end{aligned} \tag{3.29}$$

(see for instance [TidA11, Identity (2.2)]). The set $\mathbb{C} \setminus [\zeta, +\infty)$ is included in the resolvent set of H^- defined on $(I-P)\,\mathrm{Dom}(H^-)$. Similarly, it is included in the resolvent set of H^+ defined on $\mathrm{Dom}(H^+)$. Then,

$$\mathbb{C} \setminus \Big\{\Big(-\infty, -\sqrt{m^2+\zeta}\Big] \cup \Big[\sqrt{m^2+\zeta}, +\infty\Big)\Big\} \ni z \longmapsto \big(D_m(b,0)^2 - z^2\big)^{-1}\mathbf{Q} \tag{3.30}$$

is well defined and holomorphic. Therefore, so is the operator-valued function (3.23) thanks to (3.27) and (3.28).

It remains to prove the bound (3.24). As in the proof of the previous lemma, the constants change from a relation to another. First, we prove that (3.24) is true for q even.

Let us focus on the second term of the RHS of (3.28). According to (3.27) and (3.29), we have

$$\begin{aligned}&\|gL_3(z)\mathbf{Q}\|_{\mathbf{S}_q}^q\\ &\le C\big(|z|+|z|^2\big)^q\\ &\quad\times\left(\Big\|g\big(H^-+m^2-z^2\big)^{-1}(I-P)\Big\|_{\mathbf{S}_q}^q + \Big\|g\big(H^++m^2-z^2\big)^{-1}\Big\|_{\mathbf{S}_q}^q\right).\end{aligned} \tag{3.31}$$

One has

$$\begin{aligned}&\Big\|g\big(H^-+m^2-z^2\big)^{-1}(I-P)\Big\|_{\mathbf{S}_q}^q\\ &\le \big\|g(H^-+1)^{-1}\big\|_{\mathbf{S}_q}^q \Big\|(H^-+1)\big(H^-+m^2-z^2\big)^{-1}(I-P)\Big\|^q.\end{aligned} \tag{3.32}$$

The Spectral mapping theorem implies that

$$\Big\|(H^-+1)\big(H^-+m^2-z^2\big)^{-1}(I-P)\Big\|^q \le \sup\nolimits^q_{s\in[\zeta,+\infty)} \left|\frac{s+1}{s+m^2-z^2}\right|. \tag{3.33}$$

Exploiting the resolvent equation, the boundedness of b, and the diamagnetic inequality (see [AHS78, Theorem 2.3] and [Sim79, Theorem 2.13], which is only valid when q is even), we obtain

$$\begin{aligned}\left\|g\big(H^- + 1\big)^{-1}\right\|_{\mathbf{S}_q}^q &\leq \left\|I + (H^- + 1)^{-1} b\right\|^q \left\|g\big((-i\nabla - \mathbf{A})^2 + 1\big)^{-1}\right\|_{\mathbf{S}_q}^q \\ &\leq C \left\|g(-\Delta + 1)^{-1}\right\|_{\mathbf{S}_q}^q .\end{aligned} \tag{3.34}$$

The standard criterion [Sim79, Theorem 4.1] implies that

$$\left\|g(-\Delta + 1|)^{-1}\right\|_{\mathbf{S}_q}^q \leq C \|g\|_{L^q}^q \left\|\Big(| \cdot |^2 + 1\Big)^{-1}\right\|_{L^q}^q . \tag{3.35}$$

The bound (3.32) together with (3.33), (3.34) and (3.35) give

$$\left\|g\big(H^- + m^2 - z^2\big)^{-1}(I - P)\right\|_{\mathbf{S}_q}^q \leq C \|g\|_{L^q}^q \sup\nolimits_{s \in [\zeta, +\infty)}^q \left|\frac{s+1}{s + m^2 - z^2}\right| . \tag{3.36}$$

Similarly, it can be shown that

$$\left\|g\big(H^+ + m^2 - z^2\big)^{-1}\right\|_{\mathbf{S}_q}^q \leq C \|g\|_{L^q}^q \sup\nolimits_{s \in [\zeta, +\infty)}^q \left|\frac{s+1}{s + m^2 - z^2}\right| . \tag{3.37}$$

This together with (3.31) and (3.36) give

$$\|g L_3(z) \mathbf{Q}\|_{\mathbf{S}_q}^q \leq C \|g\|_{L^q}^q \big(|z| + |z|^2\big)^q \sup\nolimits_{s \in [\zeta, +\infty)}^q \left|\frac{s+1}{s + m^2 - z^2}\right| . \tag{3.38}$$

Now, we focus on the first term $g D_m(b, 0)^{-1} \mathbf{Q}$ of the RHS of (3.28). For $\gamma > 0$, as in (3.29), we have

$$\begin{aligned} &D_m(b, 0)^{-\gamma} \mathbf{Q} \\ &= \begin{pmatrix} \big(H^- + m^2\big)^{-\frac{\gamma}{2}} (I - P) & 0 & 0 & 0 \\ 0 & \big(H^+ + m^2\big)^{-\frac{\gamma}{2}} & 0 & 0 \\ 0 & 0 & \big(H^- + m^2\big)^{-\frac{\gamma}{2}} (I - P) & 0 \\ 0 & 0 & 0 & \big(H^+ + m^2\big)^{-\frac{\gamma}{2}} \end{pmatrix} . \end{aligned} \tag{3.39}$$

Therefore, arguing as above $\big($(3.31)–(3.37)$\big)$, it can be proved that

$$\|g D_m(b, 0)^{-\gamma} \mathbf{Q}\|_{\mathbf{S}_q}^q \leq C(q, \gamma) \|g\|_{L^q}^q \sup\nolimits_{s \in [\zeta, +\infty)}^q \left|\frac{s+1}{s + m^2}\right|^{\frac{\gamma}{2}}, \qquad \gamma q > 3. \tag{3.40}$$

Then, for q even, (3.24) follows by putting together (3.28), (3.38), and (3.40) with $\gamma = 1$.

We get the general case $q \geq 4$ with the help of interpolation methods.

