t-Reductions and *t*-integral closure of ideals in Noetherian domains

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ABSTRACT. This paper studies *t*-reductions and *t*-integral closure of ideals in Noetherian domains. The main objective is to establish satisfactory *t*-analogues for well-known results in the literature on reductions and integral closure of ideals in Noetherian rings. Namely, Section 2 investigates *t*-reductions of ideals subject to *t*-invertibility and localization in Noetherian domains. Section 3 investigates the *t*-integral closure of ideals and its correlation with *t*-reductions in Noetherian domains of Krull dimension one. Section 4 studies the *t*-analogue of Hays' classic notion of C-ideal and its correlation to the integral closure.

CONTENTS

1. Introduction

Throughout, all rings considered are commutative with identity. Let R be a ring and I a proper ideal of R. An ideal $J \subseteq I$ is a reduction of I if $JI^n = I^{n+1}$ for some positive integer n. An ideal which has no reduction other than itself is called a basic ideal [13, 28]. The notion of reduction was introduced by Northcott and Rees to contribute to the analytic theory of ideals in Noetherian (local) rings via minimal reductions. In [13, 14], Hays investigated reductions of ideals in more general settings of commutative rings (i.e., not necessarily local or Noetherian); particularly, Noetherian rings and Prüfer domains. He provided several sufficient conditions for an

ideal to be basic. For instance, in Noetherian rings, an ideal is basic if and only if it is locally basic. He also introduced and studied the dual notion of a basic ideal; namely, an ideal is a C-ideal if it is not a reduction of any larger ideal. Several results about C-ideals are proved; including the fact that this notion is local for regular ideals in Noetherian rings.

It is well-known that an element $x \in R$ is integral over I if and only if I is a reduction of $I + Rx$; and if I is finitely generated, then $J \subseteq I \subseteq \overline{J}$ if and only if J is a reduction of I, where $\overline{J}$ denotes the integral closure of J. This correlation allowed to prove a number of crucial results in the theory including the fact that the integral closure of an ideal is an ideal. For a full treatment of this topic, we refer the reader to Huneke and Swanson's book "Integral closure of ideals, rings, and modules" [21].

Let R be a domain, K its quotient field, I a nonzero fractional ideal of R, and $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$. The v- and t-closures of I are defined, respectively, by $I_v := (I^{-1})^{-1}$ and $I_t := \cup J_v$, where J ranges over the set of finitely generated subideals of I. The ideal I is a v-ideal (or divisorial) if $I_v = I$ and a t-ideal if $I_t = I$. Under the ideal t-multiplication $(I, J) \mapsto (IJ)_t$ the set $F_t(R)$ of fractional t-ideals of R is a semigroup with unit R. Ideal t-multiplication converts notions such as principal, Dedekind, Bézout, and Prüfer domains to factorial domains, Krull domains, GCDs, and PvMDs, respectively. We also recall the w-operation: for a nonzero fractional ideal I of R, $I_w = \bigcup(I : J)$, where the union is taken over all finitely generated ideals J of R that satisfy $J_v = R$; equivalently, $I_w = \bigcap IR_M$, where M ranges over the set of all maximal t-ideals of R. We always have $I \subseteq I_w \subseteq I_t \subseteq I_v$. We shall be using the v-, t-, and w-operations freely, and for more details, the reader may consult Gilmer's book [12] and also [1, 2, 3, 4, 6, 8, 10, 19, 27, 29, 30].

Let I be a nonzero ideal of R. An ideal $J \subseteq I$ is a t-reduction of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$. An element $x \in R$ is t-integral over I if there is an equation $x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n = 0$ with $a_i \in (I^i)_t$ for $i = 1, ..., n$. The set of all elements that are t-integral over I is called the t-integral closure of I. In [22], the authors investigated the t-reductions and t-integral closure of ideals with the aim of establishing satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions. Two of their main results assert that "the t-integral closure of an ideal is an integrally closed ideal which is not t-integrally closed in general" and " the t-integral closure coincides with the t-closure in the class of integrally closed domains." In [17], the authors investigated $\star$-reductions of ideals in Prüfer v-multiplication domains (PvMDs). One of their main results asserts that "a domain has the finite w-basic ideal property (resp., w-basic ideal property) if and only if it is a PvMD (resp., a PvMD of t-dimension one)." In [23], the authors investigated t-reductions of ideals in pullback constructions, where the main result established the transfer of the finite t-basic ideal property to pullbacks in

line with Fontana-Gabelli's result on PvMDs [9, Theorem 4.1] and Gabelli-Houston's result on v-domains [11, Theorem 4.15]. They also solved an open problem on whether the finite t-basic and v-basic ideal properties are distinct; they proved indeed that these two notions coincide in any arbitrary domain.

