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Normalizers and centralizers of cyclic subgroups generated by lone axis fully irreducible outer automorphisms

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Abstract. We let φ be an ageometric fully irreducible outer automorphism so that its Handel–Mosher axis bundle (Handel and Mosher, 2011) consists of a unique axis (as in Mosher and Pfaff, 2016). We show that the centralizer $\mathrm{Cen}(\langle\varphi\rangle)$ of the cyclic subgroup generated by φ equals the stabilizer $\mathrm{Stab}(\Lambda^+_\varphi)$ of the attracting lamination Λ^+_φ and is isomorphic to $\mathbb{Z}$. We further show, via an analogous result about the commensurator, that the normalizer $N(\langle\varphi\rangle)$ of $\langle\varphi\rangle$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$.

Contents

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1. Introduction

It is well known [McC94] that, given a pseudo-Anosov mapping class φ of a compact, connected and orientable surface of negative Euler characteristic, the centralizer $\mathrm{Cen}(\langle\varphi\rangle)$ and normalizer $N(\langle\varphi\rangle)$ of the cyclic subgroup $\langle\varphi\rangle$ are virtually cyclic.

We recall some history surrounding this problem for the outer automorphism groups $\mathrm{Out}(F_r)$. In [BFH97], Bestvina, Feighn, and Handel constructed for $\varphi \in \mathrm{Out}(F_r)$, a fully irreducible outer automorphism, the attracting and repelling laminations $\Lambda^+_\varphi, \Lambda^-_\varphi$. They proved that the stabilizer $\mathrm{Stab}(\Lambda^+_\varphi)$ of Λ^+_φ in $\mathrm{Out}(F_r)$ is virtually cyclic (see also [KL11, Theorem 4.4]). Let $\mathrm{Comm}(\langle\varphi\rangle)$ denote the commensurator of $\langle\varphi\rangle$. Whenever the lamination Λ^+_φ is defined we have[1]

$$\langle\varphi\rangle \leq \mathrm{Cen}(\langle\varphi\rangle) \leq \mathrm{Stab}(\Lambda^+_\varphi),$$

$$\mathrm{Cen}(\langle\varphi\rangle) \leq N(\langle\varphi\rangle) \leq \mathrm{Comm}(\langle\varphi\rangle) \leq \mathrm{Stab}(\{\Lambda^+_\varphi, \Lambda^-_\varphi\}).$$

In the fully irreducible case, the groups appearing above are all finite index subgroups of one another, and each of the inclusions may be strict (see Examples 3.4, 3.5, and 3.6).

This article is concerned with identifying the centralizer and normalizer of $\langle\varphi\rangle$ when φ is an ageometric lone axis fully irreducible outer automorphism, as defined in Subsection 2.6. Briefly, the term "lone axis" is used when the axis bundle, defined by Handel and Mosher [HM11], consists of a unique axis. The axis bundle is an analogue of the axis of a pseudo-Anosov, but in general contains many fold lines.

Theorem A. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_\varphi$ consists of a unique axis, then* $\mathrm{Cen}(\langle\varphi\rangle) = \mathrm{Stab}(\Lambda^+_\varphi) \cong \mathbb{Z}$.

We additionally prove the following group theoretic corollary.

Theorem B. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_\varphi$ consists of a unique axis, then either*

(1) $\mathrm{Cen}(\langle\varphi\rangle) = N(\langle\varphi\rangle) = \mathrm{Comm}(\langle\varphi\rangle) \cong \mathbb{Z}$, *or*
(2) $\mathrm{Cen}(\langle\varphi\rangle) \cong \mathbb{Z}$ *and* $N(\langle\varphi\rangle) = \mathrm{Comm}(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

Further, in the second case, we have that φ^{-1} is also an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_{\varphi^{-1}}$ consists of a unique axis.

Since the outer automorphisms constructed in [Pfa13] satisfy the conditions of [MP16, Theorem 4.7], ageometric lone axis fully irreducible outer automorphisms exist in each rank. Moreover, it is proved in [KP15] that this

[1] See (1), (3), and Definition 2.22.

situation is generic along a specific "train track directed" random walk. It is noteworthy that the conditions for an outer automorphism to be an ageometric lone axis fully irreducible can be checked via the Coulbois computer package.[2]

Understanding what properties transfer to inverses of outer automorphisms is in general elusive. Theorem B gives a condition which guarantees that φ^{-1} also admits a lone axis. However, we do not know if the latter case in fact occurs, prompting the following question.

Question 1.1. *Does there exist some ageometric lone axis fully irreducible outer automorphism such that* $\mathrm{Comm}(\langle\varphi\rangle) = N(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$*?*

We pose one further question.

Question 1.2. *Can one give a concrete description of* $\mathrm{Cen}(\langle\varphi\rangle)$ *and* $N(\langle\varphi\rangle)$ *when* φ *is not an ageometric lone axis fully irreducible outer automorphism?*

The idea of the proof follows. The results of Bestvina–Feighn–Handel [BFH97] (see also Kapovich–Lustig [KL11]) provide us with a homomorphism $\rho\colon \mathrm{Stab}(\Lambda^+_\varphi) \to (\mathbb{R}, +)$ which we interpret as the signed translation distance along an axis of φ in Outer Space (Lemma 4.2). The situation is analogous to the action of the centralizer in $\mathrm{PSL}(2,\mathbb{Z})$ of a cyclic subgroup $\langle\varphi\rangle$ on the the geodesic axis of φ in the hyperbolic plane (see Example 3.4). In the case of $\mathrm{PSL}(2,\mathbb{Z})$ we know that an element in the kernel of this action fixes more than two points in the complex plane hence, by the properties of Möbius transformations, it must be the identity. The corresponding fact is false for a general fully irreducible $\varphi \in \mathrm{Out}(F_r)$. The main work in proving Theorem A is to prove that $\ker(\rho)$ is trivial (Proposition 4.5) when φ is a lone axis fully irreducible. To do so we appeal to [MP16] (see Propostion 2.21) to use a particularly nice topological representative of φ. After Theorem A is established, we prove Theorem B by analyzing the short exact sequence of $\mathrm{Cen}(\langle\varphi\rangle) \le \mathrm{Comm}(\langle\varphi\rangle)$.

To give a more complete picture of the context of this work we mention the following additional results. Given any element $\varphi \in \mathrm{Out}(F_r)$, using the machinery of completely split relative train track maps, Feighn and Handel [FH09] present an algorithm that computes a finite index subgroup of the weak center of the centralizer of $\langle\varphi\rangle$, i.e., the set of elements that commute with some power of each element of $\mathrm{Cen}(\langle\varphi\rangle)$. When φ is a Dehn twist, Rodenhausen and Wade [RW15] give an algorithm determining a presentation of $\mathrm{Cen}(\langle\varphi\rangle)$. They use this to compute a presentation of the centralizer of a Nielsen generator.

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[2]The Coulbois computer package is available at [Cou14].

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2. Preliminary definitions and notation

To keep this section at a reasonable length, in some cases we will provide only references for better known definitions.

2.1. Train track maps, Nielsen paths, and principal vertices. Fully irreducible elements of $\mathrm{Out}(F_r)$ are those such that no power fixes the conjugacy class of a proper free factor. Every fully irreducible outer automorphism can be represented by a special graph map called a train track map, as defined in [BH92]. In particular, we will require that vertices map to vertices. Moreover, we can also choose these maps so that they are defined on graphs with no valence-1 or valence-2 vertices (from the proof of [BH92] Theorem 1.7).

Definition 2.1 (Directions)**.** Let $g\colon \Gamma \to \Gamma$ be a graph map. A *direction* at $x \in \Gamma$ is a germs of initial segments of an edge emanating from x. For each edge $e \in E(\Gamma)$, we let $D(e)$ denote the initial direction of e. For an edge-path $\gamma = e_1 \dots e_k$, we let $D\gamma = D(e_1)$. We denote by Dg the map of directions induced by g, i.e., $Dg(d) = D(g(e))$ for $d = D(e)$. A direction d is *periodic* if $Dg^k(d) = d$ for some $k > 0$ and *fixed* when $k = 1$.

