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Cellular automata, duality and sofic groups

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To Tullio G. Ceccherini-Silberstein, on the occasion of his fiftieth anniversary

Abstract. We produce for arbitrary nonamenable group G and field $\mathbb{K}$ a nonpreinjective, surjective linear cellular automaton. This answers positively Open Problem (OP-14) in Ceccherini-Silberstein and Coornaert's monograph "Cellular Automata and Groups".

We also reprove in a direct manner, for linear cellular automata, the result by Capobianco, Kari and Taati that cellular automata over sofic groups are injective if and only if they are postsurjective.

These results come from considerations related to matrices over group rings: we prove that a matrix's kernel and the image of its adjoint are mutual orthogonals of each other. This gives rise to a notion of "dual cellular automaton", which is preinjective if and only if the original cellular automaton is surjective, and is injective if and only if the original cellular automaton is postsurjective.

Contents

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1. Introduction

1.1. Cellular automata. Let G be a group and let $\mathbb{K}$ be a field. A *linear cellular automaton* on G is — no more, no less — a square matrix with entries in the group ring $\mathbb{K}G$.

The interpretation of a linear cellular automaton $\Theta \in M_n(\mathbb{K}G)$ is as follows, see [7, Corollary 8.7.8]. Let S be a finite subset of G such that all entries of Θ are in the $\mathbb{K}$-span of S. Construct the graph $\mathscr{G}$ with vertex set G, and with an edge from g to gs for all $g \in G, s \in S$. Put a copy of the vector space $V := \mathbb{K}^n$ at each vertex of $\mathscr{G}$. Elements of the vector space $V^G = \{c\colon G \to V\}$ are called *configurations*. Then Θ defines a one-step evolution rule still written Θ on the space of configurations, in which each vertex of $\mathscr{G}$ inherits a new value in V depending on the values at its neighbours: one may write $\Theta = \sum_{s\in S} \Theta_s s$ for $\mathbb{K}$-matrices Θ_s, and then every configuration $c \in V^G$ evolves under Θ to the configuration taking at every $g \in G$ the value $\sum_{s\in S} \Theta_s(c(s^{-1}g))$. More concisely, c evolves to $\Theta \cdot c$. For more information on linear cellular automata, we defer to [7, Chapter 8].

Linear cellular automata are natural linear analogues of classical cellular automata, in which each vertex of $\mathscr{G}$ takes a value in a finite set A, which evolves according to the values at its neighbours. The cellular automaton is thus a locally-defined evolution rule on the compact space A^G. In particular, if $\mathbb{K}$ is a finite field, then every linear cellular automaton is also a classical cellular automaton.

The converse, however, is far from true: linear cellular automata are extremely restricted computational models, and there is no clear way of converting a classical cellular automaton into a linear one. Every self-map of a finite set A induces a self-map of the finite-dimensional vector space $V := \mathbb{K}A$, so cellular automata acting on A^G induce linear self-maps on $\mathbb{K}^{A^G}$, but this space is much larger than $V^G \cong \mathbb{K}^{A\times G}$: in pedantic terms, the former is a completion of the tensor power $\bigotimes_G V$ (the "measuring coalgebra" $\mathbb{K}G \rightharpoonup V$), while the latter is a completion of the direct sum $\bigoplus_G V$.

1.2. Sofic groups and surjunctivity. How are algebraic properties of the group G reflected in the cellular automata carried by $\mathscr{G}$? We single out some properties of cellular automata which have received particular attention: let us write $x \sim y$ for $x, y \in A^G$ when $\{g \in G \mid x(g) \neq y(g)\}$ is finite. A cellular automaton $\Theta\colon A^G \circlearrowleft$ is

- **injective** if $\Theta(x) = \Theta(y)$ implies $x = y$;
- **preinjective** if $\Theta(x) = \Theta(y) \wedge x \sim y$ implies $x = y$; when $x \neq y$ and $\Theta(x) = \Theta(y)$ and $x \sim y$ one calls such x, y *Mutually Eraseable Configurations*;
- **surjective** if $\Theta(A^G) = A^G$; when $x \in A^G \backslash \Theta(A^G)$ one calls x a *Garden of Eden*;
- **postsurjective** if $y \sim \Theta(x)$ implies $\exists z \sim x : \Theta(z) = y$.