If q satisfies $q > 4$, then, there exists even integers $q_0 < q_1$ such that $q \in (q_0, q_1)$ with $q_0 \geq 4$. Let $\beta \in (0, 1)$ satisfy $\frac{1}{q} = \frac{1-\beta}{q_0} + \frac{\beta}{q_1}$ and consider the operator

$$L^{q_i}\big(\mathbb{R}^3\big) \ni g \xmapsto{T} g\big(D_m(b, 0) - z\big)^{-1} \mathbf{Q} \in \mathbf{S}_{q_i}\big(L^2(\mathbb{R}^3)\big), \qquad i = 0, 1.$$

Let $C_i = C(q_i)$, $i = 0$, 1, denote the constant appearing in (3.24) and set

$$C(z, q_i) := C_i^{\frac{1}{q_i}} \widetilde{M}(z, m).$$

From (3.24), we know that $\|T\| \leq C(z, q_i)$, $i = 0, 1$. Now, we use the Riesz–Thorin Theorem (see for instance [Fol84, Sub. 5 of Chap. 6], [Rie26, Tho39], [Lun09, Chap. 2]) to interpolate between q_0 and q_1. We obtain the extension $T : L^q(\mathbb{R}^2) \longrightarrow \mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$ with

$$\|T\| \leq C(z, q_0)^{1-\beta} C(\gamma, q_1)^{\beta} \leq C(q)^{\frac{1}{q}} \widetilde{M}(z, m).$$

In particular, for any $g \in L^q(\mathbb{R}^3)$, we have

$$\|T(g)\|_{\mathbf{S}_q} \leq C(q)^{\frac{1}{q}} \widetilde{M}(z, m) \|g\|_{L^q},$$

which is equivalent to (3.24). This completes the proof. □

Assumption 1.1 ensures the existence of $\mathscr{V} \in \mathscr{L}\big(L^2(\mathbb{R}^3)\big)$ such that for any $x \in \mathbb{R}^3$,

$$|V|^{\frac{1}{2}}(x) = \mathscr{V} F_{\perp}^{\frac{1}{2}}(x_1, x_2) G^{\frac{1}{2}}(x_3). \tag{3.41}$$

Therefore, the boundedness of V together with Lemmas 3.1–3.2, (3.2), and (3.41), imply that V is relatively compact with respect to $D_m(b, 0)$.

Since for $k \in \mathcal{D}_{\pm}^{*}(\epsilon)$ we have

$$z_{\widetilde{m}}(k) = \frac{\widetilde{m}(1 + k^2)}{1 - k^2} \in \mathbb{C} \setminus \big\{(-\infty, -m] \cup [m, +\infty)\big\},$$

where $\widetilde{m} \in \{\pm m\}$, then this together with Lemmas 3.1–3.2, (3.2) and (3.41) give the following:

Lemma 3.3. *For* $\widetilde{m} \in \{\pm m\}$ *and* $z_{\widetilde{m}}(k) = \frac{\widetilde{m}(1+k^2)}{1-k^2}$, *the operator-valued functions*

$$\mathcal{D}_{\pm}^{*}(\epsilon) \ni k \longmapsto \mathcal{T}_V\big(z_{\widetilde{m}}(k)\big) := \tilde{J} |V|^{\frac{1}{2}} \big(D_m(b, 0) - z_{\widetilde{m}}(k)\big)^{-1} |V|^{\frac{1}{2}}$$

are holomorphic with values in $\mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$, $\tilde{J}$ *being defined by the polar decomposition* $V = \tilde{J}|V|$.

3.2. Reduction of the problem.

We show how we can reduce the investigation of the discrete spectrum of $D_m(b, V)$ to that of zeros of holomorphic functions.

In the sequel, the definition of the q-regularized determinant $\det_{\lceil q \rceil}(\cdot)$ is recalled in Appendix A by (A.2). As in Lemma 3.3, the operator-valued function $V\big(D_m(b, 0) - \cdot\big)^{-1}$ is analytic on $\mathcal{D}_{\widetilde{m}}^{\pm}(\eta)$ with values in $\mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$. Hence, the following characterisation

$$z \in \mathrm{sp}_{\mathrm{disc}}\big(D_m(b, V)\big) \Leftrightarrow f(z) := {\det}_{\lceil q \rceil} \Big(I + V\big(D_m(b, 0) - z\big)^{-1}\Big) = 0 \tag{3.42}$$

holds; see for instance [Sim79, Chap. 9] for more details. The fact that the operator-valued function $V\big(D_m(b, 0) - \cdot\big)$ is holomorphic on $\mathcal{D}_{\widetilde{m}}^{\pm}(\eta)$ implies that the same happens for the function $f(\cdot)$ by Property (d) of Appendix A. Furthermore, the algebraic multiplicity of z as discrete eigenvalue of

$D_m(b, V)$ is equal to its order as zero of $f(\cdot)$ (this claim is a well known fact, see for instance [Han15, Proof of Theorem 4.10 (v)] for an idea of proof).

In the next proposition, the quantity $\mathrm{Ind}_{\mathscr{C}}(\cdot)$ in the RHS of (3.43) is recalled in Appendix B by (B.2).

Proposition 3.4. *The following assertions are equivalent:*

(i) $z_{\widetilde{m}}(k_0) = \frac{\widetilde{m}(1+k_0^2)}{1-k_0^2} \in \mathcal{D}_{\widetilde{m}}^{\pm}(\eta)$ *is a discrete eigenvalue of* $D_m(b, V)$.

(ii) $\det_{\lceil q \rceil}\big(I + \mathcal{T}_V\big(z_{\widetilde{m}}(k_0)\big)\big) = 0$.

(iii) -1 *is an eigenvalue of* $\mathcal{T}_V\big(z_{\widetilde{m}}(k_0)\big)$.

Moreover,

$$\mathrm{mult}\big(z_{\widetilde{m}}(k_0)\big) = \mathrm{Ind}_{\mathscr{C}}\Big(I + \mathcal{T}_V\big(z_{\widetilde{m}}(\cdot)\big)\Big), \tag{3.43}$$

$\mathscr{C}$ *being a small contour positively oriented, containing* k_0 *as the unique point* $k \in \mathcal{D}_{\pm}^{*}(\epsilon)$ *verifying* $z_{\widetilde{m}}(k) \in \mathcal{D}_{\widetilde{m}}^{\pm}(\eta)$ *is a discrete eigenvalue of* $D_m(b, V)$.

Proof. The equivalence (i)⇔(ii) follows obviously from (3.42) and the equality

$$\det_{\lceil q \rceil}\Big(I + V\big(D_m(b,0) - z\big)^{-1}\Big) = \det_{\lceil q \rceil}\Big(I + \tilde{J}|V|^{\frac{1}{2}}\big(D_m(b,0) - z\big)^{-1}|V|^{\frac{1}{2}}\Big),$$

see Property (b) of Appendix A.

The equivalence (ii)⇔(iii) is a direct consequence of Property (c) of Appendix A.

It only remains to prove (3.43). According to the discussion just after (3.42), for $\mathscr{C}'$ a small contour positively oriented containing $z_{\widetilde{m}}(k_0)$ as the unique discrete eigenvalue of $D_m(b, V)$, we have

$$\mathrm{mult}\big(z_{\widetilde{m}}(k_0)\big) = \mathrm{Ind}_{\mathscr{C}'} f, \tag{3.44}$$

f being the function defined by (3.42). The RHS of (3.44) is the index defined by (B.1), of the holomorphic function f with respect to $\mathscr{C}'$. Now, (3.43) follows directly from the equality

$$\mathrm{Ind}_{\mathscr{C}'} f = \mathrm{Ind}_{\mathscr{C}}\Big(I + \mathcal{T}_V\big(z_{\widetilde{m}}(\cdot)\big)\Big),$$

see for instance [BBR14, (2.6)] for more details. □

4. Study of the (weighted) free resolvent

We split $\mathcal{T}_V\big(z_{\widetilde{m}}(k)\big)$ into a singular part at $k = 0$, and an analytic part in $\mathcal{D}_{\pm}^{*}(\epsilon)$ which is continuous on $\overline{\mathcal{D}_{\pm}^{*}(\epsilon)}$, with values in $\mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$.