Definition 2.2 (Turns)**.** Let $g\colon \Gamma \to \Gamma$ be a graph map. We call an unordered pair of directions $\{d_i, d_j\}$ a *turn* and a *degenerate turn* if $d_i = d_j$. The turn is called an *illegal turn* for g if $Dg^k(d_i) = Dg^k(d_j)$ for some k and a *legal turn* otherwise.

Definition 2.3 (Nielsen paths)**.** Let $g\colon \Gamma \to \Gamma$ be an expanding irreducible train track map. Bestvina and Handel [BH92] define a nontrivial tight path ρ in Γ to be a *periodic Nielsen path (PNP)* if, for some power $R \geq 1$, we have $g^R(\rho) \cong \rho$ rel endpoints (and just a *Nielsen path (NP)* if $R = 1$). A (periodic) Nielsen path is *indivisible* if it cannot be decomposed as a concatenation of (periodic) Nielsen paths.

Definition 2.4 (Ageometric)**.** A fully irreducible outer automorphism is called *ageometric* if it has a train track representative with no PNPs. Further characterizations can be found in [MP16, §2.9].

Definition 2.5 (Principal points)**.** Given a train track map $g\colon \Gamma \to \Gamma$, following [HM11], we call a point *principal* that is either the endpoint of a PNP or is a periodic vertex with ≥ 3 periodic directions. Thus, in the absence of PNPs, a point is principal if and only if it is a periodic vertex with ≥ 3 periodic directions.

Definition 2.6 (Rotationless)**.** An expanding irreducible train track map is called *rotationless* if each periodic direction is fixed and each PNP is of period one. By [FH11, Proposition 3.24], one can define a fully irreducible $\varphi \in Out(F_r)$ to be *rotationless* if one (hence all) of its train track representatives is rotationless.

2.2. Outer Space CV_r and the attracting tree T_+^φ for a fully irreducible $\varphi \in \mathrm{Out}(F_r)$. Let CV_r denote the Culler–Vogtmann Outer Space in rank r, as defined in [CV86], with the asymmetric Lipschitz metric [FM11]. The group $\mathrm{Out}(F_r)$ acts naturally on CV_r on the right by homeomorphisms. An element $\varphi \in CV_r$ sends a point $X = (\Gamma, m, \ell) \in CV_r$ to the point $X \cdot \varphi = (\Gamma, m \circ \Phi, \ell)$, where Φ is a lift in $\mathrm{Aut}(F_r)$ of φ. Let $\overline{CV_r}$ denote the compactification of CV_r, as defined in [CM87] (and later identified with the space of very small $\mathbb{R}$-trees in [CL95] and [BF94]). The action of $\mathrm{Out}(F_r)$ on CV_r extends to an action on $\overline{CV_r}$ by homeomorphisms.

Definition 2.7 (Attracting tree T_+^φ)**.** Let $\varphi \in \mathrm{Out}(F_r)$ be a fully irreducible outer automorphism. Then φ acts on $\overline{CV_r}$ with North-South dynamics (see [LL03]). We denote by T_+^φ the attracting fixed point of this action and by T_-^φ the repelling fixed point of this action.

2.3. The attracting lamination Λ_φ^+ for a fully irreducible outer automorphism. We give a concrete description of Λ_φ^+ using a particular train track representative $g\colon \Gamma \to \Gamma$. This is the original definition appearing in [BFH97]. Note that apriori it is not clear that it does not depend on the train track representative.

Definition 2.8 (Iterating neighborhoods)**.** Let $g\colon \Gamma \to \Gamma$ be an affine irreducible train track map so that, in particular, there has been an identification of each edge e of Γ with an open interval of its length $\ell(e)$ determined by the Perron–Frobenius eigenvector. Let $\lambda = \lambda(\varphi)$ be its stretch factor and assume $\lambda > 1$. Let x be a periodic point which is not a vertex (such points are dense in each edge). Let $\varepsilon > 0$ be sufficiently small so that the ε-neighborhood of x, denoted U, is contained in the interior of an edge. There exists an $N > 0$ such that x is fixed, $U \subset g^N(U)$, and Dg^N fixes the directions at x. We choose an isometry $\ell\colon (-\varepsilon, \varepsilon) \to U$ and extend it to the unique locally isometric immersion $\ell\colon \mathbb{R} \to \Gamma$ so that $\ell(\lambda^N t) = g^N(\ell(t))$. We then say that ℓ is *obtained by iterating a neighborhood of x*.

Definition 2.9 (Leaf segments, equivalent isometric immersions)**.** We call isometric immersions $\gamma_1\colon [a,b] \to \Gamma$ and $\gamma_2\colon [c,d] \to \Gamma$ *equivalent* when there exists an isometry $h\colon [a,b] \to [c,d]$ so that $\gamma_1 = \gamma_2 \circ h$. Let $\ell\colon \mathbb{R} \to \Gamma$ be an isometric immersion. *A leaf segment* of ℓ is the equivalence class of the restriction to a finite interval of $\mathbb{R}$. Two isometric immersions ℓ and ℓ' are equivalent if each leaf segment of ℓ is a leaf segment of ℓ' and vice versa.

Definition 2.10 (The realization in Γ of the attracting lamination $\Lambda_\varphi^+(\Gamma)$)**.** The *attracting lamination realized in* Γ, denoted $\Lambda_\varphi^+(\Gamma)$, is the equivalence class of a line ℓ obtained by iterating a neighborhood of a periodic point in Γ (as in Definition 2.8). The periodic point chosen is inconsequential. An element of $\Lambda^+_\varphi(\Gamma)$ is called a leaf. Notice that $\Lambda^+_\varphi(\Gamma)$ can be realized as an F_r-invariant set of bi-infinite geodesics in $\widetilde{\Gamma}$, the universal cover of Γ. We shall denote this set by $\Lambda^+_\varphi(\widetilde{\Gamma})$.

The marking of Γ induces an identification of $\partial\Gamma$ with ∂F_r. The attracting lamination Λ^+_φ is the image of $\Lambda^+_\varphi(\widetilde{\Gamma})$ under this identification. In [BFH97] it is proved that this set is independent of the choice of g.

Definition 2.11 (The action of $\mathrm{Out}(F_r)$ on the set of laminations $\Lambda^\pm_\varphi$)**.** Let $\psi \in \mathrm{Out}(F_r)$, then

$$\psi \cdot (\Lambda^+_\varphi, \Lambda^-_\varphi) = (\Lambda^+_{\psi\varphi\psi^{-1}}, \Lambda^-_{\psi\varphi\psi^{-1}}) \tag{1}$$

(see [BFH97, pg. 223]).

2.4. Whitehead graphs. The following definitions originate in [HM11] but are further explained in [MP16]. The versions given here reference those in [MP16, Definition 2.32].

Definition 2.12 (Stable Whitehead graphs and local Whitehead graphs)**.** Let $g\colon \Gamma \to \Gamma$ be a train track map. The *local Whitehead graph* $LW(g, v)$ at a point $v \in \Gamma$ has a vertex for each direction at v and an edge connecting the vertices corresponding to the pair of directions $\{d_1, d_2\}$ if the turn $\{d_1, d_2\}$ is taken by an image of an edge. The *stable Whitehead graph* $SW(g, v)$ at a principal point v is then the subgraph of $LW(g, v)$ obtained by restricting to the periodic direction vertices.

Remark 2.13. Given a train track map $g\colon \Gamma \to \Gamma$, we have

$$SW(g, v) \cong SW(g^k, v)$$

for all $k \in \mathbb{N}$.

