CP-stability and the local lifting property

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ABSTRACT. The purpose of this note is to discuss the local lifting property in terms of an equivalent approximation-type property, CP-stability, which was formulated by the author and Isaac Goldbring for the purposes of studying the continuous model theory of C*-algebras and operator systems.

CONTENTS

1. Statement of the main results

The following definition first appears in [GS15].

Definition 1.1. An operator system X is said to be *CP-stable* if for any finite dimensional subsystem $E \subset X$ and $\delta > 0$ there is a finite-dimensional subsystem $E \subset S \subset X$ and $k, \epsilon > 0$ so that for every C*-algebra A and any unital linear map $\phi : S \to A$ with $\|\phi\|_k < 1 + \epsilon$ there exists a u.c.p. map $\psi : E \to A$ so that $\|\phi|_E - \psi\| < \delta$.

Let $u_1, \ldots, u_n$ be the canonical generators of $C^*(\mathbb{F}_n)$ and let W_n be the operator system spanned by the set $\{u_i^* u_j : 1 \le i, j \le n+1\}$ where $u_{n+1} := 1$. The first result gives a quantitative version of CP-stability for the operator systems W_n using work of Farenick and Paulsen [FP12].

Theorem A. *The operator system W_n is CP-stable. In particular, for any $\delta > 0$ there exists $\epsilon > 0$ so that for any unital linear map $\phi : W_n \to A$ into an arbitrary unital C^*-algebra with $\|\phi\|_{n+1} < 1 + \epsilon$, there is a u.c.p. map $\psi : W_n \to A$ so that $\|\psi - \phi\| < \delta$.*

Note that by [CH85, Corollary 4.7] no such result can hold for the related generator subsystem of the *reduced* C*-algebra $C^*_\lambda(\mathbb{F}_n)$.

We say an operator system X has the *local lifting property* (LLP) of Kirchberg if for every unital C*-algebra A, every ideal J of A, and every u.c.p. map $\varphi : X \to A/J$ and every finite-dimensional subsystem $E \subset X$ the restricted map $\varphi|_E$ admits a u.c.p. lifting $\tilde{\varphi} : E \to A$.[1] It was shown in [GS17] that for C*-algebras CP-stability is equivalent to the local lifting property.

Using operator system tensor product characterizations for exactness and the LLP (see [KP+13]), Kavruk showed that a finite-dimensional operator system has the LLP if and only if its dual system is exact [Ka14, Theorem 6.6]. We show that conversely, one can use the fact that the dual is exact (in the sense of admitting a nuclear embedding), i.e., that the operator system is CP-stable, to recover Kirchberg's tensor characterization of the LLP [KP+13, Ki93].

Theorem B. *If E is a finite-dimensional operator system which is CP-stable, then $E \otimes_{\min} \mathcal{B}(\ell^2) \cong E \otimes_{\max} \mathcal{B}(\ell^2)$ as operator systems.*

Using techniques from [Pi96] or [Ka14] Theorem A and Theorem B give a new proof of the fact (due to Kirchberg [Ki94]) that

$$C^*(\mathbb{F}_n) \otimes_{\min} \mathcal{B}(\ell^2) \cong C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{B}(\ell^2).$$

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2. Proofs of the main results

The following result is due to Farenick and Paulsen [FP12]: see the remarks after Definition 2.1 therein.

Lemma 2.1. *The "covering" map $\gamma_n : M_{n+1} \to W_n$ defined by*

$$\gamma_n(e_{ij}) = \frac{1}{n+1} u_i^* u_j,$$

where $u_1, \ldots, u_{n+1}$ are defined as above, is u.c.p. and the kernel J_{n+1} consists of all diagonal matrices in M_{n+1} of trace zero.

Remarkably, Farenick and Paulsen [FP12, Theorem 2.4] go on to show that:

Theorem 2.2 (Farenick–Paulsen). *The map $\overline{\gamma_n} : M_{n+1}/J_{n+1} \to W_n$ is a complete order isomorphism where the quotient space M_{n+1}/J_{n+1} is equipped with its canonical operator system structure as defined in [KP+13, Section 3].*

The strategy of our proof of Theorem A will be to make use of the fact that matrix algebras are CP-stable.

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[1]The definition we give here is termed the *operator system local lifting property* (OSLLP) in [Ka14, KP+13] though for our purposes we will not make a distinction.