For $z := z_{\pm m}(k)$, set

$$\mathcal{T}_1^V\big(z_{\pm m}(k)\big) := \tilde{J}|V|^{1/2}\left[p \otimes \mathscr{R}\left(k^2(z \pm m)^2\right)\right]\begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}|V|^{1/2}, \tag{4.1}$$

$$\begin{aligned}\mathcal{T}_2^V\big(z_{\pm m}(k)\big) := {} & \tilde{J}|V|^{1/2}\left[p \otimes (-i\partial_{x_3})\mathscr{R}\left(k^2(z \pm m)^2\right)\right]\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}|V|^{1/2} \\ & + \tilde{J}|V|^{1/2}\big(D_m(b,0) - z\big)^{-1}\mathbf{Q}|V|^{1/2}.\end{aligned} \tag{4.2}$$

Then, (3.2) combined with (3.3) imply that

$$\mathcal{T}_V\big(z_{\pm m}(k)\big) = \mathcal{T}_1^V\big(z_{\pm m}(k)\big) + \mathcal{T}_2^V\big(z_{\pm m}(k)\big). \tag{4.3}$$

Remark 4.1.

(i) For $z = z_m(k)$, we have

$$\operatorname{Im}\big(k(z+m)\big) = \frac{2m(1+|k|^2)\operatorname{Im}(k)}{|1+k^2|^2}. \tag{4.4}$$

Therefore, according to the choice (2.5) of the complex square root, we have respectively

$$\sqrt{k^2(z+m)^2} = \pm k(z+m) \quad for \quad k \in \mathcal{D}_\pm^*(\epsilon). \tag{4.5}$$

(ii) In the case $z = z_{-m}(k)$, we have

$$\operatorname{Im}\big(k(z-m)\big) = -\frac{2m(1+|k|^2)\operatorname{Im}(k)}{|1+k^2|^2}, \tag{4.6}$$

so that

$$\sqrt{k^2(z-m)^2} = \mp k(z-m) \quad for \quad k \in \mathcal{D}_\pm^*(\epsilon). \tag{4.7}$$

In what follows below, we focus on the study of the operator $\mathcal{T}_V\big(z_m(k)\big)$, i.e., near m. The same arguments yield that of the operator $\mathcal{T}_V\big(z_{-m}(k)\big)$ associated to $-m$ (see Remark 4.3).

Defining $G_\pm$ as the multiplication operators by the functions

$$G_\pm : \mathbb{R} \ni x_3 \mapsto G^{\pm\frac{1}{2}}(x_3),$$

we have

$$\begin{aligned}\mathcal{T}_1^V\big(z_m(k)\big) = {} & \\ & \tilde{J}|V|^{1/2}G_-\Big[p \otimes G_+\mathscr{R}\big(k^2(z+m)^2\big)G_+\Big]\begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}G_-|V|^{1/2}.\end{aligned} \tag{4.8}$$

Remark 4.1(i) together with (3.4) imply that $G_+\mathscr{R}\left(k^2(z+m)^2\right)G_+$ admits the integral kernel

$$\pm\, G^{\frac{1}{2}}(x_3)\frac{ie^{\pm ik(z+m)|x_3-x_3'|}}{2k(z+m)}G^{\frac{1}{2}}(x_3'), \quad k \in \mathcal{D}_\pm^*(\epsilon). \tag{4.9}$$

Then, from (4.9) we deduce that

$$G_+ \mathscr{R}\left(k^2(z+m)^2\right) G_+ = \pm \frac{1}{k(z+m)} a + b_m(k), \quad k \in \mathcal{D}_\pm^*(\epsilon), \tag{4.10}$$

where $a : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is the rank-one operator given by

$$a(u) := \frac{i}{2} \langle u, G_+ \rangle G_+, \tag{4.11}$$

and $b_m(k)$ is the operator with integral kernel

$$\pm\, G^{\frac{1}{2}}(x_3) i \frac{\mathrm{e}^{\pm ik(z+m)|x_3 - x_3'|} - 1}{2k(z+m)} G^{\frac{1}{2}}(x_3'). \tag{4.12}$$

Note that $-2ia = c^* c$, where $c : L^2(\mathbb{R}) \longrightarrow \mathbb{C}$ satisfies $c(u) := \langle u, G_+ \rangle$ and $c^* : \mathbb{C} \longrightarrow L^2(\mathbb{R})$ verifies $c^*(\lambda) = \lambda G_+$. Therefore, by combining (4.10), (4.11) with (4.12), we get for $k \in \mathcal{D}_\pm^*(\epsilon)$

$$p \otimes G_+ \mathscr{R}\left(k^2(z+m)^2\right) G_+ = \pm \frac{i}{2k(z+m)} p \otimes c^* c + p \otimes s_m(k), \tag{4.13}$$

where $s_m(k)$ is the operator acting from $G^{\frac{1}{2}}(x_3) L^2(\mathbb{R})$ to $G^{-\frac{1}{2}}(x_3) L^2(\mathbb{R})$ with integral kernel

$$\pm\, \frac{1 - \mathrm{e}^{\pm ik(z+m)|x_3 - x_3'|}}{2ik(z+m)}. \tag{4.14}$$

In Remark 4.3, $s_{-m}(k)$ is the corresponding operator with m replaced by $-m$ and ± replaced by ∓ in (4.14). Now, putting together (4.8) and (4.13), we get for $k \in \mathcal{D}_\pm^*(\epsilon)$

(4.15)

$$\begin{aligned} \mathcal{T}_1^V\left(z_m(k)\right) = {} & \pm \frac{i\tilde{J}}{2k(z+m)} |V|^{\frac{1}{2}} G_- (p \otimes c^* c) \begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G_- |V|^{\frac{1}{2}} \\ & + \tilde{J} |V|^{\frac{1}{2}} G_- p \otimes s_m(k) \begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G_- |V|^{\frac{1}{2}}. \end{aligned}$$

Introduce the operator

$$K_{\pm m} := \frac{1}{\sqrt{2}} (p \otimes c) \begin{pmatrix} 1-1_\mp & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-1_\pm & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G_- |V|^{\frac{1}{2}}, \quad 1_- = 0, \quad 1_+ = 1. \tag{4.16}$$