A train track map g induces a simplicial (hence continuous) map

$$Dg\colon LW(g, v) \to LW(g, g(v))$$

extending the map of vertices defined by the direction map Dg. When g is rotationless and v a principal vertex (hence, in particular, is fixed), the map $Dg\colon LW(g, v) \to LW(g, v)$ has image in $SW(g, v)$. Since Dg acts as the identity on $SW(g, v)$, when viewed as a subgraph of $LW(g, v)$, this map is in fact a surjection $Dg\colon LW(g, v) \to SW(g, v)$.

Recall that for a train track representative of a fully irreducible outer automorphism the local Whitehead graph at each vertex is connected. Hence:

Lemma 2.14. *If $g\colon \Gamma \to \Gamma$ is a train track map representing a fully irreducible outer automorphism φ and $v \in \Gamma$ is a principal vertex, then $SW(g, v)$ is connected.*

Lemma 2.15. *Let $g\colon \Gamma \to \Gamma$ be a rotationless PNP-free train track representative of an ageometric fully irreducible $\varphi \in \mathrm{Out}(F_r)$. Let $\widetilde{\Gamma}$ be the universal cover of Γ and $\widetilde{v} \in \widetilde{\Gamma}$ a vertex that projects to a principal vertex $v \in \Gamma$. Then there exist two leaves ℓ_1, ℓ_2 of the lamination $\Lambda^+_\varphi(\widetilde{\Gamma})$ so that $\ell_1 \cup \ell_2$ is a tripod whose vertex is $\widetilde{v}$.*

Proof. Since v is a principal vertex and there are no PNPs, we know $SW(g, v)$ has ≥ 3 vertices. Since $SW(g, v)$ is connected, one of these vertices d_1 will belong to at least 2 edges ϵ_1, ϵ_2. Let d_2, d_3 be the directions corresponding to the other vertices of these edges. Since g is rotationless, periodic directions are in fact fixed directions. We may lift g to a map $\widetilde{g}\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ that fixes $\widetilde{v}$. Iterating the lifts of the edges that correspond to d_1, d_2, d_3 will give us three eigenrays R_1, R_2, R_3 initiating at $\widetilde{v}$. The 2 edges ϵ_1, ϵ_2 correspond to 2 leaves ℓ_1 and ℓ_2 of $\Lambda^+_\varphi(\widetilde{\Gamma})$ [HM11, Subsection 3.1]. We have $\ell_1 \cup \ell_2 = R_1 \cup R_2 \cup R_3$. Hence, as desired, $\ell_1 \cup \ell_2$ is a tripod whose vertex is $\widetilde{v}$. □

2.5. Axis bundles. Three equivalent definitions of the axis bundle $\mathcal{A}_\varphi$ for a nongeometric fully irreducible $\varphi \in \mathrm{Out}(F_r)$ are given in [HM11]. We include only the definition that we use.

Definition 2.16 (Fold lines)**.** A *fold line* in CV_r is a continuous, injective, proper function $\mathbb{R} \to CV_r$ defined by

(1) a continuous 1-parameter family of marked graphs $t \to \Gamma_t$, and
(2) a family of homotopy equivalences $h_{ts}\colon \Gamma_s \to \Gamma_t$ defined for $s \leq t \in \mathbb{R}$, each marking-preserving,

and satisfying:

Train track property: h_{ts} is a local isometry on each edge for all $s \leq t \in \mathbb{R}$.
Semiflow property: $h_{ut} \circ h_{ts} = h_{us}$ for all $s \leq t \leq u \in \mathbb{R}$ and $h_{ss}\colon \Gamma_s \to \Gamma_s$ is the identity for all $s \in \mathbb{R}$.

Definition 2.17 (Axis Bundle)**.** $\mathcal{A}_\varphi$ is the union of the images of all fold lines $\mathcal{F}\colon \mathbb{R} \to CV_r$ such that $\mathcal{F}(\mathrm{t})$ converges in $\overline{CV_r}$ to T^φ_- as $t \to -\infty$ and to T^φ_+ as $t \to +\infty$.

Definition 2.18 (Axes)**.** We call the fold lines in Definition 2.17 the *axes* of the axis bundle.

2.6. Lone axis fully irreducibles outer automorphisms.

Definition 2.19 (Lone axis fully irreducibles)**.** A fully irreducible $\varphi \in \mathrm{Out}(F_r)$ will be called a *lone axis fully irreducible outer automorphism* if $\mathcal{A}_\varphi$ consists of a unique axis.

[MP16, Theorem 4.7] gives necessary and sufficient conditions on an ageometric fully irreducible outer automorphism $\varphi \in \mathrm{Out}(F_r)$ that ensure $\mathcal{A}_\varphi$

consists of a unique axis. It is also proved there that, under these conditions, the axis will be the periodic fold line for a (in fact any) train track representative of φ. In particular, as is always true for axis bundles, $\mathcal{A}_\varphi$ contains each point in Outer Space on which there exists an affine train track representative of a power of φ.

Remark 2.20. It will be important for our purposes that no train track representative of an ageometric lone axis fully irreducible φ has a periodic Nielsen path. We recall from [BH92], by considering a rotationless power, that a stable train track representative is one with the minimum number of indivisible PNPs. The statement then follows from [MP16, Lemma 4.5], as it shows that each train track representative of each power of φ is stable, hence (in the case of an ageometric fully irreducible outer automorphism) has no PNPs.

The following proposition is a direct consequence of [MP16, Corollary 3.8].

Proposition 2.21 ([MP16])**.** *Let φ be an ageometric lone axis fully irreducible outer automorphism. Then there exists a train track representative $g\colon \Gamma \to \Gamma$ of some power φ^R of φ so that all vertices of Γ are principal, and fixed, and all but one direction is fixed.*

2.7. The stabilizer $\mathrm{Stab}(\Lambda^+_\varphi)$ of the lamination.

Definition 2.22 ($\mathrm{Stab}(\Lambda^+_\varphi), \mathrm{Stab}(\{\Lambda^+_\varphi, \Lambda^+_\varphi\})$)**.** Given a fully irreducible $\varphi \in \mathrm{Out}(F_r)$, we let $\mathrm{Stab}(\Lambda^+_\varphi)$ denote the subgroup of $\mathrm{Out}(F_r)$ fixing Λ^+_φ setwise, i.e., sending leaves of Λ^+_φ to leaves of Λ^+_φ. Denote by $\mathrm{Stab}(\{\Lambda^+_\varphi, \Lambda^+_\varphi\})$ the subgroup of $\mathrm{Out}(F_r)$ consisting of all ψ so that

$$\psi(\Lambda^+_\varphi) = \Lambda^\pm_\varphi \quad \text{and} \quad \psi(\Lambda^-_\varphi) = \Lambda^\mp_\varphi.$$

Lemma 2.23. *Given a fully irreducible $\varphi \in \mathrm{Out}(F_r)$:*

(1) $\mathrm{Stab}(\Lambda^+_\varphi) = \mathrm{Stab}(\Lambda^+_\varphi, \Lambda^-_\varphi) = \mathrm{Stab}(T^+_\varphi, T^-_\varphi) = \mathrm{Stab}(T^+_\varphi)$ *and*
(2) $\mathrm{Stab}(\{\Lambda^+_\varphi, \Lambda^-_\varphi\}) = \mathrm{Stab}(\{T^+_\varphi, T^-_\varphi\})$.