Lemma 2.3 (Proposition 2.40 in [GS15]). *Given* k, *for any* $\delta > 0$ *there exists* $\epsilon > 0$ *so that for any* C*-*algebra* A *and any unital linear map* $\phi : M_k \to A$ *with* $\|\phi\|_k < 1 + \epsilon$, *there exists a u.c.p. map* $\tilde{\phi} : M_k \to A$ *so that* $\|\tilde{\phi} - \phi\| < \delta$.

For the reader's convenience we provide a streamlined proof.

Proof. Suppose by contradiction that there is some $\delta > 0$ so that for every n there is some unital linear map $\phi_n : M_k \to A_n$ into some C*-algebra A_n so that $\|\phi\|_k < 1 + \frac{1}{n}$ so that $\|\psi - \phi\| \geq \delta$ for any u.c.p. map $\psi : M_k \to A_n$. Fix an nonprincipal ultrafilter ω on $\mathbb{N}$ and define $\mathcal{A} := (A_n)_\omega$ to be the ultrapower C*-algebra associated to the sequence (A_n) and ω and $\widetilde{\mathcal{A}} := \prod_n A_n$. Consider the map $\phi := (\phi_\bullet) : M_k \to \mathcal{A}$. Clearly ϕ is unital and $\|\phi\|_k = 1$ whence by [Pa03, Proposition 2.11] ϕ is k-positive. By Choi's theorem [Pa03, Theorem 3.14] ϕ is therefore u.c.p. and the proof of the Choi+Effros lifting theorem [BrO08, Theorem C.3] shows there is thus a u.c.p. lift $\tilde{\phi} : M_k \to \widetilde{\mathcal{A}}$. However, this shows that the sequence (ϕ_n) is well-approximated by u.c.p. maps for $n \in \omega$ generic, a contradiction. $\square$

Proof of Theorem A. We begin by fixing $\delta > 0$ and a unital C*-algebra A. Suppose we have a unital linear map $\phi : W_n \to A$ with $\|\phi\|_{n+1} < 1 + \epsilon$ for some $\epsilon > 0$ sufficiently small and to be determined later. We will show that we can find a u.c.p. map $\psi : W_n \to A$ so that $\|\psi - \phi\| < \delta$.

Let $\phi' := \phi \circ \gamma_n : M_{n+1} \to A$ which is again unital and linear with $\|\phi\|_{n+1} < 1 + \epsilon$. By Lemma 2.3 we may choose $\epsilon > 0$ sufficiently small so that there is a u.c.p. map $\psi' : M_{n+1} \to A$ so that $\|\psi' - \phi'\| < \delta/16n^4$. Since $\phi'(e_{ii}) = \frac{1}{n+1}1_A$, we have that $\|\psi'(e_{ii}) - \frac{1}{n+1}1_A\| < \delta/16n^4$ whence $b_i := \psi'(e_{ii})$ is uniformly invertible and positive. Let $B \in M_{n+1}(A)$ be the diagonal matrix such that $B_{ii} := b_i^{-1/2}$. Let $\Psi' := [\psi'(e_{ij})] \in M_{n+1}(A)$ be the Choi matrix associated to ψ'. Since Ψ' is positive, so is $\Psi'' := B\Psi'B$, whence it defines a c.p. map $\psi'' : M_{n+1} \to A$ via the reverse correspondence $\psi''(e_{ij}) := \Psi''_{ij}$. We can see manifestly that $\psi''(e_{ii}) = \frac{1}{n+1}1_A$ whence ψ'' is unital, $J_{n+1} \subset \ker(\psi'')$, and $\|\psi'' - \psi'\| < \delta/4n^2$.

Identifying W_n with the quotient operator system M_{n+1}/J_{n+1} by Theorem 2.2, since $J_{n+1} \subset \ker(\psi'')$ it follows by [KP+13, Proposition 3.6] that there is u.c.p. map $\psi : W_n \to A$ so that $\sup_{i,j}\|\psi(u_i^* u_j) - \phi(u_i^* u_j)\| < \delta/2n^2$. Alternatively, this is not difficult to see by setting $\psi := \psi'' \circ \gamma_n^{-1}$ and unraveling the definition of the quotient operator system structure via the identification given by Theorem 2.2. In any case it follows by the small perturbation argument that $\|\psi - \phi\| < \delta$, and we are done. $\square$

Remark 2.4. For a finite-dimensional operator system E, we say that a kernel $J \subset E$ is *stable* if for any δ there exists $\epsilon > 0$ such that whenever $\phi : E \to A$ is a u.c.p. map into a C*-algebra A with $\|\phi|_J\| < \epsilon$ there is a u.c.p. map $\phi' : E \to A$ with $J \subset \ker(\phi')$ and $\|\phi - \phi'\| < \delta$. The proof of the previous proposition transfers to this context verbatim to show that whenever E is CP-stable and J is stable, then E/J is again CP-stable. A

standard ultraproduct argument as in Lemma 2.3 above (i.e., an argument by model theoretic compactness), together with the next proposition, shows that if E/J is CP-stable, then J is stable.