It is well known from [Hal98, Theorem 2.3] that p admits a continuous integral kernel $\mathcal{P}(x_\perp, x_\perp')$, $x_\perp$, $x_\perp' \in \mathbb{R}^2$. Then, we have

$$K_{\pm m} : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^2)$$

with

$$
\begin{aligned}
&(K_{\pm m}\psi)(x_\perp) \\
&\quad = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \mathcal{P}(x_\perp, x_\perp') \begin{pmatrix} 1-1_\mp & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-1_\pm & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} |V|^{\frac{1}{2}}(x_\perp', x_3')\psi(x_\perp', x_3')dx_\perp' dx_3'.
\end{aligned}
$$

Obviously, the operator $K_{\pm m}^* : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^3)$ satisfies

$$
(K_{\pm m}^*\varphi)(x_\perp, x_3) = \frac{1}{\sqrt{2}}|V|^{\frac{1}{2}}(x_\perp, x_3) \begin{pmatrix} 1-1_\mp & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-1_\pm & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} (p\varphi)(x_\perp).
$$

Noting that $K_{\pm m}K_{\pm m}^* : L^2(\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}^2)$ verifies

$$
K_{\pm m}K_{\pm m}^* = \begin{pmatrix} 1-1_\mp & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-1_\pm & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} p\mathbf{V}_{\pm m}p, \tag{4.17}
$$

$\mathbf{V}_{\pm m}$ being the multiplication operators by the functions (also noted) $\mathbf{V}_{\pm m}$ defined by (2.2). Thus, by combining (4.15) and (4.16) we obtain for $k \in \mathcal{D}_\pm^*(\epsilon)$

$$
\begin{aligned}
\mathcal{T}_1^V\big(z_m(k)\big) = \pm\frac{i\tilde{J}}{k}K_m^*K_m + i\tilde{J}kK_{-m}^*K_{-m} \\
+ \tilde{J}|V|^{\frac{1}{2}}G_-p \otimes s_m(k) \begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G_-|V|^{\frac{1}{2}}.
\end{aligned} \tag{4.18}
$$

Now, for $\lambda \in \mathbb{R}_+^*$, we define $\big(-\partial_{x_3}^2 - \lambda\big)^{-1}$ as the operator with integral kernel

$$
I_\lambda(x_3, x_3') := \lim_{\delta \downarrow 0} I_{\lambda + i\delta}(x_3, x_3') = \frac{ie^{i\sqrt{\lambda}|x_3 - x_3'|}}{2\sqrt{\lambda}}. \tag{4.19}
$$

Here $I_z(\cdot)$ is given by (3.4). Therefore, it can be proved, using a limiting absorption principle, that the operator-valued functions

$$
\overline{\mathcal{D}_\pm^*(\epsilon)} \ni k \mapsto G_+s_m(k)G_+ \in \mathbf{S}_2\big(L^2(\mathbb{R})\big)
$$

are well defined and continuous similarly to [Rai10, Proposition 4.2]. We thus arrive to the following:

Proposition 4.2. *Let $k \in \mathcal{D}_\pm^*(\epsilon)$. Then,*

$$
\mathcal{T}_V\big(z_m(k)\big) = \pm\frac{i\tilde{J}}{k}\mathscr{B}_m + \mathscr{A}_m(k), \quad \mathscr{B}_m := K_m^*K_m, \tag{4.20}
$$

where $\mathscr{A}_m(k) \in \mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$ given by

$$
\begin{aligned}
\mathscr{A}_m(k) := i\tilde{J}kK_{-m}^*K_{-m} + \tilde{J}|V|^{\frac{1}{2}}G_-p \otimes s_m(k) \\
\times \begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G_-|V|^{\frac{1}{2}} + \mathcal{T}_2^V\big(z_m(k)\big)
\end{aligned}
$$

is holomorphic in $\mathcal{D}_\pm^*(\epsilon)$ and continuous on $\overline{\mathcal{D}_\pm^*(\epsilon)}$.

Remark 4.3.

(i) Identity (4.17) implies that for any $r > 0$, we have

$$\operatorname{Tr} \mathbf{1}_{(r,\infty)} \left(K_{\pm m}^* K_{\pm m} \right) = \operatorname{Tr} \mathbf{1}_{(r,\infty)} \left(K_{\pm m} K_{\pm m}^* \right) = \operatorname{Tr} \mathbf{1}_{(r,\infty)} \big(p \mathbf{V}_{\pm m} p \big). \tag{4.21}$$

(ii) For V verifying Assumption 2.3, Proposition 4.2 holds with $\tilde{J}$ replaced by $J\Phi$, $J := sign(W)$, and in (4.21) $\mathbf{V}_{\pm m}$ replaced by $\mathbf{W}_{\pm m}$.

(iii) Near $-m$, taking into account Remark 4.1(ii), Proposition 4.2 holds with

$$\mathcal{T}_V \big(z_{-m}(k) \big) = \mp \frac{i\tilde{J}}{k} \mathscr{B}_{-m} + \mathscr{A}_{-m}(k), \quad \mathscr{B}_{-m} := K_{-m}^* K_{-m}, \tag{4.22}$$

and

$$\begin{aligned} \mathscr{A}_{-m}(k) := i\tilde{J}kK_m^*K_m + &\tilde{J}|V|^{\frac{1}{2}} G_- p \otimes s_{-m}(k) \\ &\times \begin{pmatrix} z+m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & z-m & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G_- |V|^{\frac{1}{2}} + \mathcal{T}_2^V \big(z_{-m}(k) \big). \end{aligned}$$

5. Proof of the main results

5.1. Proof of Theorem 2.2. It suffices to prove that both sums in the LHS of (2.14) are bounded by the RHS. We only give the proof for the first sum. For the second one, the estimate follows similarly by using Remark 2.1(iii),(iv), Proposition 4.2, and Remark 4.3(i),(iii).

In what follows below,

$$N\big(D_m(b,V)\big) := \big\{ \langle D_m(b,V)f, f \rangle : f \in \operatorname{Dom}\big(D_m(b,V)\big), \|f\|_{L^2} = 1 \big\}$$

denotes the numerical range of the operator $D_m(b,V)$. It satisfies the inclusion $\operatorname{sp}\big(D_m(b,V)\big) \subseteq \overline{N\big(D_m(b,V)\big)}$, see, e.g., [Dav07, Lemma 9.3.14].

The proof of the theorem uses the following:

Proposition 5.1. *Let $0 < s_0 < \epsilon$ be small enough. Then, for any*

$$k \in \{0 < s < |k| < s_0\} \cap \mathcal{D}_\pm^*(\epsilon),$$

the following properties hold:

(i) *$z_{\pm m}(k) \in \operatorname{sp}_{\mathrm{disc}}^+\big(D_m(b,V)\big)$ near $\pm m$ if and only if k is a zero of the determinants*

$$\mathscr{D}_{\pm m}(k,s) := \det \big(I + \mathscr{K}_{\pm m}(k,s) \big), \tag{5.1}$$

where $\mathscr{K}_{\pm m}(k,s)$ are finite-rank operators analytic with respect to k such that

$$\operatorname{rank} \mathscr{K}_{\pm m}(k,s) = \mathcal{O}\Big(\operatorname{Tr} \mathbf{1}_{(s,\infty)} \big(p \mathbf{V}_{\pm m} p \big) + 1 \Big), \quad \| \mathscr{K}_{\pm m}(k,s) \| = \mathcal{O}\left(s^{-1} \right), \tag{5.2}$$

uniformly with respect to $s < |k| < s_0$.