Proof. That $\mathrm{Stab}(\Lambda^+_\varphi) = \mathrm{Stab}(T^-_\varphi)$ and $\mathrm{Stab}(\Lambda^-_\varphi) = \mathrm{Stab}(T^+_\varphi)$ follows from [BFH97, Corollary 3.6]. $\psi \in \mathrm{Stab}(\Lambda^+_\varphi)$ if and only if $\Lambda^+_{\psi\varphi\psi^-} = \Lambda^+_\varphi$. By [BFH97, Proposition 2.16.] this occurs if an only if we have $\Lambda^-_{\psi\varphi\psi^-} = \Lambda^-_\varphi$ and this occurs if and only if $\psi \in \mathrm{Stab}(\Lambda^-_\varphi)$. □

Bestvina, Feighn, and Handel [BFH97] define a homomorphism (related to the expansion factor)

$$\sigma\colon \mathrm{Stab}(\Lambda^+_\varphi) \to (\mathbb{R}_{>0}, \cdot). \tag{2}$$

They use σ to prove the following theorem ([BFH97, Theorem 2.14]):

Theorem 2.24 ([BFH97, Theorem 2.14] or [KL11, Theorem 4.4])**.** *For each fully irreducible $\varphi \in \mathrm{Out}(F_r)$, we have that $\mathrm{Stab}(\Lambda^+_\varphi)$ is virtually cyclic.*

2.8. Commensurators. Throughout this subsection let G be a group and H a subgroup of G.

Definition 2.25 (Commensurator $\mathrm{Comm}_G(H)$)**.** The *commensurator*, or *virtual normalizer*, of H in G is defined as

$$\begin{aligned}&\mathrm{Comm}_G(H)\\ &:= \{g \in G \mid [H : H \cap g^{-1}Hg] < \infty \text{ and } [g^{-1}Hg : H \cap g^{-1}Hg] < \infty\}.\end{aligned}$$

Notice that when $H = \langle a \rangle$, for some a, we have

$$\mathrm{Comm}_G(\langle a \rangle) := \{g \in G \mid \exists\, m, n \in \mathbb{Z} \text{ so that } ga^n g^{-1} = a^m\}. \tag{3}$$

Remark 2.26. $N_G(H) \leq \mathrm{Comm}_G(H)$.

Proposition 2.27. *Let $a \in G$. If $\mathrm{Cen}_G(\langle a \rangle) \leq H$ and H is cyclic, then $\mathrm{Cen}_G(\langle a \rangle) = H$.*

Proof. Since H is cyclic, $H = \langle b \rangle$ for some $b \in H$. Then $a = b^k$ for some k and hence b and a commute. □

Proposition 2.28. *If $a \in G$ and $\mathrm{Comm}_G(\langle a \rangle)$ is virtually cyclic, then for some $k \in \mathbb{Z}$ we have $N_G(\langle a^k \rangle) = \mathrm{Comm}_G(\langle a \rangle)$. Thus, in this case, $N_G(\langle a \rangle) \leq \mathrm{Comm}_G(\langle a \rangle) = N_G(\langle a^k \rangle)$.*

Proof. First notice that

$$N_G(\langle a^k \rangle) \leq \mathrm{Comm}_G(\langle a^k \rangle) \leq \mathrm{Comm}_G(\langle a \rangle).$$

Hence, we are left to show $\mathrm{Comm}_G(\langle a \rangle) \leq N_G(\langle a^k \rangle)$. Let $\langle a \rangle$ have index n in the group $\mathrm{Comm}_G(\langle a \rangle)$. Let $b \in \mathrm{Comm}_G(\langle a \rangle)$ and let $\omega_b \in Aut(G)$ denote conjugation by b. By (3), there exist $k, m \in \mathbb{Z}$ so that $ba^k b^{-1} = a^m$, hence $\omega_b(\langle a^k \rangle) = \langle a^m \rangle$. Now

$$\begin{aligned} n|m| &= [\mathrm{Comm}_G(\langle a \rangle) : \langle a^m \rangle] \\ &= [\mathrm{Comm}_G(\langle a \rangle) : \omega_b(\langle a^k \rangle)] \\ &= [\omega_b(\mathrm{Comm}_G(\langle a \rangle)) : \omega_b(\langle a^k \rangle)] = n|k|. \end{aligned}$$

Hence, $|m| = |k|$ and so $b \in N_G(\langle a^k \rangle)$. □

3. The sequence of inclusions for a fully irreducible outer automorphism.

Convention 3.1 ($\langle \varphi \rangle$, $\mathrm{Cen}(\langle \varphi \rangle)$, $N(\langle \varphi \rangle)$)**.** Given an element $\varphi \in \mathrm{Out}(F_r)$, we let $\langle \varphi \rangle$ denote the cyclic subgroup generated by φ, we let $\mathrm{Cen}(\langle \varphi \rangle)$ denote its centralizer in $\mathrm{Out}(F_r)$, and we let $N(\langle \varphi \rangle)$ denote its normalizer in $\mathrm{Out}(F_r)$.

Lemma 3.2. *Let $\varphi \in Out(F_r)$ be fully irreducible. Then:*

1. *Each element $\psi \in \mathrm{Cen}(\langle \varphi \rangle)$ fixes the ordered pair $(T_\varphi^+, T_\varphi^-)$ and the ordered pair $(\Lambda_\varphi^+, \Lambda_\varphi^-)$. In particular, $\mathrm{Cen}(\langle \varphi \rangle) \leq \mathrm{Stab}(\Lambda_\varphi^+)$.*

(2) $\mathrm{Comm}(\langle\varphi\rangle) = \mathrm{Stab}(\{\Lambda^+_\varphi, \Lambda^-_\varphi\}) = \mathrm{Stab}(\{T^+_\varphi, T^-_\varphi\})$. *And, in particular, each element* $\psi \in N(\langle\varphi\rangle)$ *fixes the unordered pair* $\{T^+_\varphi, T^-_\varphi\}$ *and the unordered pair* $\{\Lambda^+_\varphi, \Lambda^-_\varphi\}$.

Proof. (1) Let $\psi \in \mathrm{Cen}(\langle\varphi\rangle)$. Then by Equation (1) we have

$$\psi \cdot (\Lambda^+_\varphi, \Lambda^-_\varphi) = (\Lambda^+_\varphi, \Lambda^-_\varphi).$$

That ψ fixes the ordered pair $(T^+_\varphi, T^-_\varphi)$ now follows from Lemma 2.23.

(2) By Equations (1) and (3) we have

$$\mathrm{Comm}(\langle\varphi\rangle) \leq \mathrm{Stab}(\{\Lambda^+_\varphi, \Lambda^-_\varphi\}).$$

By Lemma 2.23 this implies that $\mathrm{Comm}(\langle\varphi\rangle) \leq \mathrm{Stab}(\{T^+_\varphi, T^-_\varphi\})$. Now suppose $\psi \in \mathrm{Stab}(\{\Lambda^+_\varphi, \Lambda^-_\varphi\})$. By Equation (1) and [BFH97, Proposition 2.16], we know $\psi\varphi^k\psi^{-1} = \varphi^n$ for some $k, n \in \mathbb{Z}$. So $\psi \in \mathrm{Comm}(\langle\varphi\rangle)$. □

Corollary 3.3. *If* $\varphi \in \mathrm{Out}(F_r)$ *is fully irreducible then there exists an integer* $k \in \mathbb{Z}$ *so that*

$$\mathrm{Cen}(\langle\varphi\rangle) \leq \mathrm{Stab}(\Lambda^+_\varphi) \leq \mathrm{Comm}(\langle\varphi\rangle) = N(\langle\varphi^k\rangle).$$

Moreover:

(1) *The subgroup in the right-hand inequality has index* ≤ 2.
(2) *If* $\mathrm{Stab}(\Lambda^+_\varphi)$ *is cyclic then the left-hand inequality is an equality.*