This paper studies t-reductions and t-integral closure of ideals in Noetherian domains. The main objective is to establish satisfactory t-analogues for well-known results in the literature on reductions and integral closure of ideals in Noetherian rings. Namely, Section 2 investigates t-reductions of ideals subject to t-invertibility and localization in Noetherian domains. Section 3 investigates the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one. Section 4 studies the t-analogue of Hays' classic notion of C-ideal and its correlation to the integral closure.

2. t-reductions subject to t-invertibility and localization

This section investigates t-reductions of ideals subject to t-invertibility and localization in Noetherian domains. The first objective is to establish a t-analogue for Hays' result on the correlation between invertible reductions and the Krull dimension of a Noetherian domain [13, Theorem 4.4]. The second objective is to reach a satisfactory t-analogue for Hays' global-local result on the basic property in Noetherian rings [13, Theorem 3.6].

Definition 2.1 ([17, 22, 23]). Let R be a domain and I a nonzero ideal of R.

(1) An ideal $J \subseteq I$ is a *t-reduction* of I if $(JI^n)_t = (I^{n+1})_t$ for some integer $n \geq 0$. The ideal J is a *trivial t-reduction* of I if $J_t = I_t$.
(2) I is *t-basic* if it has no t-reduction other than the trivial t-reductions.
(3) R has the *t-basic ideal property* if every nonzero ideal of R is t-basic.

Clearly, the notion of t-reduction extends naturally to fractional ideals. Also, notice that a reduction is necessarily a t-reduction; and the converse is not true, in general. Each of [22, Example 2.2] and [17, Example 1.5] exhibits a Noetherian domain R with two t-ideals $J \subsetneqq I$ such that J is a t-reduction but not a reduction of I.

In 1973, Hays proved the following result:

Theorem 2.2 ([13, Theorem 4.4]). *Let R be a Noetherian domain such that R/M is infinite for every maximal ideal M of R. Then, each nonzero ideal has an invertible reduction if and only if $\dim(R) \leq 1$.*

Next, we establish a t-analogue for this result. To this end, recall that the t-dimension of a domain R, denoted t-$\dim(R)$, is the supremum of the lengths of chains of prime t-ideals in R (and, for the purpose of this definition, (0) is considered as a prime t-ideal although technically it is not); and

we always have t-$\dim(R) \leq \dim(R)$ [16]. Throughout, $\mathrm{Max}_t(R)$ will denote the set of maximal t-ideals of R.

Theorem 2.3. *Let R be a Noetherian domain such that the residue field of each maximal t-ideal of R is infinite. Then, the following statements are equivalent:*

(1) *Each t-ideal of R has a t-invertible t-reduction;*
(2) *Each maximal t-ideal of R has a t-invertible t-reduction;*
(3) t-$\dim(R) \leq 1$.

The following lemma proves the implication $(2) \Rightarrow (3)$ without the infinite residue field assumption.

Lemma 2.4. *Let R be a Noetherian domain. If every maximal t-ideal of R has a t-invertible t-reduction, then t-$\dim(R) \leq 1$.*

Proof. Assume that every maximal t-ideal has a t-invertible t-reduction. We may suppose that R is not a field and will prove that t-$\dim(R) = 1$. Let $M \in \mathrm{Max}_t(R)$ and let $J = J_t$ be a t-invertible t-reduction of M. Then $(M^{n+1})_t = (JM^n)_t$ for some positive integer n and hence $M^{n+1} \subseteq J \subseteq M$. Now If D is a Noetherian domain and P is a prime t-ideal of D, then PD_P is a prime t-ideal of D_P. This follows from the discussion after Proposition 1.4 of [31]. Thus MR_M is a t-ideal of R_M. Therefore, JR_M is invertible and hence principal in R_M. Moreover, M is minimal over J, and so is MR_M over JR_M. Since R_M is Noetherian, $\mathrm{ht}(M) = \mathrm{ht}(MR_M) = 1$ by the Principal Ideal Theorem. Consequently, t-$\dim(R) = 1$, as desired. $\qquad\square$

The converse of Lemma 2.4 is not true in general. For, let R be an almost Dedekind domain which is not Dedekind. Then R is a one-dimensional locally Noetherian Prüfer domain (i.e., the d- and t-operations coincide). Hence R has the basic ideal property [13, Theorem 6.1]. But R is not Dedekind, so it posses a non-invertible maximal ideal M which has no reduction other than itself.