Two formal weakenings of the LLP were introduced by the author and Isaac Goldbring: the *local ultrapower lifting property* (LULP) [GS15, Proposition 2.42] and the *approximate local lifting property* (ALLP) [GS17, Definition 7.3].[2] Both definitions carry over straightforwardly to the category of operator systems. For instance, an operator system X can be said to have the LULP if every u.c.p. map $\phi : X \to A_\omega$ admits local u.c.p. lifts to $\ell^\infty(A)$. The following proposition is essentially contained in [GS15, GS17]. We provide a sketch of the proof for the convenience of the reader.

Proposition 2.5. *For an operator system X the following statements are equivalent:*

(1) X *has the LLP;*
(2) X *has the ALLP;*
(3) X *has the LULP;*
(4) X *is CP-stable.*

The equivalence of the first two statements essentially appears in the work of Effros and Haagerup [EH85, Theorem 3.2]. We also remark that using the equivalence with the ALLP, it is easy to see that the LLP passes to inductive limits, noting that it suffices to check the ALLP only on a dense subalgebra.

Proof. The equivalence of (3) and (4) is proved in [GS15, Proposition 2.42]. The implication $(1) \Rightarrow (2)$ is straightforward. For $(2) \Rightarrow (3)$, we note that by the small perturbation we can require the approximate lifts to be unital, and we may also assume they are $*$-linear. In conjunction with [BrO08, Corollary B.11] which shows that we can correct such an approximate lift to a u.c.p. map a controlled distance away (depending on the dimension of the domain), we can thus assume that the approximate lifts are u.c.p. from which the implication follows easily. We include a proof of $(4) \Rightarrow (2)$, though it closely follows the reasoning given in [GS17, Proposition 7.7].

To this end, note that by the main result of [RS89] that for any finite-dimensional operator system, any u.c.p. map $\phi : E \to A/J$ admits n-positive unital liftings $\tilde{\phi}_n : E \to A$ for every n. Hence if E was a finite dimensional subsystem of a CP-stable system X and $\phi : X \to A/J$ was u.c.p. it would follow that for every n there is a u.c.p. map $\psi : E \to A$ so that $\|\pi_J \circ \psi - \phi|_E\| < \frac{1}{n}$, where $\pi_J : A \to A/J$ is the quotient $*$-epimorphism. Hence X has the ALLP.

Finally, the implication $(2) \Rightarrow (1)$ follows from a foundational result of Arveson that liftable u.c.p. maps are closed in the point-norm topology: see [BrO08, Lemma C.2]. □

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[2]The ALLP is implicitly formulated in the work of Effros and Haagerup [EH85], where it is shown to be equivalent to the LLP.

Let $\mathcal{OS}_n$ be the set of all complete isomorphism classes of n-dimensional operator systems. The set $\mathcal{OS}_n$ is naturally equipped with two complete metrics, the cb-Banach distance and the weak metric: see [GS17] for details. With the equivalence of LLP and CP-stability in hand, we give a new proof of a result of Kavruk [Ka14].

Proposition 2.6 (Kavruk). *A finite-dimensional operator system* E *is exact if and only if the dual system* E^* *is CP-stable.*

Proof. It is well known (see [Pi95, GS17]) that E is an exact n-dimensional operator system if and only if for any sequence $\phi_\alpha : \mathsf{E}_\alpha \to \mathsf{E}$ of unital $*$-linear maps such that $\|\phi_\alpha\|_k \to 1$ for all k, there is a sequence of unital maps $\psi_\alpha : \mathsf{E}_\alpha \to \mathsf{E}$ with $\|\psi_\alpha\|_{cb} \to 1$ with $\|\psi_\alpha - \phi_\alpha\| \to 0$. Dualizing (noting by [JP95, Proposition 2.1] or [BlP91] that this is a well behaved operation) and applying a standard compactness argument, we see that this is implies that E^* is CP-stable. The converse follows similarly by unraveling the definitions. $\square$

It follows as a consequence that for any kernel $\mathsf{J} \subset \mathsf{M}_n$ the quotient system M_n/J is CP-stable. Indeed the trace pairing gives a complete order isomorphism $(\mathsf{M}_n/\mathsf{J})^* \cong \mathsf{J}^\perp \subset \mathsf{M}_n$ [FP12, Proposition 1.8] which is matricial, hence exact. By Remark 2.4 this implies that every kernel in M_n is stable (in the sense given therein) which can be viewed as a generalization of Lemma 2.1.