(ii) *Moreover, if* $z_{\pm m}(k_0) \in \mathrm{sp}^+_{\mathrm{disc}}\big(D_m(b,V)\big)$ *near* $\pm m$, *then*

$$\mathrm{mult}\big(z_{\pm m}(k_0)\big) = \mathrm{Ind}_{\mathscr{C}}\ \big(I + \mathscr{K}_{\pm m}(\cdot, s)\big) = \mathrm{mult}(k_0), \tag{5.3}$$

where $\mathscr{C}$ *is chosen as in* (3.43), *and* $\mathrm{mult}(k_0)$ *is the multiplicity of* k_0 *as zero of* $\mathscr{D}_{\pm m}(\cdot, s)$.

(iii) *If* $z_{\pm m}(k)$ *satisfies*

$$\mathrm{dist}\big(z_{\pm m}(k), \overline{N\big(D_m(b,V)\big)}\big) > \varsigma > 0,$$

then $I + \mathscr{K}_{\pm m}(k,s)$ *are invertible and satisfy*

$$\Big\|\big(I + \mathscr{K}_{\pm m}(k,s)\big)^{-1}\Big\| = \mathcal{O}\left(\varsigma^{-1}\right)$$

uniformly with respect to $s < |k| < s_0$.

Proof. (i)–(ii) By Proposition 4.2 and Remark 4.3(iii), the operator-valued functions

$$k \mapsto \mathscr{A}_{\pm m}(k) \in \mathbf{S}_q\big(L^2(\mathbb{R}^3)\big) \tag{5.4}$$

are continuous near zero. Then, for s_0 small enough, there exists $\mathscr{A}_{0,\pm m}$ finite-rank operators which do not depend on k, and $\widetilde{\mathscr{A}_{\pm m}}(k) \in \mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$ continuous near zero satisfying $\|\widetilde{\mathscr{A}_{\pm m}}(k)\| < \frac{1}{4}$ for $|k| \le s_0$, such that

$$\mathscr{A}_{\pm m}(k) = \mathscr{A}_{0,\pm m} + \widetilde{\mathscr{A}_{\pm m}}(k). \tag{5.5}$$

Let $\mathscr{B}_{\pm m}$ be the operators defined respectively by (4.20) and (4.22). Then, with the help of the decomposition

$$\mathscr{B}_{\pm m} = \mathscr{B}_{\pm m}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}_{\pm m}) + \mathscr{B}_{\pm m}\mathbf{1}_{]\frac{1}{2}s,\infty[}(\mathscr{B}_{\pm m}), \tag{5.6}$$

and using that $\Big\|\pm\frac{i\tilde{J}}{k}\mathscr{B}_{\pm m}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}_{\pm m}) + \widetilde{\mathscr{A}_{\pm m}}(k)\Big\| < \frac{3}{4}$ for $0 < s < |k| < s_0$, we obtain

$$\begin{aligned}\big(I + \mathcal{T}_V\big(z_{\pm m}(k)\big)\big) \\ = \big(I + \mathscr{K}_{\pm m}(k,s)\big)\left(I \pm \frac{i\tilde{J}}{k}\mathscr{B}_{\pm m}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}_{\pm m}) + \widetilde{\mathscr{A}_{\pm m}}(k)\right),\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}\mathscr{K}_{\pm m}(k,s) := \left(\pm\frac{i\tilde{J}}{k}\mathscr{B}_{\pm m}\mathbf{1}_{]\frac{1}{2}s,\infty[}(\mathscr{B}_{\pm m}) + \mathscr{A}_{0,\pm m}\right) \\ \times\left(I \pm \frac{i\tilde{J}}{k}\mathscr{B}_{\pm m}\mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}_{\pm m}) + \widetilde{\mathscr{A}_{\pm m}}(k)\right)^{-1}.\end{aligned} \tag{5.8}$$

Observe that $\mathscr{K}_{\pm m}(k,s)$ are finite-rank operators with ranks of order

$$\mathcal{O}\Big(\mathrm{Tr}\,\mathbf{1}_{(\frac{1}{2}s,\infty)}(\mathscr{B}_{\pm m}) + 1\Big) = \mathcal{O}\Big(\mathrm{Tr}\,\mathbf{1}_{(s,\infty)}\big(p\mathbf{V}_{\pm m}p\big) + 1\Big), \tag{5.9}$$

according to (4.21). Moreover, their norms are of order $\mathcal{O}\big(|k|^{-1}\big) = \mathcal{O}\big(s^{-1}\big)$. From above we know that for $0 < s < |k| < s_0$ we have

$$\left\| \pm \frac{i\tilde{J}}{k} \mathscr{B}_{\pm m} \mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}_{\pm m}) + \widetilde{\mathscr{A}_{\pm m}}(k) \right\| < \frac{3}{4} < 1.$$

Then, we obtain

$$\mathrm{Ind}_{\mathscr{C}} \left(I \pm \frac{i\tilde{J}}{k} \mathscr{B}_{\pm m} \mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}_{\pm m}) + \widetilde{\mathscr{A}}(k) \right) = 0 \tag{5.10}$$

from [GohGK90, Theorem 4.4.3]. Therefore, equalities (5.3) follow by applying to (5.7) the properties of the index of a finite meromorphic operator-valued function recalled in the Appendix B. Proposition 3.4 together with (5.7) imply that $z_{\pm m}(k)$ belongs to $\mathrm{sp}_{\mathrm{disc}}^{+}\big(D_m(b,V)\big)$ near $\pm m$ if and only if k is a zero of the determinants $\mathscr{D}_{\pm m}(k,s)$ defined by (5.1).