Proof of Theorem 2.3. $(1) \Rightarrow (2)$ is trivial, and $(2) \Rightarrow (3)$ is handled by Lemma 2.4. It remains to prove $(3) \Rightarrow (1)$. Suppose that t-$\dim(R) = 1$ and let I be a t-ideal of R. Clearly, $\mathrm{ht}(I) = 1$. Since R is Noetherian, it is a TV-domain and hence has finite t-character by [19, Theorem 1.3]. Let $M_1, \ldots, M_n$ be all the maximal t-ideals of R containing I. Let $i \in \{1, \ldots, n\}$. Since R_{M_i} is a one-dimensional Noetherian domain, by [13, Theorem 4.4], IR_{M_i} has an invertible (so principal) reduction, say $a_i R_{M_i}$. Clearly, $\sqrt{a_i R_{M_i}} = \sqrt{IR_{M_i}} = M_i R_{M_i}$, and so $M_i{}^r R_{M_i} \subseteq a_i R_{M_i}$ for some integer r. Let $A_i := a_i R_{M_i} \cap R$. We have

$$M_i^r \subseteq M_i{}^r R_{M_i} \cap R \subseteq a_i R_{M_i} \cap R = A_i \subseteq M_i.$$

Hence M_i is the only maximal t-ideal of R containing A_i. It follows that $A_i R_M = R_M$ for any $M \in \mathrm{Max}_t(R) \setminus \{M_i\}$. Let $J := \prod_{i=1}^n A_i$. Then, we claim that J is a t-invertible t-reduction of I. First, we show that $J \subseteq I$.

Indeed, one can check that $M_1, \ldots, M_n$ are the only maximal t-ideals of R containing J and let $\mathfrak{M} := \mathrm{Max}_t(R) \setminus \{M_1, \ldots, M_n\}$. So

$$
\begin{aligned}
J_w &= \bigcap_{M \in \mathrm{Max}_t(R)} JR_M \\
&= \left(\bigcap_{1 \le i \le n} A_i R_{M_i} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&= \left(\bigcap_{1 \le i \le n} a_i R_{M_i} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&\subseteq \left(\bigcap_{1 \le i \le n} IR_{M_i} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&= \bigcap_{M \in \mathrm{Max}_t(R)} IR_M \\
&= I
\end{aligned}
$$

and thus $J \subseteq I$. Second, we show that J is a t-reduction of I. Indeed, let m be a positive integer such that $a_i I^m R_{M_i} = I^{m+1} R_{M_i}$ for all $i = 1, \ldots, n$. Notice also that $M_1, \ldots, M_n$ are the only maximal t-ideals of R containing JI^m and I^{m+1}. So

$$
\begin{aligned}
(JI^m)_w &= \bigcap_{M \in \mathrm{Max}_t(R)} (JI^m) R_M \\
&= \left(\bigcap_{1 \le i \le n} a_i I^m R_{M_i} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&\subseteq \left(\bigcap_{1 \le i \le n} I^{m+1} R_{M_i} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&= \bigcap_{M \in \mathrm{Max}_t(R)} I^{m+1} R_M \\
&= (I^{m+1})_w
\end{aligned}
$$

and thus $(JI^m)_t = (I^{m+1})_t$ since t is coarser than w. Finally, we show that J is t-invertible. Indeed, we have

$$
\begin{aligned}
(JJ^{-1})_w &= \bigcap_{M \in \mathrm{Max}_t(R)} (JJ^{-1}) R_M \\
&= \left(\bigcap_{1 \le i \le n} (JJ^{-1}) R_{M_i} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&= \left(\bigcap_{1 \le i \le n} JR_{M_i} J^{-1} R_{M_i} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&= \left(\bigcap_{1 \le i \le n} JR_{M_i} (JR_{M_i})^{-1} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&= \left(\bigcap_{1 \le i \le n} JR_{M_i} (a_i R_{M_i})^{-1} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&= \left(\bigcap_{1 \le i \le n} a_i R_{M_i} a_i^{-1} R_{M_i} \right) \cap \left(\bigcap_{M \in \mathfrak{M}} R_M \right) \\
&= \bigcap_{M \in \mathrm{Max}_t(R)} R_M \\
&= R
\end{aligned}
$$

and so J is t-invertible, completing the proof of the theorem. $\qquad \square$

Next, we examine the global-local transfer of the t-basic ideal property. Throughout, an ideal I is locally basic (resp., t-locally t-basic) if IR_M is basic (resp., t-basic) for each maximal ideal (resp., maximal t-ideal) M of R containing I. In 1973, Hays proved the following result:

Theorem 2.5 ([13, Theorem 3.6]). *In a Noetherian ring, an ideal is basic if and only if it is locally basic.*

Next, we establish a t-analogue for the "if" assertion of this result.