Proof. (1) $\mathrm{Stab}(\Lambda^+_\varphi) = \mathrm{Stab}(\Lambda^+_\varphi, \Lambda^-_\varphi)$, see Lemma 2.23. By Lemma 3.2(2), $\mathrm{Comm}(\langle\varphi\rangle) = \mathrm{Stab}(\{\Lambda^+_\varphi, \Lambda^-_\varphi\})$. Thus, $|\mathrm{Comm}(\langle\varphi\rangle) : \mathrm{Stab}(\Lambda^+_\varphi)| \leq 2$. By Theorem 2.24 and Proposition 2.28, there exist $k \in \mathbb{Z}$ such that

$$\mathrm{Comm}(\langle\varphi\rangle) = N(\langle\varphi^k\rangle).$$

(2) follows from Proposition 2.27. □

Example 3.4. We work out an example where

$$\mathrm{Cen}(\langle\varphi\rangle) = \mathrm{Cen}(\langle\varphi^2\rangle) \lneq N(\langle\varphi^2\rangle) \leq \mathrm{Comm}(\langle\varphi\rangle).$$

Recall that $\mathrm{Out}(F_2) \cong \mathrm{GL}(2,\mathbb{Z})$ via the abelianization map. Thus, it suffices to carry out the computations in $\mathrm{GL}(2,\mathbb{Z})$. Consider,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The image $\bar{A}$ of A in $\mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ acts on the hyperbolic plane by hyperbolic isometries fixing the points $\lambda, -\frac{1}{\lambda} \in \mathbb{R}$. Each element of $\mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\lambda, -\frac{1}{\lambda})$ preserves the hyperbolic geodesic between λ and $-\frac{1}{\lambda}$, we denote this by $[\lambda, -\frac{1}{\lambda}]$. Thus, the map $\rho\colon \mathrm{Stab}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\lambda, -\frac{1}{\lambda}) \to (\mathbb{R}, +)$ sending an element to its signed translation length on $[\lambda, -\frac{1}{\lambda}]$ is a homomorphism. Moreover, its image is discrete and its kernel is trivial. Hence, $\mathrm{Stab}(\lambda, -\frac{1}{\lambda})$ is infinite cyclic and, by Proposition 2.27, $\mathrm{Stab}(\lambda, -\frac{1}{\lambda}) = \mathrm{Cen}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle\bar{A}\rangle)$. Since $\bar{A} \in \mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ is primitive, then $\mathrm{Cen}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle\overline{A}\rangle) = \langle\bar{A}\rangle$. One can check directly that $\mathrm{Cen}_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) = \langle A, -I\rangle$, where I denotes the identity matrix (this follows since $\mathrm{GL}(2,\mathbb{Z}) \to \mathbb{P}\mathrm{GL}(2,\mathbb{Z})$ is 2-to-1). Similarly, $\mathrm{Cen}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle \bar{A}^2\rangle)$ is infinite cyclic and $\bar{A}$ is a primitive element of this group, hence $\mathrm{Cen}_{\mathbb{P}\mathrm{GL}(2,\mathbb{Z})}(\langle \bar{A}^2\rangle) = \langle \bar{A}\rangle$. Again, $\mathrm{Cen}_{\mathrm{GL}(2,\mathbb{Z})}(\langle A^2\rangle) = \langle A, -I\rangle$. Moreover, one can check directly that $P \in N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A^2\rangle) - N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle)$. Hence, $\mathrm{Comm}_{\mathrm{GL}(2,\mathbb{Z})}(\langle A\rangle) \geq N_{\mathrm{GL}(2,\mathbb{Z})}(\langle A^2\rangle) \geq \langle A, -I, P\rangle$.

Example 3.5. We show that there exists an ageometric fully irreducible outer automorphism φ such that $\mathrm{Cen}(\langle\varphi\rangle) \ncong \mathbb{Z}$, and moreover

$$\mathrm{Cen}(\langle\varphi\rangle) \ncong \mathbb{Z} \times \mathbb{Z}_2$$

(as in $\mathrm{Out}(F_2)$, whose center is $\mathbb{Z}_2$). Consider $F_3 = \langle a, b, c\rangle$. Let R_3 be the 3-petaled rose and define

$$\Psi : a \to b \to c \to ab.$$

It is straight-forward (see [Pfa13, Proposition 4.1]) to check that this map represents an ageometric fully irreducible outer automorphism. Denote by Δ the 3-fold cover corresponding to the subgroup

$$\langle b, c, a^3, abA, acA, a^2bA^2, a^2cA^2\rangle.$$

We claim that Ψ^{13} lifts to Δ. Indeed, let A be the transition matrix of Ψ, then

$$A^{13} = \begin{pmatrix} 7 & 9 & 12 \\ 12 & 16 & 21 \\ 9 & 12 & 16 \end{pmatrix}.$$

In particular, both $\Psi(b)$ and $\Psi(c)$ cross a a multiple of 3 times. Thus Ψ^{13} lifts to Δ. Denote the vertices of Δ by v_0, v_1, v_2. We denote by $g\colon \Delta \to \Delta$ the lift of Ψ^{13} that sends v_0 to itself. Let $T\colon \Delta \to \Delta$ denote the deck transformation sending v_0 to v_1. The action of T on $H_1(\Delta, \mathbb{Z})$ is nontrivial, so T does not represent an inner automorphism. Moreover, we claim that $g \circ T = T \circ g$. First note that the maps $g \circ T$ and $T \circ g$ are both lifts of Ψ^{13}. Moreover, since a appears in $\Psi^{13}(a)$ 7 times (see the matrix A^{13}), $g(v_1) = v_1$. Therefore,

$$g \circ T(v_0) = g(v_1) = v_1 = T(v_0) = T \circ g(v_0).$$

Thus, $g \circ T = T \circ g$. Let $\varphi \in \mathrm{Out}(F_7)$ be the outer automorphism represented by g, and θ the outer automorphism represented by T. An elementary computation shows that g is an irreducible train track map and that each local Whitehead graph is connected. Moreover, a PNP for g would descend to a PNP for Ψ. Since Ψ contains no such paths, there are no PNPs for g. Thus, the outer automorphism φ is ageometric and fully irreducible (see [Pfa13, Proposition 4.1]). In conclusion, θ is an order-3 element in $\mathrm{Cen}_{\mathrm{Out}(F_7)}(\langle\varphi\rangle)$, in contrast to the conclusion of our theorem for a lone axis ageometric fully irreducible outer automorphism.

Example 3.6. In this example $\mathrm{Cen}(\langle\varphi\rangle) \lneq \mathrm{Stab}(\Lambda_\varphi^+)$. Consider Ψ as in Example 3.5 with its transition matrix A. We have:

$$A^{16} = \begin{pmatrix} 16 & 21 & 28 \\ 28 & 37 & 49 \\ 21 & 28 & 37 \end{pmatrix}.$$

Thus, Ψ^{16} lifts to a cover Δ corresponding to the index 7 subgroup of the free group

$$\langle b, c, aba^{-1}, aca^{-1}, a^2ba^{-2}, a^2ca^{-2}, \ldots, a^6ba^{-6}, a^6ca^{-6}, a^7\rangle.$$

Number the vertices of Δ by $v_0, \ldots v_6$. Let $g : \Delta \to \Delta$ be the lift of Ψ^{16} fixing v_0. Since $\Psi^{16}(a)$ crosses a a multiple of 16 times, $g(v_1) = v_2$. Thus, if T is an order 7 deck transformation such that $T(v_i) = v_{i+1 \mod 7}$ then $g \circ T \neq T \circ g$. Let φ denote the automorphism represented by g. Then, as in the previous example, φ is an ageometric fully irreducible outer automorphism. The lamination Λ_φ^+ is a lift of the lamination Λ_ψ^+ and therefore it is preserved by T. Thus θ, the automorphism represented by T, is contained in $\mathrm{Stab}(\Lambda_\varphi^+)$. But $\theta \notin \mathrm{Cen}(\langle\varphi\rangle)$.