In the category of operator systems, the correct treatment of tensor products has only recently appeared in the work of Kavruk, Paulsen, Todorov, and Tomforde [KP+11, KP+13]. We refer to these works for the basic definitions and properties of various operator system tensor products. Using these ideas we give a new proof of a famous and difficult theorem of Kirchberg [Ki94]. A short and particularly elegant proof of the same result in the context of operator spaces was given by Pisier [Pi96]. A second elementary proof was recently discovered by Farenick and Paulsen [FP12]. (See also [Ha14, Oz13].)

Theorem 2.7 (Kirchberg). *If* E *is a finite-dimensional operator system which is CP-stable, then*

$$\mathsf{E} \otimes_{\min} \mathcal{B}(\ell^2) = \mathsf{E} \otimes_{\max} \mathcal{B}(\ell^2).$$

Lemma 2.8. *If* E *is a finite-dimensional operator system such that*

$$\mathsf{E} \otimes_{\min} \mathsf{F} \cong \mathsf{E} \otimes_{\max} \mathsf{F}$$

for all finite-dimensional operator systems F, *then*

$$\mathsf{E}^* \otimes_{\min} \mathcal{B}(\ell^2) = \mathsf{E}^* \otimes_{\max} \mathcal{B}(\ell^2).$$

Proof. Since $\mathcal{B}(\ell^2)$ has the WEP, by [KP+13, Lemma 6.1 and Theorem 6.7] it suffices to check that $\mathsf{E}^* \otimes_{\min} \mathsf{F} = \mathsf{E}^* \otimes_{\max} \mathsf{E}$ for any finite-dimensional

operator system F. Using [FP12, Proposition 1.9] we have that

$$(\mathsf{E}^* \otimes_{\min} \mathsf{F})^* \cong \mathsf{E} \otimes_{\max} \mathsf{F}^* \cong \mathsf{E} \otimes_{\min} \mathsf{F}^*$$

as operator systems. By the same

$$\mathsf{E}^* \otimes_{\min} \mathsf{F} \cong (\mathsf{E}^* \otimes_{\min} \mathsf{F})^{**} \cong (\mathsf{E} \otimes_{\min} \mathsf{F}^*)^* \cong \mathsf{E}^* \otimes_{\max} \mathsf{F},$$

and we are done.   □

Specializing to $\mathsf{E} = \mathsf{M}_\mathfrak{n}$, we have that

$$\mathsf{M}_\mathfrak{n}^* \otimes_{\min} \mathcal{B}(\ell^2) = \mathsf{M}_\mathfrak{n}^* \otimes_{\max} \mathcal{B}(\ell^2).$$

However, we note that it is well-known that $\mathsf{M}_\mathfrak{n}$ and $\mathsf{M}_\mathfrak{n}^*$ are isomorphic as operator systems via the trace pairing, and that the min- and max-tensors of operator systems coincide with the usual definitions for C*-algebras. For the sake of clarity we maintain the distinction between $\mathsf{M}_\mathfrak{n}$ and $\mathsf{M}_\mathfrak{n}^*$.

Lemma 2.9. *Let* E *be a finite-dimensional operator system with the LLP. Then for every* $\epsilon > 0, \mathsf{k}$ *there is* $\mathfrak{n}$ *and a u.c.p. map* $\phi : \mathsf{M}_\mathfrak{n}^* \to \mathsf{E}$ *so that for any positive* $\mathsf{x} \in \mathsf{M}_\mathsf{k}(\mathsf{E} \otimes_{\min} \mathcal{B}(\ell^2))^+$ *there is* $\hat{\mathsf{x}} \in \mathsf{M}_\mathsf{k}(\mathsf{M}_\mathfrak{n}^* \otimes \mathcal{B}(\ell^2))^+$ *positive so that* $\|\mathsf{x} - (\phi \otimes \mathrm{id})_\mathsf{k}(\hat{\mathsf{x}})\| < \epsilon$.