Theorem 2.6. *In a Noetherian domain, if an ideal is t-locally t-basic, then it is t-basic.*

Proof. Let R be a Noetherian domain and let I be a t-locally t-basic ideal of R. Let $J \subseteq I$ be a t-reduction of I; that is, $(JI^n)_t = (I^{n+1})_t$, for some positive integer n. Next, we prove that $J_t = I_t$. Since $(JI^n)_t = (J_tI^n)_t$, we may assume, without loss of generality, that J is a t-ideal. Let $M \in \mathrm{Max}_t(R)$ such that $I \subseteq M$, and let t_M and v_M denote the t- and v- operations with respect to R_M, respectively. By [24, Lemma 2.18], we get

$$
\begin{aligned}
\left(JR_M I^n R_M\right)_{t_M} &= \left((JI^n)_t R_M\right)_{t_M} \\
&= \left((I^{n+1})_t R_M\right)_{t_M} \\
&= \left(I^{n+1} R_M\right)_{t_M}
\end{aligned}
$$

and the t-locally t-basic assumption yields

$$
\begin{aligned}
(JR_M)^{-1} &= \left((JR_M)_{v_M}\right)^{-1} \\
&= \left((JR_M)_{t_M}\right)^{-1} \\
&= \left((IR_M)_{t_M}\right)^{-1} \\
&= \left((IR_M)_{v_M}\right)^{-1} \\
&= (IR_M)^{-1}.
\end{aligned}
$$

Moreover, since $I^{n+1} \subseteq J_t = J \subseteq I$, then a maximal t-ideal contains I if and only if it contains J. It follows that

$$
J^{-1}R_M = (JR_M)^{-1} = (IR_M)^{-1} = I^{-1}R_M
$$

for all maximal t-ideals of R. Therefore, we obtain

$$
\begin{aligned}
(J^{-1})_w &= \bigcap_{M \in \mathrm{Max}_t(R)} J^{-1}R_M \\
&= \bigcap_{M \in \mathrm{Max}_t(R)} I^{-1}R_M \\
&= (I^{-1})_w.
\end{aligned}
$$

Consequently, $J^{-1} = (J^{-1})_v = (I^{-1})_v = I^{-1}$ and thus $J = J_v = I_v = I_t$, as desired. $\qquad\square$

It is worthwhile noting that, in his proof of the implication "basic $\Rightarrow$ locally basic" (Theorem 2.5), Hays used two basic facts; the first of which asserts that $(J \cap I) + IM$ is a reduction of I whenever JR_M is a reduction of IR_M in an arbitrary ring R. A t-analogue for this result is proved below in Proposition 2.7. But, the second fact was Nakayama's lemma, which ensures that $J \subseteq I \subseteq J + IM$ in a local Noetherian ring (R, M) forces $J = I$; and a t-analogue for this Nakayama property is not true in general. For instance, consider the local Noetherian ring $R := k + M^2 \subseteq k[x, y]$, where $M = (x, y)$ and $(M^2)_t = (M^3)_t$ [17, Example 1.5].

Proposition 2.7. *Let R be a domain, M a maximal t-ideal of R, and $I \subseteq M$ a nonzero ideal of R. If J is an ideal of R such that JR_M is a t-reduction of IR_M, then $(J \cap I) + IM$ is a t-reduction of I.*

Proof. Let J be an ideal of R such that JR_M is a t-reduction of IR_M, say, $(JR_M I^n R_M)_{t_M} = (I^{n+1} R_M)_{t_M}$, for some positive integer n and where t_M denotes the t-operation with respect to R_M. Let $Q \in \mathrm{Max}_t(R)$ with $Q \neq M$. Then, $(J \cap I + IM)R_Q = IR_Q$ yielding $(J \cap I + IM)I^n R_Q = I^{n+1}R_Q$. Whence, $\left((J \cap I + IM)I^n\right)^{-1} R_Q = \left(I^{n+1}\right)^{-1} R_Q$. On the other hand, we have

$$
\begin{aligned}
\left((J \cap I + IM)I^n R_M\right)_{t_M} &= \left((JR_M \cap IR_M + IR_M MR_M)I^n R_M\right)_{t_M} \\
&= \left((JR_M + IR_M MR_M)I^n R_M\right)_{t_M} \\
&= \left(JR_M I^n R_M + I^{n+1} R_M MR_M\right)_{t_M} \\
&= \left(I^{n+1} R_M\right)_{t_M}
\end{aligned}
$$

and thus

$$
\left((J \cap I + IM)I^n\right)^{-1} R_M = \left(I^{n+1}\right)^{-1} R_M.
$$

Therefore, we obtain

$$
\begin{aligned}
\left((I^{n+1})^{-1}\right)_w &= \bigcap_{N \in \mathrm{Max}_t(R)} (I^{n+1})^{-1} R_N \\
&= \bigcap_{N \in \mathrm{Max}_t(R)} \left((J \cap I + IM)I^n\right)^{-1} R_N \\
&= \left(\left((J \cap I + IM)I^n\right)^{-1}\right)_w
\end{aligned}
$$