4. Proof of the main theorems

Lemma 4.1. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric lone axis fully irreducible outer automorphism. If $\psi \in \mathrm{Out}(F_r)$ is an outer automorphism fixing the pair $(T_\varphi^+, T_\varphi^-)$, then ψ fixes $\mathcal{A}_\varphi$ as a set, and also preserves its orientation.*

Proof. $\mathcal{A}_\varphi$ consists precisely of all fold lines $\mathcal{F} \colon \mathbb{R} \to CV_r$ such that $\mathcal{F}(t)$ converges in $\overline{CV_r}$ to T_φ^- as $t \to -\infty$ and to T_φ^+ as $t \to +\infty$. Further, since $\varphi \in \mathrm{Out}(F_r)$ is a lone axis fully irreducible outer automorphism, there is only one such fold line. Hence, since ψ fixes $(T_\varphi^+, T_\varphi^-)$, it suffices to show that the image of the single fold line $\mathcal{A}_\varphi$ under ψ is a fold line. Indeed given the fold line $t \to \Gamma_t$ with the semi-flow family $\{h_{ts}\}$, the new fold line is just $t \to \Gamma_t \cdot \psi$ with the same family of homotopy equivalences $\{h_{ts}\}$. Hence the properties of Definition 2.16 still hold. ☐

Recall that $\mathcal{A}_\varphi$ is a directed geodesic and suppose that the map $t \to \Gamma_t$ is a parametrization of $\mathcal{A}_\varphi$ according to arc-length with respect to the Lipschitz metric, i.e.,

$$d(\Gamma_t, \Gamma_{t'}) = t' - t \qquad \text{for } t' > t. \tag{4}$$

Lemma 4.2. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric lone axis fully irreducible outer automorphism and $\psi \in \mathrm{Stab}(\Lambda_\varphi^+)$. Then there exists a number $\rho(\psi) \in \mathbb{R}$ so that for all $t \in \mathbb{R}$, we have $\psi(\Gamma_t) = \Gamma_{\rho(\psi)+t}$.*

Proof. $\mathrm{Stab}(\Lambda_\varphi^+) = \mathrm{Stab}(T_\varphi^+, T_\varphi^-)$ by Lemma 2.23, and by Lemma 4.1, $\psi(\mathcal{A}_\varphi) = \mathcal{A}_\varphi$ and ψ preserves the direction of the fold line. Therefore, there exists a strictly monotonically increasing surjective function $f \colon \mathbb{R} \to \mathbb{R}$ so that $\psi(\Gamma_t) = \Gamma_{f(t)}$. Moreover, since ψ is an isometry with respect to the Lipschitz metric, for $t < t'$, since $f(t) < f(t')$, Equation (4) implies

$$f(t') - f(t) = d(\Gamma_{f(t)}, \Gamma_{f(t')}) = d(\psi(\Gamma_t), \psi(\Gamma_{t'})) = d(\Gamma_t, \Gamma_{t'}) = t' - t.$$

Hence $f(t') = f(t) + t' - t$. This implies that for all $s \in \mathbb{R}$, we have $f(s) = f(0) + s$. Define $\rho(\psi) = f(0)$, then

$$\psi(\Gamma_t) = \Gamma_{f(t)} = \Gamma_{f(0)+t} = \Gamma_{\rho(\psi)+t}.$$

□

Lemma 4.3. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric lone axis fully irreducible outer automorphism. Then the map $\rho\colon \mathrm{Stab}(\Lambda_\varphi^+) \to (\mathbb{R}, +)$ is a homomorphism.*

Proof. For each $t \in \mathbb{R}$,

$$\Gamma_t = \psi^{-1}\psi(\Gamma_t) = \psi^{-1}(\Gamma_{\rho(\psi)+t}) = \Gamma_{\rho(\psi^{-1})+\rho(\psi)+t}.$$

Thus, $t = \rho(\psi^{-1}) + \rho(\psi) + t$, i.e., $\rho(\psi^{-1}) = -\rho(\psi)$. Moreover, let $\psi, \nu \in \mathrm{Stab}(\Lambda_\varphi^+)$. Then

$$\Gamma_{\rho(\psi\circ\nu)+t} = \psi \circ \nu(\Gamma_t) = \psi(\nu(\Gamma_t)) = \psi(\Gamma_{\rho(\nu)+t}) = \Gamma_{\rho(\psi)+\rho(\nu)+t}.$$

Thus,

$$\rho(\psi \circ \nu) = \rho(\psi) + \rho(\nu).$$

We therefore obtain that ρ is a homomorphism. □

Since $\mathrm{Stab}(\Lambda_\varphi^+)$ is virtually cyclic and $\rho(\varphi) \neq 0$, the image of $\mathrm{Stab}(\Lambda_\varphi^+)$ under ρ is infinite cyclic. Thus it gives rise to a surjective homomorphism

$$\tau\colon \mathrm{Stab}(\Lambda_\varphi^+) \to \mathbb{Z} \tag{5}$$

with a finite kernel. Note that the kernel consists precisely of those elements of $\mathrm{Out}(F_r)$ that, when acting on CV_r, fix the axis $\mathcal{A}_\varphi$ pointwise. We show in Corollary 4.6 that $ker(\tau) = id$.

Proposition 4.4. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric lone axis fully irreducible outer automorphism and let $\psi \in \mathrm{Stab}(\Lambda_\varphi^+)$ be an outer automorphism that fixes $\mathcal{A}_\varphi$ pointwise. Let $f\colon \Gamma \to \Gamma$ be an affine train track representative of some power φ^R of φ such that all vertices of Γ are principal and all directions but one are fixed (guaranteed by Proposition 2.21). Let $h\colon \Gamma \to \Gamma$ be any isometry representing ψ. Then h permutes the f-fixed directions and hence fixes the (unique) nonfixed direction.*

Proof. ψ fixes the points Γ and $\Gamma\varphi$. Thus there exist isometries $h\colon \Gamma \to \Gamma$ and $h'\colon \Gamma\varphi \to \Gamma\varphi$ that represent an automorphism Ψ in the outer automorphism class of ψ, i.e., the following diagrams commute up to homotopy

$$\begin{array}{ccc} R_r & \xrightarrow{\Psi} & R_r \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ \Gamma & \xrightarrow{h} & \Gamma \end{array} \qquad \begin{array}{ccc} R_r & \xrightarrow{\Psi} & R_r \\ \downarrow{\scriptstyle f\circ m} & & \downarrow{\scriptstyle f\circ m} \\ \Gamma & \xrightarrow{h'} & \Gamma \end{array}$$

Therefore, the following diagram commutes up to homotopy

$$\begin{array}{ccc}\Gamma & \xrightarrow{h} & \Gamma \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \Gamma & \xrightarrow{h'} & \Gamma\end{array}$$

We will show that this diagram commutes and in fact that $h' = h$. Let $H\colon \Gamma \times I \to \Gamma$ be the homotopy so that $H(x, 0) = f \circ h(x)$ and $H(x, 1) = h' \circ f(x)$. Choose a lift $\widetilde{f}$ of f and a lift $\widetilde{h}$ of h to $\widetilde{\Gamma}$. Note that $\widetilde{f} \circ \widetilde{h}$ is a lift of $f \circ h$. Let $\widetilde{H}$ be a lift of H that starts with the lift $\widetilde{f} \circ \widetilde{h}$. Then $\widetilde{H}(x, 1)$ is a lift of $h' \circ f$, which we denote by $\widetilde{h' \circ f}$. This in turn determines a lift $\widetilde{h}'$ of h' so that $\widetilde{h' \circ f} = \widetilde{h}' \circ \widetilde{f}$. There exists a constant M so that for all $x \in \widetilde{\Gamma}$, we have $d(\widetilde{f} \circ \widetilde{h}(x), \widetilde{h}' \circ \widetilde{f}(x)) \leq M$, hence $\widetilde{f} \circ \widetilde{h}$ and $\widetilde{h}' \circ \widetilde{f}$ induce the same homeomorphism on $\partial\widetilde{\Gamma}$. Let $v \in \widetilde{\Gamma}$ be any vertex. By Lemma 2.15 there exist leaves ℓ_1, ℓ_2 of $\Lambda_+(\widetilde{\Gamma})$ that form a tripod whose vertex is v. Then $\widetilde{f} \circ \widetilde{h}(\ell_1), \widetilde{f} \circ \widetilde{h}(\ell_2), \widetilde{f} \circ \widetilde{h}(\ell_3)$ are embedded lines forming a tripod, as are $\widetilde{h}' \circ \widetilde{f}(\ell_1), \widetilde{h}' \circ \widetilde{f}(\ell_2), \widetilde{h}' \circ \widetilde{f}(\ell_3)$. Moreover, the ends of the two tripods coincide. Thus, $\widetilde{f} \circ \widetilde{h}(v) = \widetilde{h}' \circ \widetilde{f}(v)$. Since v was arbitrary and the maps are linear, we have $\widetilde{f} \circ \widetilde{h} = \widetilde{h}' \circ \widetilde{f}$ and $f \circ h = h' \circ f$.