Moore and Myhill's celebrated "Garden of Eden" theorem asserts that, if $G = \mathbb{Z}^d$, then cellular automata are preinjective if and only if they are surjective [11,12]. This has been extended to *amenable* groups G by Ceccherini-Silberstein, Machì and Scarabotti [5], and I proved in [2,3] that both results may fail as soon as G is not amenable. We shall not need the precise definition of amenable groups; suffice it to say that one of the equivalent definitions states that G contains finite subsets that are arbitrarily close to invariant under translation, in the sense that for every finite $S \subseteq G$ and every $\epsilon > 0$ there exists a finite subset $F \subseteq G$ with $\#(FS \setminus F) < \epsilon \# F$. For our purpose, the main result of [3] may be formulated as:

Theorem 1.1. *For a group G, the following are equivalent:*

(1) *G is nonamenable.*
(2) *For some integer n and every (equivalently, some) field $\mathbb{K}$, there is an injective $\mathbb{K}G$-linear map $(\mathbb{K}G)^n \to (\mathbb{K}G)^{n-1}$.*

This $\mathbb{K}G$-linear map is nothing more than an $(n-1) \times n$ matrix with entries in $\mathbb{K}G$. We shall sketch Theorem 1.1's proof in §3.

We shall not need the precise definition of *sofic groups*, a common generalization of amenable and residually finite groups; we refer to the original article [16]. Suffice it to say that it is at present unknown whether nonsofic groups exist, and that if G is sofic then it satisfies Gottschalk's "Surjunctivity Conjecture" from [10], namely every injective cellular automaton is surjective [16, §3]. Capobianco, Kaari and Taati show in [4] that, when G is sofic, every postsurjective cellular automaton is preinjective. Thus

$$\begin{tikzcd}
\text{injective} \arrow[rr, Rightarrow] \arrow[ddrr, Rightarrow] & & \text{preinjective} \arrow[dd, Leftrightarrow, "\boxed{\text{iff } G \text{ amenable}}"] \\
& \boxed{\text{if } G \text{ sofic}} & \\
\text{postsurjective} \arrow[rr, Rightarrow] \arrow[uurr, Rightarrow] & & \text{surjective}
\end{tikzcd}$$

We remark that if a cellular automaton is injective and surjective, then its inverse is also a cellular automaton (the oldest reference seems to be [14, Corollary 4]; see also [7, Theorem 1.10.2]). Similarly, if a cellular automaton is preinjective and postsurjective, then it is bijective and its inverse is also a cellular automaton, see [4, Theorem 1].

The notions of (pre)injectivity and (post)surjectivity become substantially simpler in the context of linear cellular automata, and exhibit more clearly the duality:

Lemma 1.2. *A linear cellular automaton $\Theta\colon V^G \circlearrowleft$ is preinjective, respectively postsurjective if and only if its restriction $\Theta\restriction \bigoplus_G V$ to $\bigoplus_G V$ is injective, respectively surjective.* $\square$

Note that linear cellular automata have *closed image* (see Proposition 2.3), so nonsurjective linear cellular automata $\Theta\colon V^G \circlearrowleft$ avoid a nonempty open subset of V^G, namely there exists a finite subset $F \subseteq G$ and $x \in V^F$ such that $\Theta(y)$ never restricts to x on F.

1.3. A problem of Ceccherini-Silberstein and Coornaert. Ceccherini-Silberstein and Coornaert prove in [6] that if G is an amenable group then a linear cellular automaton on G is preinjective if and only if it is surjective, and ask if this is also a characterization of amenability in the restricted context of linear cellular automata.

I gave in [3] a construction, for every nonamenable group G, of a preinjective, nonsurjective cellular automaton on G; and noted that it is in fact a linear cellular automaton. Ceccherini-Silberstein and Coornaert ask in [7, Open Problem 14]:

Problem 1.3. *Let G be a nonamenable group and let $\mathbb{K}$ be a field. Does there exist a finite-dimensional $\mathbb{K}$-vector space V and a linear cellular automaton $\Theta\colon V^G \circlearrowleft$ which is surjective but not preinjective?*

The group ring $\mathbb{K}G$ admits an anti-involution $*$, defined on basis elements $g \in G$ by $g^* := g^{-1}$ and extended by linearity. It induces an anti-involution on $M_n(\mathbb{K}G)$ as follows: for $\Theta \in M_n(\mathbb{K}G)$, set $(\Theta^*)_{ij} = \Theta^*_{ji}$ for all $i, j \in \{1, \dots, n\}$; namely, Θ^* is computed from Θ by transposing the matrix and applying $*$ to all its entries. Clearly $\Theta^{**} = \Theta$. There is a natural bilinear pairing $(\mathbb{K}G)^n \times (\mathbb{K}^n)^G \to \mathbb{K}$, given by

$$\langle v|\xi\rangle := \sum_{g\in G} v(g)\cdot\xi(g). \tag{1}$$

I shall prove in §4 the following:

Theorem 1.4. *Let G be a group, let $\mathbb{K}$ be a field, and let $\Theta \in M_n(\mathbb{K}G)$ be a linear cellular automaton. Then, with respect to the pairing (1),*

$$\ker(\Theta\restriction(\mathbb{K}G)^n)^\perp = \operatorname{im}(\Theta^*\restriction(\mathbb{K}^n)^G), \tag{2}$$

$$\ker(\Theta\restriction(\mathbb{K}^n)^G)^\perp = \operatorname{im}(\Theta^*\restriction(\mathbb{K}G)^n), \tag{3}$$

$$\operatorname{im}(\Theta\restriction(\mathbb{K}G)^n)^\perp = \ker(\Theta^*\restriction(\mathbb{K}^n)^G), \tag{4}$$

$$\operatorname{im}(\Theta\restriction(\mathbb{K}^n)^G)^\perp = \ker(\Theta^*\restriction(\mathbb{K}G)^n). \tag{5}$$

In particular, Θ is preinjective if and only if Θ^* is surjective, and Θ is injective if and only if Θ^* is postsurjective.

This answers positively Problem 1.3:

Corollary 1.5. *Let G be a nonamenable group and let $\mathbb{K}$ be an arbitrary (possibly finite) field. Then there exist surjective, nonpreinjective linear cellular automata on G.*

Proof. Let $\Theta \in M_n(\mathbb{K}G)$ be a preinjective, nonsurjective linear cellular automaton, obtained e.g. by adding a full row of 0's to the matrix given by Theorem 1.1. Then Θ^* is the required example. □

1.4. Capobianco, Kari and Taati's result. From this duality of linear cellular automata, one also deduces an immediate proof of Capobianco, Kari and Taati's main result, when restricted to linear cellular automata:

Theorem 1.6 (See [4, Theorem 2])**.** *Let G be a sofic group. Then every postsurjective linear cellular automaton is preinjective.*

Proof. Let Θ be a postsurjective linear cellular automaton. By Theorem 1.4, Θ^* is injective, so Θ^* is surjective by [8, Theorem 1.2], so Θ is preinjective again by Theorem 1.4. $\square$

1.5. Reddite ergo quæ Cæsaris sunt. The notion of dual linear cellular automata is quite natural, but its first appearance seems only to be a passing remark in [13]. The last line of Theorem 1.4 has been proven, in the setting of locally finite graphs, by Matthew Tointon in [15]. I am indebted to Professor Coornaert for having pointed out that reference to me when I shared this note with him.

In a recent article [9], Gaboriau and Seward study the sofic entropy of algebraic actions, and note the following consequence of Corollary 1.5: if G is sofic but not amenable, then the Yuzvinsky addition formula for entropy $h(G \looparrowright A) = h(G \looparrowright B) + h(G \looparrowright A/B)$ fails for some G-modules $B \leqslant A$. Indeed take $A = (\mathbb{K}^n)^G$ and $B = \ker(\Theta)$ for a surjective, nonpreinjective cellular automaton Θ. I am grateful to Messrs. Gaboriau and Seward for having communicated their note to me ahead of its publication.

2. Linear cellular automata

We start with a field $\mathbb{K}$ and an integer n. We write $V := \mathbb{K}^n$, and identify V with V^*. There is a natural bilinear, nondegenerate pairing $V^* \times V \to \mathbb{K}$ given by

$$\langle \phi | v \rangle = \phi(v) = \sum_{i=1}^{n} \phi_i v_i.$$

Let G be a group. We denote by V^G the vector space of functions $G \to V$, and define its topology by taking as base of open sets the

$$B_{S,\mathcal{O}} := \left\{ c \in V^G \;\middle|\; c \restriction S \in \mathcal{O} \right\}$$

for all finite $S \subseteq G$ and all Zariski-open $\mathcal{O} \subseteq V^S$. We note the easy:

Lemma 2.1. *The restriction maps $\pi_S \colon V^G \to V^S$ are continuous for all finite $S \subseteq G$, and V^G is compact (but not Hausdorff).*

Proof. π_S is continuous by construction. To show that V^G is compact, most proofs of Tychonoff's theorem adapt *verbatim.* For example, by Alexander's subbase theorem [1, Theorem 1], it suffices to show that every cover by the $B_{S,\mathcal{O}}$ admits a finite subcover. Let therefore $(B_{S_i,\mathcal{O}_i})_{i \in I}$ be a cover. In particular, $I \neq \varnothing$, so one may choose $j \in I$. Consider the projected cover $(B_{S_i,\mathcal{O}_i} \restriction S_j)_{i \in I}$ of V^{S_j}. It is a cover by Zariski-open subsets of V^{S_j}, and the