(iii) From (5.7), we deduce that

$$\begin{aligned} I + \mathscr{K}_{\pm m}(k,s) \\ = \Big(I + \mathcal{T}_V\big(z_{\pm m}(k)\big)\Big) \left(I + \frac{\tilde{J}}{k} \mathscr{B}_{\pm m} \mathbf{1}_{[0,\frac{1}{2}s]}(\mathscr{B}_{\pm m}) + \widetilde{\mathscr{A}_{\pm m}}(k) \right)^{-1}, \end{aligned} \tag{5.11}$$

for $0 < s < |k| < s_0$. With the help of the resolvent equation, it can be shown that

$$\begin{aligned} I = \\ \Big(I + \tilde{J}|V|^{1/2}\big(D_m(b,0) - z\big)^{-1}|V|^{1/2}\Big)\Big(I - \tilde{J}|V|^{1/2}\big(D_m(b,V) - z\big)^{-1}|V|^{1/2}\Big). \end{aligned} \tag{5.12}$$

Then, for $z_{\pm m}(k) \in \rho\big(D_m(b,V)\big)$, obviously we have

$$\Big(I + \mathcal{T}_V\big(z_{\pm m}(k)\big)\Big)^{-1} = I - \tilde{J}|V|^{1/2}\big(D_m(b,V) - z_{\pm m}(k)\big)^{-1}|V|^{1/2}. \tag{5.13}$$

This together with (5.11) imply the invertibility of $I + \mathscr{K}_{\pm m}(k,s)$ for $0 < s < |k| < s_0$, and according to [Dav07, Lemma 9.3.14] it satisfies

$$\begin{aligned} \Big\| \big(I + \mathscr{K}_{\pm m}(k,s)\big)^{-1} \Big\| &= \mathcal{O}\Big(1 + \Big\| |V|^{1/2}\big(D_m(b,V) - z_{\pm m}(k)\big)^{-1}|W|^{1/2} \Big\| \Big) \\ &= \mathcal{O}\left(1 + \mathrm{dist}\big(z_{\pm m}(k), \overline{N\big(D_m(b,V)\big)}\big)^{-1}\right) \\ &= \mathcal{O}\left(\varsigma^{-1}\right), \end{aligned}$$

for $\mathrm{dist}\big(z_{\pm m}(k), \overline{N\big(D_m(b,V)\big)}\big) > \varsigma > 0$. This completes the proof. □

End of the proof of Theorem 2.2. Now, from Proposition 5.1(i), we obtain for $0 < s < |k| < s_0$

$$\mathscr{D}_{\pm m}(k,s) = \prod_{j=1}^{\mathcal{O}\left(\mathrm{Tr}\,\mathbf{1}_{(s,\infty)}(p\mathbf{V}_{\pm m}p)+1\right)} \big(1+\lambda_{j,\pm m}(k,s)\big) \tag{5.14}$$
$$= \mathcal{O}(1)\exp\Big(\mathcal{O}\big(\mathrm{Tr}\,\mathbf{1}_{(s,\infty)}\big(p\mathbf{V}_{\pm m}p\big)+1\big)|\ln s|\Big),$$

where the $\lambda_{j,\pm m}(k,s)$ are the eigenvalues of $\mathscr{K}_{\pm m} := \mathscr{K}_{\pm m}(k,s)$ which satisfy $|\lambda_{j,\pm m}(k,s)| = \mathcal{O}\left(s^{-1}\right)$ so that $\ln\big|1+\lambda_{j,\pm m}(k,s)\big| = \mathcal{O}\big(|\ln s|\big)$ (for s small enough). Otherwise, we have for $0 < s < |k| < s_0$

$$\mathscr{D}_{\pm m}(k,s)^{-1} = \det\big(I+\mathscr{K}_{\pm m}\big)^{-1} = \det\big(I-\mathscr{K}_{\pm m}(I+\mathscr{K}_{\pm m})^{-1}\big)$$

if $\mathrm{dist}\big(z_{\pm m}(k), \overline{N\big(D_m(b,V)\big)}\big) > \varsigma > 0$. Then, as in (5.14), it can be shown that

$$|\mathscr{D}_{\pm m}(k,s)| \geq C\exp\Big(-C\big(\mathrm{Tr}\,\mathbf{1}_{(s,\infty)}\big(p\mathbf{V}_{\pm m}p\big)+1\big)\big(|\ln\varsigma|+|\ln s|\big)\Big). \tag{5.15}$$

To conclude, we need the following Jensen lemma (see for instance [BBR07, Lemma 6] for a simple proof).

Lemma 5.2. *Let Δ be a simply connected sub-domain of $\mathbb{C}$ and let g be holomorphic in Δ with continuous extension to $\overline{\Delta}$. Assume that there exists $\lambda_0 \in \Delta$ such that $g(\lambda_0) \neq 0$ and $g(\lambda) \neq 0$ for $\lambda \in \partial\Delta$ (the boundary of Δ). Let $\lambda_1, \lambda_2, \dots, \lambda_N \in \Delta$ be the zeros of g repeated according to their multiplicity. For any domain $\Delta' \Subset \Delta$, there exists $C' > 0$ such that $N(\Delta', g)$, the number of zeros λ_j of g contained in Δ' satisfies*

$$N(\Delta', g) \leq C'\left(\int_{\partial\Delta}\ln|g(\lambda)|d\lambda - \ln|g(\lambda_0)|\right). \tag{5.16}$$

Consider the domains

$$\Delta_\pm := \big\{r < |k| < 2r : |\mathrm{Re}(k)| > \sqrt{\nu} : |\mathrm{Im}(k)| > \sqrt{\nu} : \nu > 0\big\} \cap \mathcal{D}_\pm^*(\epsilon)$$

with $0 < r < \epsilon/2$, and some $k_0 \in \Delta_\pm$ satisfying

$$\mathrm{dist}\big(z_{\pm m}(k_0), \overline{N\big(D_m(b,V)\big)}\big) > \varsigma > 0.$$

Then, we get that the first sum in the LHS of (2.14) is bounded by the RHS by using Lemma 5.2 with the functions $g = g_{\pm m}(k) := \mathscr{D}_{\pm m}(k,r)$, together with (5.14) and (5.15). This concludes the proof of Theorem 2.2. □

5.2. Proof of Theorem 2.5. It will only be given for the case $\mathrm{Arg}\,\Phi \in (0,\pi)$. To prove the case $\mathrm{Arg}\,\Phi \in -(0,\pi)$, it suffices to argue similarly by replacing k by $-k$.

Remark 2.4(i), together with Proposition 4.2 and Remark 4.3(ii),(iii), imply that

$$\mathcal{T}_{\varepsilon V}\big(z_{\pm m}(k)\big) = \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} + \varepsilon\mathscr{A}_{\pm m}(k), \qquad k \in \mathcal{D}_\pm^*(\epsilon), \tag{5.17}$$

where $\mathscr{B}_{\pm m}$ are positive self-adjoint operators which do not depend on k, and $\mathscr{A}_{\pm m}(k) \in \mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$ are holomorphic in $\mathcal{D}_\pm^*(\epsilon)$ and continuous on $\overline{\mathcal{D}_\pm^*(\epsilon)}$. Noting that

$$I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} = \frac{iJ\Phi}{k}(\varepsilon\mathscr{B}_{\pm m} - iJk\Phi^{-1}), \tag{5.18}$$

it is easy to see that $I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m}$ are invertible whenever $iJk\Phi^{-1} \notin \mathrm{sp}\,(\varepsilon\mathscr{B}_{\pm m})$. Moreover, we have

$$\left\| \left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} \right)^{-1} \right\| \le \frac{|k|}{\sqrt{\big(J\,\mathrm{Im}(k\Phi^{-1})\big)_+^2 + |\,\mathrm{Re}(k\Phi^{-1})|^2}}. \tag{5.19}$$