Consequently, $\left((J \cap I + IM)I^n\right)_t = \left(I^{n+1}\right)_t$. That is, $(J \cap I) + IM$ is a t-reduction of I, completing the proof of the proposition. $\qquad\square$

3. t-reductions and t-integral closure in one-dimensional Noetherian domains

This section investigates the t-integral closure of ideals and its correlation with t-reductions in Noetherian domains of Krull dimension one. Our objective is to establish satisfactory t-analogues of well-known results, in the literature, on the integral closure of ideals and its correlation with reductions of ideals in Noetherian rings.

From [22, 23], let R be a domain and I a nonzero ideal of R. An element $x \in R$ is t-integral over I if there is an equation

$$
x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0 \quad \text{with} \quad a_i \in (I^i)_t \ \forall i = 1, \dots, n.
$$

The set of all elements that are t-integral over I is called the t-integral closure of I, and is denoted by $\widetilde{I}$. If $I = \widetilde{I}$, then I is said to be t-integrally closed. Recall that "$\widetilde{I}$ *is an integrally closed ideal which is not t-integrally closed in general*" [22, Theorem 3.2]. Several ideal-theoretic properties of $\widetilde{I}$ are collected in [22, Remark 3.8], including the basic inclusions

$$
I \subseteq \overline{I} \subseteq \widetilde{I} \subseteq \sqrt{I_t}.
$$

Next, consider the two sets:

$$\widehat{I}^{\,d} := \big\{x \in R \mid I \text{ is a reduction of } (I, x)\big\}$$

$$\widehat{I}^{\,t} := \big\{x \in R \mid I \text{ is a } t\text{-reduction of } (I, x)\big\}$$

For the trivial operation, it is well-known that the equality $\overline{I} = \widehat{I}^{\,d}$ always holds [21, Corollary 1.2.2]. This is the very fact which was used to show that $\overline{I}$ is an ideal [21, Corollary 1.3.1]. However, it is still an open problem of whether $\widehat{I}^{\,t}$ is an ideal in general [23, Question 3.5]. We always have

$$I_t \subseteq \widetilde{I} \subseteq \widehat{I}^{\,t}$$

where the second containment is proved in [22, Proposition 3.7] and can be strict as shown by [22, Example 3.10(a)]. Moreover, "$I_t = \widetilde{I}$ *for each nonzero ideal I if and only if R is integrally closed*" [22, Theorem 3.5], and "$I_t = \widehat{I}^{\,t}$ *for each nonzero ideal I if and only if R has the finite t-basic ideal property*" [23, Theorem 3.2].

The class of Prüfer domains is the only known class of domains, so far, where the two notions of reduction and t-reduction coincide (since the t- and trivial operations coincide). The next result shows that such coincidence also occurs in one-dimensional Noetherian domains (where the t- and trivial operations are not necessarily the same).

Theorem 3.1. *In a one-dimensional Noetherian domain, the notions of reduction and t-reduction coincide. Moreover, $\overline{I} = \widetilde{I} = \widehat{I}^{\,t}$ for any nonzero ideal I.*

The proof draws on the following lemma, which is of independent interest.

Recall from [4], an extension of domains $R \subseteq T$ is t-compatible if $I_t T \subseteq (IT)_{t_1}$ for every nonzero ideal I of R, where t_1 denotes the t-operation with respect to T. Throughout, for a domain R, we will denote by $\overline{R}$ the integral closure of R in its quotient field.