Proof. Let us fix $\epsilon, \mathsf{k} > 0$. Using [KP+13, Lemma 8.5] we may identify the positive cone $\mathsf{M}_\mathsf{k}(\mathsf{E} \otimes_{\min} \mathcal{B}(\ell^2))^+$ with the space $\mathrm{CP}(\mathsf{E}^*, \mathsf{M}_\mathsf{k}(\mathcal{B}(\ell^2)))$ of completely positive maps $\phi : \mathsf{E}^* \to \mathsf{M}_\mathsf{k}(\mathcal{B}(\ell^2))$. By Proposition 2.6 E^* is exact, so there are matricial operator systems $\mathsf{E}_\mathfrak{m} \subset \mathsf{M}_{\ell_\mathfrak{m}}$ and u.c.p. bijections $\phi_\mathfrak{m} : \mathsf{E}^* \to \mathsf{E}_\mathfrak{m}$ with $\|\phi_\mathfrak{m}^{-1}\|_{\mathrm{cb}} \to 1$.

By pre-composition each map $\phi_\mathfrak{m}$ induces a map

$$\Phi_{\mathfrak{m},\mathsf{k}} : \mathrm{CP}(\mathsf{M}_{\ell_\mathfrak{m}}, \mathsf{M}_\mathsf{k}(\mathcal{B}(\ell^2))) \to \mathrm{CP}(\mathsf{E}^*, \mathsf{M}_\mathsf{k}(\mathcal{B}(\ell^2)))$$

which preserves unitality and is easily identified with the u.c.p. map

$$(\phi_\mathfrak{m}^* \otimes \mathrm{id})_\mathsf{k} : \mathsf{M}_\mathsf{k}(\mathsf{M}_{\ell_\mathfrak{m}}^* \otimes \mathcal{B}(\ell^2)) \to \mathsf{M}_\mathsf{k}(\mathsf{E} \otimes_{\min} \mathcal{B}(\ell^2)).$$

(We are using that the minimal operator system tensor product is functorial: see [KP+11, Theorem 4.6].)

Given a u.c.p. map $\psi : \mathsf{E}^* \to \mathcal{B}(\mathsf{H})$ we may pre-compose with $\phi_\mathfrak{m}^{-1}$ to obtain a unital, self-adjoint map $\psi_\mathfrak{m} : \mathsf{E}_\mathfrak{m} \to \mathcal{B}(\mathsf{H})$ which we may isometrically extend to $\hat{\psi}_\mathfrak{m} : \mathsf{M}_\mathfrak{m} \to \mathcal{B}(\mathsf{H})$. As $\|\psi_\mathfrak{m}\|_{\mathrm{cb}} \leq \|\phi_\mathfrak{m}^{-1}\|_{\mathrm{cb}} \to 1$, by [BrO08, Corollary B.9] there is an approximating sequence to $(\hat{\psi}_\mathfrak{m})$ consisting of u.c.p. maps $\upsilon_\mathfrak{m} : \mathsf{M}_\mathfrak{m} \to \mathcal{B}(\mathsf{H})$ with $\|\hat{\psi}_\mathfrak{m} - \upsilon_\mathfrak{m}\| \leq 2(\|\phi_\mathfrak{m}^{-1}\|_{\mathrm{cb}} - 1)$. Via the identification with $\Phi_{\mathfrak{m},\mathsf{k}}$ it therefore follows that $(\phi_\mathfrak{m}^* \otimes \mathrm{id})_\mathsf{k}$ restricted to $\mathsf{M}_\mathsf{k}(\mathsf{M}_{\ell_\mathfrak{m}}^* \otimes \mathcal{B}(\ell^2))^+$ is ϵ-surjective into $\mathsf{M}_\mathsf{k}(\mathsf{E} \otimes_{\min} \mathcal{B}(\ell^2))^+$ for $\mathfrak{m}$ sufficiently large.   □

Proof of Theorem 2.7. Given $\epsilon, \mathsf{k} > 0$, by Lemma 2.9 we can find $\mathfrak{n}$ such that there is a u.c.p. map $\phi : \mathsf{M}_\mathfrak{n}^* \to \mathsf{E}$ so that

$$(\phi \otimes \mathrm{id})_\mathsf{k} : \mathsf{M}_\mathsf{k}(\mathsf{M}_\mathfrak{n}^* \otimes_{\min} \mathcal{B}(\ell^2))^+ \to \mathsf{M}_\mathsf{k}(\mathsf{E} \otimes_{\min} \mathcal{B}(\ell^2))^+$$

is ϵ-surjective. By Lemma 2.8 we have that

$$M_k(M_n^* \otimes_{\min} \mathcal{B}(\ell^2))^+ = M_k(M_n^* \otimes_{\max} \mathcal{B}(\ell^2))^+.$$