Therefore,

$$\left\| \left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} \right)^{-1} \right\| \le \sqrt{1+\delta^{-2}} \tag{5.20}$$

for $k \in \Phi\mathcal{C}_\delta(J)$, uniformly with respect to $0 < |k| < r_0$. Then, we deduce from (5.17) that

$$I + \mathcal{T}_{\varepsilon V}\big(z_{\pm m}(k)\big) = \big(I + A_{\pm m}(k)\big)\left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} \right), \tag{5.21}$$

where

$$A_{\pm m}(k) := \varepsilon\mathscr{A}_{\pm m}(k)\left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} \right)^{-1} \in \mathbf{S}_q\big(L^2(\mathbb{R}^3)\big). \tag{5.22}$$

Now, by exploiting the continuity of $\mathscr{A}_{\pm m}(k) \in \mathbf{S}_q\big(L^2(\mathbb{R}^3)\big)$ near $k = 0$, it can be proved that $\|\mathscr{A}_{\pm m}(k)\| \le C$ for some $C > 0$ constant (not depending on k). This together with (5.20) and (5.22) imply clearly the invertibility of $I + \mathcal{T}_{\varepsilon V}\big(z_{\pm m}(k)\big)$ for $k \in \Phi\mathcal{C}_\delta(J)$ and $\varepsilon < \big(C\sqrt{1+\delta^{-2}}\big)^{-1}$. Thus, $z_{\pm m}(k)$ is not a discrete eigenvalue near $\pm m$.

5.3. Proof of Theorem 2.8. Denote $(\mu_j^{\pm m})_j$ the sequences of the decreasing nonzero eigenvalues of $p\mathbf{W}_{\pm m}p$ taking into account the multiplicity. If Assumption 2.3 holds, it can be proved that there exists a constant $\nu_{\pm m} > 0$ such that

$$\#\big\{ j : \mu_j^{\pm m} - \mu_{j+1}^{\pm m} > \nu_{\pm m}\mu_j^{\pm m} \big\} = \infty. \tag{5.23}$$

Since $\mathscr{B}_{\pm m}$ and $p\mathbf{W}_{\pm m}p$ have the same nonzero eigenvalues, then, there exist decreasing sequences $(r_\ell^{\pm m})_\ell$, $r_\ell^{\pm m} \searrow 0$ with $r_\ell^{\pm m} > 0$, such that

$$\mathrm{dist}\big(r_\ell^{\pm m}, \mathrm{sp}\,(\mathscr{B}_{\pm m})\big) \ge \frac{\nu r_\ell^{\pm m}}{2}, \quad \ell \in \mathbb{N}. \tag{5.24}$$

Furthermore, there exists paths $\widetilde{\Sigma}_\ell^{\pm m} := \partial\Lambda_\ell^{\pm m}$ with

$$\Lambda_\ell^{\pm m} := \big\{ \tilde{k} \in \mathbb{C} : 0 < |\tilde{k}| < r_0 : |\,\mathrm{Im}(\tilde{k})| \le \delta\,\mathrm{Re}(\tilde{k}) : r_{\ell+1}^{\pm m} \le \mathrm{Re}(\tilde{k}) \le r_\ell^{\pm m} \big\}, \tag{5.25}$$

(see Figure 5.1) enclosing the eigenvalues of $\mathscr{B}_{\pm m}$ lying in $[r_{\ell+1}^{\pm m}, r_\ell^{\pm m}]$.

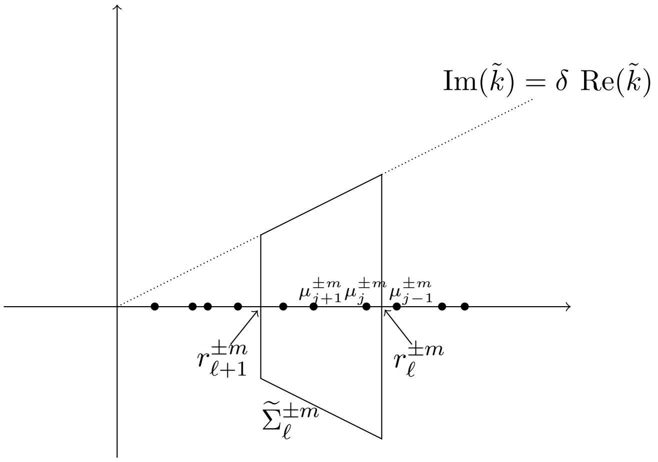

FIGURE 5.1. Representation of the paths $\widetilde{\Sigma}_\ell^{\pm m} = \partial \Lambda_\ell^{\pm m}$, $\ell \in \mathbb{N}$.

Clearly, the operators $\tilde{k} - \mathscr{B}_{\pm m}$ are invertible for $\tilde{k} \in \widetilde{\Sigma}_\ell^{\pm m}$. Moreover, it can be easily checked that

$$\left\| (\tilde{k} - \mathscr{B}_{\pm m})^{-1} \right\| \leq \frac{\max\left(\delta^{-1}\sqrt{1+\delta^2}, \min^{-1}(\frac{1}{4}\nu_{\pm m}^2, 1)\right)}{|\tilde{k}|}. \tag{5.26}$$

Set $\Sigma_\ell^{\pm m} := -iJ\varepsilon\Phi\widetilde{\Sigma}_\ell^{\pm m}$. The construction of the paths $\Sigma_\ell^{\pm m}$ together with (5.26) imply that $I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m}$ are invertible for $k \in \Sigma_\ell^{\pm m}$ with

$$\left\| \left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} \right)^{-1} \right\| \leq \max\left(\delta^{-1}\sqrt{1+\delta^2}, \overset{-1}{\min}\left(\frac{1}{4}\nu_{\pm m}^2, 1\right)\right). \tag{5.27}$$

Hence, we have

$$\begin{aligned} I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} + \varepsilon\mathscr{A}_{\pm m}(k) \\ = \left(I + \varepsilon\mathscr{A}_{\pm m}(k)\left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} \right)^{-1} \right)\left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} \right), \end{aligned} \tag{5.28}$$

for $k \in \Sigma_\ell^{\pm m}$. Now, if we choose $0 < \varepsilon \leq \varepsilon_0$ small enough and use Property (e) given by (A.4), we get for any $k \in \Sigma_\ell^{\pm m}$

$$\left| \det_{\lceil q \rceil}\left[I + \varepsilon\mathscr{A}_{\pm m}(k)\left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} \right)^{-1} \right] - 1 \right| < 1. \tag{5.29}$$