Lemma 3.2. *Let R be a domain such that $R \subseteq \overline{R}$ is t-compatible, $\overline{R}$ has the t-basic ideal property, and $\overline{J\overline{R}} = \widetilde{J\overline{R}}$ for any nonzero ideal J of R. Then, the notions of reduction and t-reduction coincide in R.*

Proof. Let $J \subseteq I$ be nonzero ideals of R such that J is a t-reduction of I; say, $(JI^n)_t = (I^{n+1})_t$, for some positive integer n. We need to show that J is a reduction of I. Indeed, by t-compatibility, we have

$$I^{n+1}\overline{R} \subseteq (I^{n+1})_t \overline{R} = (JI^n)_t \overline{R} \subseteq (JI^n \overline{R})_{t_1}$$

yielding $(I^{n+1}\overline{R})_{t_1} \subseteq (JI^n \overline{R})_{t_1}$. The reverse inclusion is obvious. So, $J\overline{R}$ is a t-reduction of $I\overline{R}$. Hence, by hypothesis, $(J\overline{R})_{t_1} = (I\overline{R})_{t_1}$. Therefore, we

obtain

$$
\begin{aligned}
I \;\; &\subseteq \;\; (I\overline{R})_{t_1} \cap R \\
&= \;\; (J\overline{R})_{t_1} \cap R \\
&= \;\; \widetilde{J\overline{R}} \; \cap R \text{ (by [22, Theorem 3.5])} \\
&= \;\; \overline{J\overline{R}} \cap R \text{ (by hypothesis)} \\
&= \;\; \overline{J} \text{ (by [21, Proposition 1.6.1]).}
\end{aligned}
$$

It follows that J is a reduction of I by [21, Corollary 1.2.5], as desired. □

Proof of Theorem 3.1. In order to prove the first statement of the theorem, it suffices to show that R satisfies the three assumptions in Lemma 3.2. Indeed, $R \subseteq \overline{R}$ is t-compatible by [4, Lemma 2.3]. By Mori-Nagata integral closure theorem, $\overline{R}$ is Krull. Therefore, $\overline{R}$ has the t-basic ideal property by [17, Figure 2]. Moreover, since $\dim(\overline{R}) = \dim(R) = 1$ by [21, Theorem 2.2.5], then $\overline{R}$ is Dedekind by [26, Theorem 12.5]. Hence, the t- and trivial operations coincide in $\overline{R}$. Whence, $\overline{J\overline{R}} = \widetilde{J\overline{R}}$ for any nonzero ideal J of R, as desired.

Now, let I be any nonzero ideal I of R. The fact that the two notions of reduction and t-reduction coincide in R combined with [21, Corollary 1.2.2] yields

$$
\overline{I} \subseteq \widetilde{I} \subseteq \widehat{I}^{\,t} = \widehat{I}^{\,d} = \overline{I}
$$

completing the proof of the theorem.   □

As illustrative examples for Theorem 3.1, we consider one-dimensional Noetherian domains which are not divisorial (i.e., t-operation is not trivial), as shown below.

Example 3.3. Let $\mathbb{Q}$ be the field of rational numbers and X an indeterminate over $\mathbb{Q}$. Consider the pseudo-valuation domain (PVD, for short) $R := \mathbb{Q} + X\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$. Then, R, as pullback issued from the DVR $\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$, is a one-dimensional Noetherian domain. Further, R is not a divisorial domain since, otherwise, V would be a two-generated R-module by [15, Theorem 3.5] or [18, Theorem 2.4], which is absurd since $[V/M : R/M] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

One wonders whether there exist Noetherian domains of dimension > 1 where the notions of reduction and t-reduction coincide. Next, we show this cannot happen in a large class of Noetherian domains.

Proposition 3.4. *Let R be a Noetherian domain with $(R : \overline{R}) \neq 0$. Then, the notions of reduction and t-reduction coincide in R if and only if R has dimension 1.*

Proof. In view of Theorem 3.1, we only need to prove the "only if" assertion. Assume that the notions of reduction and t-reduction coincide in R. Since

R is Noetherian, $\overline{R}$ is a Krull domain (Mori-Nagata theorem). Set $A := (R : \overline{R}) \neq 0$. Clearly, we have

$$\overline{R} = (A : A) = ((R : \overline{R}) : A) = (R : \overline{R}A) = (R : A) = A^{-1}.$$

Suppose, for contradiction, that $\dim(R) = \dim(\overline{R}) \geq 2$ and let N be a maximal ideal of $\overline{R}$ with $\text{ht}(N) \geq 2$. Since $(R : \overline{R}) \neq 0$, $\overline{R}$ is a finitely generated fractional ideal of R, and hence a Noetherian ring. So, by [20, Theorem 3.0 & Proposition 2.3], we have

$$(\overline{R} : N) = (N : N) = \overline{R}$$

and then

$$(R : AN) = ((R : A) : N) = (\overline{R} : N) = \overline{R} = A^{-1}.$$

Hence

$$(AN)_t = (AN)_v = A_v = A.$$

That is, AN is a t-reduction and hence, by hypothesis, a reduction of $(AN)_t = A$. So $A^{n+1}N = (AN)A^n = A^{n+1}$, for some positive integer n. By [25, Theorem 76], $A^{n+1} = 0$, the desired contradiction. □

4. t-C-ideals

This section studies the t-analogue of Hays' classic notion of C-ideal. In a ring, an ideal I is called a *C-ideal* if it is not a reduction of any larger ideal; i.e., if $I \subseteq K$ with $IK^n = K^{n+1}$ for some positive integer n, then $I = K$ [13, 14]. Our aim is to establish satisfactory t-analogues of Hays' results on C-ideals in Noetherian rings.