We now show that $h' = h$. Let e_1 be the oriented edge representing the nonfixed direction of Df. For all $i \neq 1$, $Df(e_i) = e_i$. Let k be such that $h(e_k) = e_1$. We have $Dh' \circ Df = Df \circ Dh$. Thus for $i \neq 1, k$ we have $Dh'(e_i) = Dh(e_i)$. Since h and h' are isometries, this implies that $h'(e_i) = h(e_i)$ for $i \neq 1, k$. If $k = 1$ then h and h' agree on all but one oriented edge and therefore coincide, so we assume $k \neq 1$. If $e_1 \neq \bar{e}_k$ then $h(\bar{e}_i) = h'(\bar{e}_i)$ for both $i = 1$ and $i = k$, hence $h' = h$. Therefore we may assume that $\bar{e}_1 = e_k$. We have $h(e_k) = e_1$, hence $h(e_1) = e_k$. So $h'(\{e_1, e_k\}) = \{e_1, e_k\}$, hence we assume $h'(e_k) = e_k$ and $h'(e_1) = e_1$. Notice that the edge of e_1 must be a loop, since h and h' coincide on all other edges. Further, the orientation of the loop is preserved by h' and flipped by h. Now let $j \neq 1$ be so that $Df(e_1) = e_j$ and let u be an edge path so that $f(e_1) = e_j u e_1$. Thus, $f(e_k) = \overline{f(e_1)} = e_k \bar{u} \bar{e}_j$. We have

$$e_k \bar{u} \bar{e}_j = f(e_k) = f(h(e_1)) = h'(f(e_1)) = h'(e_j) h'(u) h'(e_1).$$

Thus $h'(e_j) = e_k$, so $j = k$. Hence $Df(e_1) = e_k = Df(e_k)$. So the unique illegal turn of f is $\{e_1, \bar{e}_1\}$. But this is impossible since f is a homotopy equivalence and must fold to the identity. Thus, $h = h'$ and so, since we have from the previous paragraph that $f \circ h = h' \circ f$, we now know that the following diagram commutes:

$$\begin{array}{ccc} \Gamma & \xrightarrow{h} & \Gamma \\ \downarrow f & & \downarrow f \\ \Gamma & \xrightarrow{h} & \Gamma. \end{array}$$

Let e be an edge so that the direction defined by e is fixed by Df. We have $Dh(e) = Dh(Df(e)) = Df(Dh(e))$, therefore $Dh(e)$ is also a fixed direction. Thus $h(e)$ defines a fixed direction, hence the f-fixed directions are permuted by h. □

Proposition 4.5. *Under the conditions of Proposition* 4.4*, h is the identity on* Γ.

Proof. Let e be the oriented edge of Γ representing the unique direction that is not f-fixed (or f-periodic). By Proposition 4.4, we know that $h(e) = e$. Let p be an f-periodic point in the interior of e. We can switch to a power of f fixing p. Let $\ell \in \Lambda^+_\varphi(\Gamma)$ be the leaf of the lamination obtained by iterating a neighborhood of p (see Definition 2.8). Denote by $\widetilde{\Gamma}$ the universal cover of Γ and let $\widetilde{p}$ be a lift of p and $\widetilde{e}$ and $\widetilde{\ell}$ be the corresponding lifts of e and ℓ. Let $\widetilde{h}$ and $\widetilde{f}$ be the respective lifts of h and f fixing the point $\widetilde{p}$. The lift $\widetilde{f}$ fixes $\widetilde{\ell}$, since this leaf is generated by $\widetilde{f}$-iterating a neighborhood of $\widetilde{p}$ contained in $\widetilde{e}$.

We first claim $\widetilde{f}$ fixes only one leaf of $\widetilde{\Lambda}^+_\varphi(\Gamma)$. Indeed, if $\widetilde{\ell}'$ is another such leaf, both ends of $\widetilde{\ell}'$ are $\widetilde{f}$-attracting, so there exists an $\widetilde{f}$-fixed point $\widetilde{q} \in \widetilde{\ell}'$. If $\widetilde{q} \neq \widetilde{p}$, then the segment between them is an NP, contradicting the fact that f has no PNPs (see Remark 2.20). Thus $\widetilde{q} = \widetilde{p}$. The intersection $\widetilde{\ell}' \cap \widetilde{\ell}$ contains $\widetilde{p}$ but since $\widetilde{p}$ is not a branch point, it must also contain $\widetilde{e}$, i.e., the edge containing $\widetilde{p}$. But since $\widetilde{\ell}$ and $\widetilde{\ell}'$ are both $\widetilde{f}$-fixed they must both contain $\widetilde{f}^k(\widetilde{e})$ for each k. Thus $\widetilde{\ell} = \widetilde{\ell}'$.

We now claim that $\widetilde{h}(\widetilde{\ell}) = \widetilde{\ell}$. By the previous paragraph, it suffices to show that $\widetilde{f}(\widetilde{h}(\widetilde{\ell})) = \widetilde{h}(\widetilde{\ell})$. We have $\widetilde{h}(\widetilde{\ell}) = \widetilde{h} \circ \widetilde{f}(\widetilde{\ell}) = \widetilde{f} \circ \widetilde{h}(\widetilde{\ell}) = \widetilde{f}(\widetilde{h}(\widetilde{\ell}))$, and our claim is proved.

Recall from before that $\widetilde{h}(\widetilde{e}) = \widetilde{e}$. Since $\widetilde{h}$ is an isometry, it restricts to the identity on $\widetilde{\ell}$. Projecting to Γ, since ℓ covers all of Γ, we get that h equals the identity on Γ. □

Recall the surjective homomorphism τ from Equation (5).

Corollary 4.6. *Let $\varphi \in \mathrm{Out}(F_r)$ be an ageometric fully irreducible outer automorphism such that the axis bundle $\mathcal{A}_\varphi$ consists of a unique axis, then $Ker(\tau) = \{id\}$.*

Proof of Theorem A. We showed in Corollary 4.6 that $Ker(\tau) = id$. It then follows from Equation (5) that $\mathrm{Stab}(\Lambda^+_\varphi) \cong \mathbb{Z}$. The rest follows from Corollary 3.3. □

Proof of Theorem B. By Corollary 3.3 and Theorem A,

$$\mathrm{Cen}(\langle\varphi\rangle) \leq \mathrm{Comm}(\langle\varphi\rangle)$$

and is of index ≤ 2. If equality holds, then we are in Case (1) and are done. Otherwise there is a short exact sequence

$$1 \to \mathrm{Cen}(\langle\varphi\rangle) \to \mathrm{Comm}(\langle\varphi\rangle) \to \mathbb{Z}_2 \to 1. \tag{6}$$