Therefore, from the Rouché Theorem, we know that the number of zeros of $\det_{\lceil q \rceil}\big(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} + \varepsilon\mathscr{A}_{\pm m}(k) \big)$ enclosed in

$$\left\{ z_{\pm m}(k) \in \mathrm{sp}_{\mathrm{disc}}^{+}\big(D_m(b, \varepsilon V) \big) : k \in \Sigma_\ell^{\pm m} \right\}$$

taking into account the multiplicity, coincides with that of

$$\det_{\lceil q \rceil}\left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m}\right)$$

enclosed in $\left\{z_{\pm m}(k) \in \mathrm{sp}^+_{\mathrm{disc}}\big(D_m(b, \varepsilon V)\big) : k \in \Sigma_\ell^{\pm m}\right\}$ taking into account the multiplicity. The number of zeros of $\det_{\lceil q \rceil}\big(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m}\big)$ enclosed in $\left\{z_{\pm m}(k) \in \mathrm{sp}^+_{\mathrm{disc}}\big(D_m(b, \varepsilon V)\big) : k \in \Sigma_\ell^{\pm m}\right\}$ taking into account the multiplicity is equal to $\mathrm{Tr}\,\mathbf{1}_{[r_{\ell+1}^{\pm m}, r_\ell^{\pm m}]}\big(p\mathbf{W}_{\pm m}p\big)$. The zeros of

$$\det_{\lceil q \rceil}\left(I + \frac{iJ\varepsilon\Phi}{k}\mathscr{B}_{\pm m} + \varepsilon\mathscr{A}_{\pm m}(k)\right)$$

are the discrete eigenvalues of $D_m(b, \varepsilon V)$ near $\pm m$ taking into account the multiplicity. Then, this together with Proposition 3.4 and Property (B.3) applied to (5.28) give estimate (2.23). Since the sequences $(r_\ell^{\pm m})_\ell$ are infinite tending to zero, then the infiniteness of the number of the discrete eigenvalues claimed follows, which completes the proof of Theorem 2.8.

Appendix A. Reminder on Schatten–von Neumann ideals and regularized determinants

Consider a separable Hilbert space $\mathscr{H}$. Let $\mathbf{S}_\infty(\mathscr{H})$ denote the set of compact linear operators on $\mathscr{H}$, and $s_k(T)$ be the k-th singular value of $T \in \mathbf{S}_\infty(\mathscr{H})$. For $q \in [1, +\infty)$, the Schatten–von Neumann classes are defined by

$$\mathbf{S}_q(\mathscr{H}) := \Big\{T \in \mathbf{S}_\infty(\mathscr{H}) : \|T\|_{\mathbf{S}_q}^q := \sum_k s_k(T)^q < +\infty\Big\}. \tag{A.1}$$

When no confusion can arise, we write $\mathbf{S}_q$ for simplicity. If $T \in \mathbf{S}_q$ with $\lceil q \rceil := \min\big\{n \in \mathbb{N} : n \geq q\big\}$, the q-regularized determinant is defined by

$$\det_{\lceil q \rceil}(I - T) := \prod_{\mu\, \in\, \sigma(T)}\left[(1-\mu)\exp\left(\sum_{k=1}^{\lceil q \rceil - 1}\frac{\mu^k}{k}\right)\right]. \tag{A.2}$$

In particular, when $q = 1$, to simplify the notation we set

$$\det(I - T) := \det_{\lceil 1 \rceil}(I - T). \tag{A.3}$$

Let us give (see for instance [Sim77]) some elementary useful properties on this determinant.

(a) We have $\det_{\lceil q \rceil}(I) = 1$.

(b) If A, $B \in \mathscr{L}(\mathscr{H})$ with AB and BA lying in $\mathbf{S}_q$, then $\det_{\lceil q \rceil}(I - AB) = \det_{\lceil q \rceil}(I - BA)$. Here, $\mathscr{L}(\mathscr{H})$ is the set of bounded linear operators on $\mathscr{H}$.

(c) $I - T$ is an invertible operator if and only if $\det_{\lceil q \rceil}(I - T) \neq 0$.

(d) If $T : \Omega \longrightarrow \mathbf{S}_q$ is a holomorphic operator-valued function on a domain Ω, then so is $\det_{\lceil q \rceil}\big(I - T(\cdot)\big)$ on Ω.

(e) As function on $\mathbf{S}_q$, $\det_{\lceil q \rceil}(I-T)$ is Lipschitz uniformly on balls. This means that

$$\begin{aligned} \left|\det_{\lceil q \rceil}(I-T_1) - \det_{\lceil q \rceil}(I-T_2)\right| \\ \leq \|T_1 - T_2\|_{\mathbf{S}_q} \exp\left(\Gamma_q\left(\|T_1\|_{\mathbf{S}_q} + \|T_2\|_{\mathbf{S}_q} + 1\right)^{\lceil q \rceil}\right), \end{aligned} \tag{A.4}$$

[Sim77, Theorem 6.5].

Appendix B. On the index of a finite meromorphic operator-valued function

We refer for instance to [GohL09] for the definition of a finite meromorphic operator-valued function.

Let $f : \Omega \to \mathbb{C}$ be a holomorphic function in a vicinity of a contour $\mathscr{C}$ positively oriented. Then, it index with respect to $\mathscr{C}$ is given by

$$\operatorname{Ind}_{\mathscr{C}} f := \frac{1}{2i\pi} \int_{\mathscr{C}} \frac{f'(z)}{f(z)} dz. \tag{B.1}$$

Observe that if $\partial\Omega = \mathscr{C}$, then $\operatorname{ind}_{\mathscr{C}} f$ is equal to the number of zeros of f lying in Ω taking into account their multiplicity (by the residues theorem).

If $D \subseteq \mathbb{C}$ is a connected domain, $Z \subset D$ a closed pure point subset, $A : \overline{D} \backslash Z \longrightarrow \mathrm{GL}(\mathscr{H})$ a finite meromorphic operator-valued function which is Fredholm at each point of Z, the index of A with respect to the contour $\partial\Omega$ is given by

$$\operatorname{Ind}_{\partial\Omega} A := \frac{1}{2i\pi} \operatorname{Tr} \int_{\partial\Omega} A'(z) A(z)^{-1} dz = \frac{1}{2i\pi} \operatorname{Tr} \int_{\partial\Omega} A(z)^{-1} A'(z) dz. \tag{B.2}$$

We have the following well known properties:

$$\operatorname{Ind}_{\partial\Omega} A_1 A_2 = \operatorname{Ind}_{\partial\Omega} A_1 + \operatorname{Ind}_{\partial\Omega} A_2; \tag{B.3}$$

if $K(z)$ is of trace-class, then

$$\operatorname{Ind}_{\partial\Omega} (I+K) = \operatorname{Ind}_{\partial\Omega} \det (I+K), \tag{B.4}$$

see for instance to [GohGK90, Chap. 4] for more details.

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