Definition 4.1. In a domain, a nonzero ideal I is called a *t-C-ideal* if it is not a non-trivial t-reduction of any larger ideal; i.e., if $I \subseteq K$ with $(IK^n)_t = (K^{n+1})_t$ for some positive integer n, then $I_t = K_t$.

Notice that a nonzero ideal I is a t-C-ideal if and only if I_t is a t-C-ideal. This fact will be used in the sequel without explicit mention.

Next, we collect some ideal-theoretic properties of t-C-ideals in an arbitrary domain (i.e, not necessarily Noetherian), as t-analogues of their respective classic counterparts [13, Section 5].

Proposition 4.2. *In a domain R, the following assertions hold:*
(1) *Every prime t-ideal is a t-C-ideal.*
(2) *Any intersection of t-C-ideals is a t-C-ideal (cf. [13, Lemma 5.2]).*
(3) *If I and J are t-comaximal t-C-ideals, then IJ is a t-C-ideal (cf. [13, Theorem 5.6]).*
(4) *Let I be a nonzero ideal and let J be a t-invertible t-C-ideal. Then, IJ is a t-C-ideal if and only if I is a t-C-ideal (cf. [13, Theorem 5.7]).*

Proof. (1) Let P be a prime t-ideal of R. Suppose $P \subseteq K$ with $(PK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Then

$$(K_t)^{n+1} \subseteq (K^{n+1})_t = (PK^n)_t \subseteq P$$

which yields $K_t \subseteq P$ and hence $P = K$. So, P is a t-C-ideal.

(2) Let $\{A_\lambda\}$ be a set of t-C-ideals of R and let $B := \cap_\lambda A_\lambda$. Suppose $B \subseteq K$ with $(BK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Then, for each λ, we have

$$(K^{n+1})_t = (K^n(\cap_\lambda A_\lambda))_t \subseteq (K^n A_\lambda)_t$$

yielding

$$((K + A_\lambda)^{n+1})_t = (A_\lambda(K + A_\lambda)^n)_t.$$

It follows that $K_t \subseteq (K + A_\lambda)_t = (A_\lambda)_t$ and thus $B_t = K_t$, as desired.

(3) Let I and J be two t-C-ideals of R and assume $IJ \subseteq K$ with $(IJK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. If $(I + J)_t = R$, then by [7, Lemma 16], $(IJ)_t = (I \cap J)_t$. It follows that $((I \cap J)_t K_t^n)_t = (K_t^{n+1})_t$. Hence $(I \cap J)_t = K_t$ since $I \cap J$ is a t-C-ideal by (2). That is, $(IJ)_t = K_t$.

(4) Let I be a nonzero ideal and J a t-invertible t-C-ideal of R. Suppose IJ is a t-C-ideal and $I \subseteq K$ with $(IK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Composing by J^{n+1} and taking the t-closure, we get

$$(IJ(KJ)^n)_t = ((KJ)^{n+1})_t.$$

Hence, $(IJ)_t = (KJ)_t$. As J is t-invertible, we get $I_t = K_t$. That is, I is a t-C-ideal.

Conversely, suppose I is a t-C-ideal and $IJ \subseteq K$ with $(IJK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Therefore, we have

$$(K^{n+1})_t \subseteq (JK^n)_t \text{ and } (K^{n+1})_t \subseteq (IK^n)_t.$$

So, one can easily check that

$$((J + K)^{n+1})_t = (K^{n+1} + J(K + I)^n)_t = (J(K + I)^n)_t.$$

It follows that $K_t \subseteq J_t$ as J is a t-C-ideal by hypothesis. Next, let $F := KJ^{-1}$. Clearly,

$$I \subseteq F \subseteq K_t J^{-1} \subseteq (JJ^{-1})_t = R.$$

Further, we have

$$(IJ(FJ)^n)_t = ((FJ)^{n+1})_t.$$

The fact that J is t-invertible yields

$$(IF^n)_t = (F^{n+1})_t.$$

Consequently, $F_t = I_t$ as I is a t-C-ideal by hypothesis. That is, $K_t = (IJ)_t$. $\qquad\square$

The next theorem completes Hays' result [13, Theorem 5.11] on C-ideals in the context of integrally closed Noetherian domains.