Since the maximal tensor norm is functorial [KP+11, Theorem 5.5], it follows that $(\phi \otimes \mathrm{id})_k$ maps $M_k(M_n^* \otimes_{\max} \mathcal{B}(\ell^2))^+$ into $M_k(E \otimes_{\max} \mathcal{B}(\ell^2))^+$. As ϵ was arbitrary this shows that $M_k(E \otimes_{\min} \mathcal{B}(\ell^2))^+ \subset M_k(E \otimes_{\max} \mathcal{B}(\ell^2))^+$, and we are done. $\square$

3. CP-stability and seminuclearity

By reframing the LLP as CP-stability, one can view the LLP and exactness as dual "rigidity" properties in the sense that they interpolate between the topologies of $\|\cdot\|_n$-convergence for every n and $\|\cdot\|_{cb}$-convergence. Consequently one may contrive (and investigate) various complimentary "soft" approximation properties which play against the LLP to produce strong approximations.

Definition 3.1. A unital C*-algebra A is said to be *seminuclear* if for every finite-dimensional operator system $E \subset A$ and $k \in \mathbb{N}$ there is a unital map $\varphi : M_n \to A$ with $\|\varphi\|_{cb} < 1 + 1/k$, $E \subset \varphi(M_n)$, and $\|\varphi^{-1}|_E\|_k < 1 + 1/k$.

The following proposition follows by standard techniques.

Proposition 3.2. *A unital C*-algebra is nuclear if and only if it is CP-stable and seminuclear.*

In this light is seems strange that (at least to my knowledge) this property has remained unexamined. Clearly, there are nonseminuclear C*-algebras, namely $C^*(\mathbb{F}_n)$ for $2 \le n \le \infty$. It is likely that $\mathcal{B}(\ell^2)$ is also not seminuclear. At present there seems to be no example of a seminuclear C*-algebra which is not nuclear. We remark that the mirror (one should not say dual) property is trivial: every finite-dimensional operator system E for any k admits some u.c.p. map $\varphi : E \to M_n$ with $\|\varphi^{-1}\|_k < 1 + 1/k$. However, Lance's weak expectation property (WEP) is implied by, and possibly equivalent to, a seemingly only slightly stronger property.

Proposition 3.3. *A unital C*-algebra A has the WEP if for every finite-dimensional operator system $E \subset A$ and $k \in \mathbb{N}$ there is a unital map*

$$\varphi : M_n \to A$$

with $\|\varphi\|_k < 1 + 1/k$, $E \subset \varphi(M_n)$, and $\|\varphi^{-1}|_E\|_{cb} < 1 + 1/k$.

Proof. Let $A \subset \mathcal{B}(H)$ be the universal representation of A. Let $\mathcal{F}$ be the net of all pairs (F, k) of finite-dimensional operator systems $F \subset \mathcal{B}(H)$ and $k \in \mathbb{N}$ with the natural product ordering. For $(F, k) \in \mathcal{B}(H)$, let $E := F \cap A$. Finding a suitable map $\varphi : M_n \to A$ as above, we can extend φ^{-1} to a unital map $\psi_{F,k} : F \to M_n$ with $\|\psi_{F,k}\|_{cb} < 1 + 1/k$. Setting $\theta_{F,k} := \varphi \circ \psi_{F,k} : F \to A$ and taking a pointwise-ultraweak cluster point along $\mathcal{F}$, we see that we have

produced a u.c.p. map $\theta : \mathcal{B}(H) \to A''$ such that $\theta|_A = \mathrm{id}_A$, whence A has the WEP. $\qquad\square$

Let us draw out a few more connections and consequences. It has been speculated that the WEP enjoys the same complimentarity property with the LLP, i.e., a C^*-algebra which has both the LLP and the WEP is necessarily nuclear. This would refute Kirchberg's conjecture that the LLP implies the WEP. Since seminuclearity superficially resembles semidiscreteness while the WEP can be viewed as a weak version of injectivity, it is tempting to think there is a possible connection between the two.

Conjecture 3.4. *If A is a separable, unital C^*-algebra, then A has the WEP if and only if A is seminuclear.*

It is known that there are uncountably many pairwise nonisomorphic, separable, unital C^*-algebras with the WEP; see for instance [GS15, Propositions 2.3 and 2.23] for an elementary proof via model theoretic techniques.

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