Zariski topology is compact, so there exists a finite subcover $(B_{S_i,\mathcal{O}_i}\restriction S_j)_{i\in J}$. Finally $(B_{S_i,\mathcal{O}_i})_{i\in J\cup\{j\}}$ is a finite cover of V^G. For the last claim, the Zariski topology itself is not Hausdorff. □

We denote by V^*G the vector subspace of finitely-supported functions in V^G. There is a left action of G on V^G by translation: for $g\in G, c\in V^G$ we define $gc\in V^G$ by $(gc)(h)=c(g^{-1}h)$. This action preserves V^*G. There is also a bilinear pairing

$$\langle -|-\rangle\colon V^*G\times V^G\to\mathbb{K},\qquad \langle\omega|c\rangle=\sum_{g\in G}\langle\omega(g)|c(g)\rangle.$$

Lemma 2.2. *$\langle -|-\rangle$ is nondegenerate in both arguments.* □

In the notation introduced above, a linear cellular automaton is both an element of $V\otimes V^*G$ and a G-equivariant, continuous self-map $\Theta\colon V^G\circlearrowleft$. Note that Θ restricts to a self-map $V^*G\circlearrowleft$.

Proposition 2.3. *Let $\Theta\colon V^G\circlearrowleft$ be a linear cellular automaton. Then $\Theta(V^G)$ is a closed subspace of V^G.*

Proof. Verbatim the proof of [7, Theorem 8.8.1]. Note that the authors prove in fact the weaker statement that $\Theta(V^G)$ is closed in the prodiscrete topology. Note also that the proposition does not follow trivially from the fact that V^G is compact, because V^G is not Hausdorff. □

Consider a linear cellular automaton $\Theta\in V\otimes V^*G$, written as

$$\Theta=\sum_i v_i\otimes\phi_i g_i$$

for finitely many $v_i\in V,\phi_i\in V^*,g_i\in G$. Then, tracing back to our original definition, its adjoint $\Theta^*\in V^*\otimes VG$ is $\Theta^*=\sum_i\phi_i\otimes v_i g_i^{-1}$.

Lemma 2.4. *Let $\Theta\in V\otimes V^*G$ be a cellular automaton, with adjoint Θ^*. Then*

$$\langle\Theta^*(\omega)|c\rangle=\langle\omega|\Theta(c)\rangle \text{ for all } \omega\in V^*G, c\in V^G.$$

Proof. Write Θ as a finite sum $\sum_i v_i\otimes\phi_i g_i$. Then the sides of the above equation are respectively

$$\sum_{g\in G}\Big\langle\Big\{\sum_i\phi_i\otimes v_i(g_i^{-1}\omega)\Big\}(g)\Big|c(g)\Big\rangle=\sum_{g\in G,i}\langle\phi_i|c(g)\rangle\,\langle\omega(g_ig)|v_i\rangle$$

and

$$\sum_{g\in G}\Big\langle\omega(g)\Big|\Big\{\sum_i v_i\otimes\phi_i(g_ic)\Big\}(g)\Big\rangle=\sum_{g\in G,i}\langle\omega(g)|v_i\rangle\,\langle\phi_i|c(g_i^{-1}g)\rangle,$$

which are just permutations of each other. □

3. Proof of Theorem 1.1

The main result of [3] is the construction, on an arbitrary nonamenable group G, of a nonsurjective, preinjective cellular automaton. This cellular automaton is in fact linear, given by an $n \times n$ matrix Θ with entries in $\mathbb{K}S$ for some n and some large enough finite subset $S \subset G$. The cellular automaton is guaranteed to be nonsurjective by imposing that Θ has a full row of 0's. On the other hand, the matrix Θ depends on $N = n(n-1)\#S$ parameters and may be thus viewed as an element still written Θ of $\mathbb{K}^N$. I show in [3] that, unless Θ belongs to a finite union of hypersurfaces of $\mathbb{K}^N$ defined over $\mathbb{Z}$, the corresponding cellular automaton is preinjective. This will be the case as soon as $\mathbb{K}$ is large enough (say of cardinality at least 2^t; in particular infinite is O.K.).