Theorem 4.3. *Let R be a Noetherian domain. The following assertions are equivalent:*

(1) *R is integrally closed;*

(2) *Each invertible ideal is a C-ideal;*

(3) *Each principal ideal is a C-ideal;*

(4) *Each nonzero ideal is a t-C-ideal;*

(5) *Each t-invertible t-ideal is a t-C-ideal;*

(6) *Each principal ideal is a t-C-ideal;*

(7) *$\overline{I} \subseteq I_t$ for each nonzero ideal I of R;*

(8) *$\widetilde{I} = I_t$ for each nonzero ideal I of R;*

(9) *$\widehat{I}^{\,t} = I_t$ for each nonzero ideal I of R;*

(10) *R has the t-basic ideal property.*

The proof of this result draws on the following elementary lemmas.

Lemma 4.4. *A domain D has the t-basic ideal property if and only if every nonzero ideal of D is a t-C-ideal.*

Proof. Straightforward. $\qquad\square$

Lemma 4.5. *In a Noetherian domain, every nonzero ideal is a reduction (resp., t-reduction) of its integral closure (resp., t-integral closure).*

Proof. Combine [21, Corollary 1.2.5] and [22, Proposition 3.7(b)] with the assumption that every ideal is finitely generated (and so is the t-integral closure of any nonzero ideal). $\qquad\square$

Proof of Theorem 4.3. $(1) \Leftrightarrow (2) \Leftrightarrow (3)$ is [13, Theorem 5.11]. Moreover, $(1) \Leftrightarrow (7) \Leftrightarrow (8)$, $(9) \Leftrightarrow (10)$, and $(10) \Leftrightarrow (4)$ hold in any arbitrary domain (i.e., not necessarily Noetherian) by [22, Theorem 3.5], [23, Theorem 3.2], and Lemma 4.4, respectively. Also, $(4) \Rightarrow (5) \Rightarrow (6)$ are trivial.

$(1) \Rightarrow (10)$ Assume R is integrally closed. Then, R is Krull and hence it has the t-basic ideal property by [17, Figure 2], as desired.

$(6) \Rightarrow (1)$ By [12, Lemma 24.6], it suffices to show that every principal ideal is integrally closed. Let (a) be a principal ideal of R and let

$$b \in \overline{(a)} \subseteq \widetilde{(a)} \subseteq \widehat{(a)}^{\,t}.$$

So, (a) is a t-reduction of (a, b). Since (a) is a t-C-ideal, $(a) = (a, b)_t$; that is, $b \in (a)$. Thus, (a) is integrally closed. $\qquad\square$

Recall that a Krull domain has the t-basic ideal property and the converse is not true in general [17, Example 3.3]. However, the two notions coincide in Noetherian domains as shown by Theorem 4.3, which also provides a t-analogue for Hays' result that "a Noetherian domain is Dedekind if and only if it has the basic ideal property" [13, Corollary 6.6]:

Corollary 4.6. *A Noetherian domain is Krull if and only if it has the t-basic ideal property.*

In [13], Hays proved that the notion of regular C-ideal is local in Noetherian rings (cf. [13, Theorem 5.8 & Theorem 5.9 & Corollary 5.10]). We close this section by establishing a satisfactory t-analogue for this result.

Proposition 4.7. *In a Noetherian domain, if an ideal is t-locally a t-C-ideal, then it is a t-C-ideal.*

Proof. Let I be a nonzero ideal of R which is t-locally a t-C-ideal. Suppose $I \subseteq K$ with $(IK^n)_t = (K^{n+1})_t$ for some ideal K of R and positive integer n. Localizing at $M \in \operatorname{Max}_t(R)$, we get

$$((IK^n)_t R_M)_{t_M} = ((K^{n+1})_t R_M)_{t_M}$$

where t_M denotes the t- operation in R_M. By [24, Lemma 2.18], we have

$$(IR_M KR_M{}^n)_{t_M} = (KR_M{}^{n+1})_{t_M}.$$

Since I is t-locally a t-C-ideal, $(IR_M)_{t_M} = (KR_M)_{t_M}$. Consequently, as all ideals are finitely generated, $I^{-1}R_M = K^{-1}R_M$, $\forall M \in \operatorname{Max}_t(R)$. It follows that

$$(I^{-1})_w = \bigcap_{M\in\operatorname{Max}_t(R)} (I^{-1})R_M = \bigcap_{M\in\operatorname{Max}_t(R)} (K^{-1})R_M = (K^{-1})_w.$$

Thus, $I_t = K_t$; that is, I is a t-C-ideal. $\qquad\square$

The converse of the above result is still elusively open.

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