There are two homomorphisms $\mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{Cen}(\langle\varphi\rangle))$. We call the one whose image is the identity in $\mathrm{Aut}(\mathrm{Cen}(\langle\varphi\rangle))$ the trivial action and we call the one mapping the identity in $\mathbb{Z}_2$ to the automorphism in $\mathrm{Aut}(\mathrm{Cen}(\langle\varphi\rangle))$ taking a generator to its inverse the nontrivial action. First suppose $\mathbb{Z}_2$ acts trivially. Let $\psi \in \mathrm{Comm}(\langle\varphi\rangle)$ be any outer automorphism mapping to $1 \in \mathbb{Z}_2$, then $\psi \notin \mathrm{Cen}(\langle\varphi\rangle)$ and $\psi\varphi\psi^{-1} = \varphi$ (because the action is trivial) and this is a contradiction. If $\mathbb{Z}_2$ acts nontrivially, then $H^2(\mathbb{Z}_2, \mathbb{Z}) \cong \{0\}$ classifies the possible group extensions in the short exact sequence (6) (see [Ben91, Proposition 3.7.3]). Hence, the only possible extension is $\mathrm{Comm}(\langle\varphi\rangle) \cong \mathrm{Cen}(\langle\varphi\rangle) \rtimes \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}_2$. Again let $\psi \notin \mathrm{Cen}(\langle\varphi\rangle)$ be any automorphism mapping to $1 \in \mathbb{Z}_2$. Since the homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathrm{Cen}(\langle\varphi\rangle))$ is mapping 1 to the automorphism taking a generator to its inverse, $\psi\varphi\psi^{-1} = \varphi^{-1}$. Hence, $\psi \in N(\langle\varphi\rangle)$. So $\mathrm{Comm}(\langle\varphi\rangle) = \langle \mathrm{Cen}(\langle\varphi\rangle), \psi\rangle \leq N(\langle\varphi\rangle)$. On the other hand, by Remark 2.26, $N(\langle\varphi\rangle) \leq \mathrm{Comm}(\langle\varphi\rangle)$. Hence, $N(\langle\varphi\rangle) = \mathrm{Comm}(\langle\varphi\rangle)$.

We now prove the last part of the theorem. If $\mathrm{Comm}(\langle\varphi\rangle) \cong \mathbb{Z}_2 * \mathbb{Z}_2$, then it contains an element ψ mapping to the nonzero element in $\mathbb{Z}_2$ (as before) so that $\psi\varphi\psi^{-1} = \varphi^{-1}$. In other words, φ^{-1} is in the conjugacy class of φ. Hence, it has the same index list and ideal Whitehead graph as φ (and is also ageometric fully irreducible). In particular, φ^{-1} satisfies the conditions to be a lone axis fully irreducible outer automorphism [MP16, Theorem 4.6]. □

References

[Ben91] Benson, David J. Representations and cohomology. I. Basic representation theory of finite groups and associative algebras. Cambridge Studies in Advanced Mathematics, 30. *Cambridge University Press, Cambridge*, 1991. xii+224 pp. ISBN: 0-521-36134-6. MR1110581 (92m:20005), Zbl 0718.20001, doi: 10.1017/CBO9780511623615.

[BF94] Bestvina, Mladen; Feighn, Mark. Outer limits. Preprint, 1994. `http://andromeda.rutgers.edu/~feighn/papers/outer.pdf`.

[BFH97] Bestvina, Miaden; Feighn, Mark; Handel, Michael. Laminations, trees, and irreducible automorphisms of free groups. *Geom. Funct. Anal.* **7** (1997), no. 2, 215–244. MR1445386 (98c:20045), Zbl 0884.57002, doi: 10.1007/PL00001618.

[BH92] Bestvina, Mladen; Handel, Michael. Train tracks and automorphisms of free groups. *Ann. of Math.* (2) **135** (1992), no. 1, 1–51. MR1147956, Zbl 0757.57004, doi: 10.2307/2946562.

[CL95] Cohen, Marshall M.; Lustig, Martin. Very small group actions on **R**-trees and Dehn twist automorphisms. *Topology* **34** (1995), no. 3, 575–617. MR1341810, Zbl 0844.20018, doi: 10.1016/0040-9383(94)00038-M.

[Cou14] Coulbois, Thierry. Free group automorphisms and train-track representative in python/sage. https://github.com/coulbois/sage-train-track, 2012–2014.

[CM87] Culler, Marc; Morgan, John W. Group actions on **R**-trees. *Proc. London Math. Soc.* (3) **55** (1987), no. 3, 571–604. MR0907233, Zbl 0658.20021, doi: 10.1112/plms/s3-55.3.571.

[CV86] Culler, Marc; Vogtmann, Karen. Moduli of graphs and automorphisms of free groups. *Invent. Math.* **84** (1986), no. 1, 91–119. MR0830040, Zbl 0589.20022, doi: 10.1007/BF01388734.

[FH09] Feighn, Mark; Handel, Michael. Abelian subgroups of Out(F_n). *Geom. Topol.* **13** (2009), no. 3, 1657–1727. MR2496054 (2010h:20068), Zbl 1201.20031, arXiv:math/0612705, doi: 10.2140/gt.2009.13.1657.

[FH11] Feighn, Mark; Handel, Michael. The recognition theorem for Out(F_n). *Groups Geom. Dyn.* **5** (2011), no. 1, 39–106. MR2763779 (2012b:20061), Zbl 1239.20036, arXiv:math/0612702, doi: 10.4171/GGD/116.

[FM11] Francaviglia, Stefano; Martino, Armando. Metric properties of outer space. *Publ. Mat.* **55** (2011), no. 2, 433–473. MR2839451, Zbl 1268.20042, arXiv:0803.0640, doi: 10.5565/PUBLMAT_55211_09.

[HM11] Handel, Michael; Mosher, Lee. Axes in outer space. *Mem. Amer. Math. Soc.* **213** (2011), no. 1004, vi+104 pp. ISBN: 978-0-8218-6927-7. MR2858636, Zbl 1238.57002, arXiv:math/0605355, doi: 10.1090/S0065-9266-2011-00620-9.

[KL11] Kapovich, Ilya; Lustig, Martin. Stabilizers of $\mathbb{R}$-trees with free isometric actions of F_N. *J. Group Theory* **14** (2011), no. 5, 673–694. MR2831965, Zbl 1262.20031, arXiv:0904.1881, doi: 10.1515/jgt.2010.070.

[KP15] Kapovich, Ilya; Pfaff, Catherine. A train track directed random walk on Out(F_r). *Internat. J. Algebra Comput.* **25** (2015), no. 5, 745–798. MR3384080, Zbl 1351.20025, arXiv:1409.8044, doi: 10.1142/S0218196715500186.

[LL03] Levitt, Gilbert; Lustig, Martin. Irreducible automorphisms of F_n have north-south dynamics on compactified outer space. *J. Inst. Math. Jussieu* **2** (2003), no. 1, 59–72. MR1955207, Zbl 1034.20038, doi: 10.1017/S1474748003000033.

[McC94] McCarthy, John D. Normalizers and centralizers of pseudo-anosov mapping classes. Preprint, 1994. http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.568.8956&rep=rep1&type=pdf.

[MP16] Mosher, Lee; Pfaff, Catherine. Lone axes in outer space. *Algebr. Geom. Topol.* **16** (2016), no. 6, 3385–3418. MR3584262, Zbl 06666104, arXiv:1311.6855, doi: 10.2140/agt.2016.16.3385.

[Pfa13] Pfaff, Catherine. Ideal Whitehead graphs in *Out*(F_r) II: the complete graph in each rank. *J. Homotopy Relat. Struct.* **10** (2015), no. 2, 275–301. MR3348263, Zbl 06448499, arXiv:1301.6645, doi: 10.1007/s40062-013-0060-5.

[RW15] Rodenhausen, Moritz; Wade, Richard D. Centralisers of Dehn twist automorphisms of free groups. *Math. Proc. Cambridge Philos. Soc.* **159** (2015), no. 1, 89–114. MR3349333, arXiv:1206.5728, doi: 10.1017/S0305004115000225.

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