The *a priori* dependency of n on the cardinality of $\mathbb{K}$ may be removed as follows. The cellular automaton Θ is preinjective when $\mathbb{K}$ is a field extension of degree at least t. For all such extensions $\mathbb{K}'$, restrict scalars to the ground field $\mathbb{K}$ so as to obtain a preinjective cellular automaton with stateset $(\mathbb{K}^t)^n$, given by an $(n-1)t \times nt$ matrix with entries in $\mathbb{K}G$. Add some rows of 0's to this matrix to obtain an $(nt-1) \times nt$ matrix still written Θ; the preinjectivity of the cellular automaton is equivalent to the injectivity of the map $\Theta \colon (\mathbb{K}G)^{nt-1} \to (\mathbb{K}G)^{nt}$, see Lemma 1.2.

4. Proof of Theorem 1.4

Let $\Theta \in M_n(\mathbb{K}G)$ be a linear cellular automaton, and as in §2 set

$$V = V^* = \mathbb{K}^n,$$

with the usual scalar product.

We begin by the inclusion $\ker(\Theta\downharpoonleft V^*G)^\perp \supseteq \operatorname{im}(\Theta^*\downharpoonleft V^G)$ from (2). Given $c \in \operatorname{im}(\Theta^*\downharpoonleft V^G)$, say $c = \Theta^*(d)$, for all $\omega \in \ker(\Theta\downharpoonleft V^*G)$ we have

$$\langle\omega|c\rangle = \langle\omega|\Theta^*(d)\rangle = \langle\Theta(\omega)|d\rangle = \langle 0|d\rangle = 0,$$

so $c \perp \ker(\Theta\downharpoonleft V^*G)$. The exact same computation gives all '$\supseteq$' inclusions from (3), (4) and (5).

We continue with the inclusion $\ker(\Theta\downharpoonleft V^*G)^\perp \subseteq \operatorname{im}(\Theta^*\downharpoonleft V^G)$ from (2). Given $c \notin \operatorname{im}(\Theta^*\downharpoonleft V^G)$, there exists by Proposition 2.3 an open neighbourhood of c in $V^G\setminus\operatorname{im}(\Theta^*\downharpoonleft V^G)$; so there exists a finite subset $S \subseteq G$ and a proper subspace $W < V^S$ such that the projection $\pi_S(V^G)$ belongs to W. Since V^S is finite-dimensional, there exists a linear form ω on V^S that vanishes on W but does not vanish on c. Note that ω, qua element of $(V^S)^*$, is canonically identified with an element of $(V^*)^S$, and therefore with an element of V^*G. From $\omega \perp \operatorname{im}(\Theta^*\downharpoonleft V^G)$ we get $\Theta(\omega) \perp V^G$ so $\Theta(\omega) = 0$ because the bilinear pairing $\langle -|-\rangle$ is nondegenerate. Therefore $c \not\perp \ker(\Theta\downharpoonleft V^*G)$ as desired.

We continue with the inclusion $\ker(\Theta\downharpoonleft V^G)^\perp \subseteq \operatorname{im}(\Theta^*\downharpoonleft V^*G)$ from (3). Given $\omega \notin \operatorname{im}(\Theta^*\downharpoonleft V^*G)$, there exists a linear form $c \in (V^*G)^*$ that vanishes on $\operatorname{im}(\Theta^*\downharpoonleft V^*G)$ but does not vanish on ω. Note that $(V^*G)^*$ canonically identifies with V^G. From $c \perp \operatorname{im}(\Theta^*\downharpoonleft V^*G)$ we get $\Theta(c) \perp V^*G$, so $\Theta(c) = 0$ because the bilinear pairing $\langle -|-\rangle$ is nondegenerate. Therefore $\omega \not\perp \ker(\Theta\downharpoonleft V^G)$ as desired.

We finally consider the inclusion $\operatorname{im}(\Theta\downharpoonleft V^*G)^\perp \subseteq \ker(\Theta^*\downharpoonleft V^G)$ from (4). Given $c \perp \operatorname{im}(\Theta\downharpoonleft V^*G)$, we have $c \perp \Theta(\omega)$ for all $\omega \in V^*G$, so $\Theta^*(c) \perp \omega$ for all $\omega \in V^*G$. Therefore $\Theta^*(c) \perp V^*G$, and $\Theta^*(c) = 0$ because the bilinear pairing $\langle -|-\rangle$ is nondegenerate. The exact same computation gives the '$\subseteq$' inclusion from (5).

Recalling that Θ is preinjective if and only if $\ker(\Theta\downharpoonleft V^*G) = 0$ and Θ is injective if and only if $\ker(\Theta\downharpoonleft V^G) = 0$ and Θ is postsurjective if and only if $\operatorname{im}(\Theta\downharpoonleft V^*G) = V^*G$ and Θ is surjective if and only if $\operatorname{im}(\Theta\downharpoonleft V^G) = V^G$, the last conclusions follow.

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