New York Journal of Mathematics

New York J. Math. **24** (2018) 419–428.

Basis properties of complex exponentials and invertibility of Toeplitz operators

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Abstract. We give a criterion for basicity of a sequence of complex exponentials in terms of the invertibility properties of a certain naturally associated Toeplitz operator. The criterion is similar to the well-known criterion of Hruschev, Nikolskii and Pavlov, the main difference being that we don't require preliminary translation of the frequency sequence to the upper half-plane.

Contents

1. Introduction

Questions about various types of expansion properties of the sequence of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^2[0,1]$ have a very long history, with origins in the work of Paley and Wiener [12], and Levinson [6]. Today, this is considered to be a classical topic with an extensive literature behind (see, e.g., [4, 5, 16, 17] and references therein.)

The idea, to use Toeplitz operators in the study of complex exponentials, was first introduced in the seventies by Douglas, Sarason, and Clark [2,3] and culminated with the remarkable paper of Hruschev, Pavlov, and Nikolski [5] where the solution of the Riesz basis problem for complex exponentials was

Received September 1, 2017.

2010 *Mathematics Subject Classification.* 30D20.

Key words and phrases. complex exponentials, Toeplitz operators, reproducing kernels, inner function.

† Research supported in part by National Science Foundation DMS grant # 1600874.

ISSN 1076-9803/2018

presented. The later paper gives a thorough, comprehensive treatment of the Riesz basis problem and shows how most of the related results obtained by that point can be derived using their approach. More recently, Makarov and Poltoratski [7, 8] used the Toeplitz operator method to give a new approach to the difficult completeness problem for complex exponentials. Their approach opened a possibility for several important applications and generalizations [1, 9, 10, 13, 14]. The goal of this note is to present yet another connection between the basis properties of a given sequence of complex exponentials $\{e^{i\lambda_n t}\}$ with a frequency sequence $\Lambda = \{\lambda_n\}$ and the invertibility properties of a certain Toeplitz operator.

To better explain our motivation we briefly recall the basic idea of the Toeplitz operator approach. Using the Fourier transform it is easy to see that studying the basis properties of complex exponentials $\{e^{i\lambda_n t}\}_{\lambda_n \in \Lambda}$ in $L^2[0, 1]$ is equivalent to studying the basis properties of the corresponding sequence of normalized reproducing kernels in the model space $\mathcal{K}_{e^{iz}} = \mathcal{H}^2 \ominus e^{iz}\mathcal{H}^2$. If the frequency sequence $\Lambda = \{\lambda_n\}$ is uniformly discrete, by the Carleson interpolation theorem, the sequence of normalized reproducing kernels in the model space $\mathcal{K}_{B_{\Lambda'}} = \mathcal{H}^2 \ominus B_{\Lambda'}\mathcal{H}^2$ forms a Riesz basis, where $B_{\Lambda'}$ is the Blaschke product with zero set $\Lambda' = \Lambda + i = \{\lambda_n + i\} \subset \mathbb{C}_+$. The crucial idea is then to observe that the basis properties of the normalized reproducing kernels in $\mathcal{K}_{e^{iz}}$ are encoded in the invertibility properties of the orthogonal projection onto $\mathcal{K}_{e^{iz}}$ restricted to $\mathcal{K}_{B_{\Lambda'}}$, which is in turn equivalent to the corresponding invertibility properties of the Toeplitz operator $T_{\bar{B}_{\Lambda'}S}$. Consequently, complex exponentials form a Riesz basis if and only if the Toeplitz operator $T_{\bar{B}_{\Lambda'}S}$ is invertible, where $S(z) = e^{iz}$ is the singular inner function.

One novelty introduced in the Makarov-Poltoratski Toeplitz approach was that they worked with the original sequence Λ with a special inner function Θ_Λ satisfying $\Lambda = \{\Theta_\Lambda = 1\}$ instead of using the translated sequence $\Lambda' = \Lambda + i$, and the corresponding Blaschke product. In a sense, they were forced to do this since they needed to handle the case when the frequency sequence Λ is not uniformly discrete, i.e., separated. It was observed slightly later by Baranov [1] that for the Riesz basis problem one needs to be careful in doing this. Namely, he showed that there exists a sequence $\Lambda \subset \mathbb{R}$ and an inner function Θ satisfying $\Lambda = \{\Theta = 1\}$ such that the Topelitz operator $T_{\bar{S}\Theta}$ is invertible, even though the corresponding sequence of complex exponentials is not a Riesz basis. Similarly, in the opposite direction, he gave an example of a Riesz basis of complex exponentials and an inner function Θ satisfying $\Lambda = \{\Theta = 1\}$ such that the Topelitz operator $T_{\bar{S}\Theta}$ is not invertible. As an alternative he provided a Riesz basis criterion in terms of naturally associated de Branges spaces, generalizing the criterion for Fourier frames obtained earlier by Seip and Ortega-Cerda [11].

The goal of this note is to show that the analog of the Hruschev, Nikolski, Pavlov criterion can be obtained without using the translated sequence. This might be slightly surprising in the view of the Baranov's counterexamples mentioned above. For this, we needed to impose a slightly stronger assumption on the frequency sequence. To state our result we need to introduce some terminology that will be also used throughout the paper.

We will say that a sequence of real numbers $\Lambda = \{\lambda_n\}$ is discrete if it has no finite accumulation points. A discrete sequence is called uniformly discrete or separated if $\inf_{m,n} |\lambda_m - \lambda_n| > 0$. The sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ will be always indexed in an increasing way, i.e., $\lambda_n < \lambda_{n+1}$ for all n. Below and throughout the paper $S(z)$ will always denote the singular inner function e^{iz}.

Theorem 1.1. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a discrete sequence of real numbers. Assume that there exists an inner function Θ such that $\{\Theta = 1\} = \Lambda$ and $|\Theta'(t)| \simeq 1, t \in \mathbb{R}$, i.e., there exist $0 < c \leq C < \infty$ such that $c < |\Theta'(t)| < C$ for all $t \in \mathbb{R}$. Then*

- (i) *the sequence of complex exponentials $\{e^{i\lambda_n t}\}_{\lambda_n\in\Lambda}$ is a Riesz sequence in $L^2[0,1]$ if and only if the Toeplitz operator $T_{\bar{\Theta}S} : \mathcal{H}^2 \to \mathcal{H}^2$ is injective with closed range;*
- (ii) *the sequence of complex exponentials $\{e^{i\lambda_n t}\}_{\lambda_n\in\Lambda}$ is a frame in $L^2[0,1]$ if and only if the Toeplitz operator $T_{\bar{\Theta}S} : \mathcal{H}^2 \to \mathcal{H}^2$ is surjective;*
- (iii) *the sequence of complex exponentials $\{e^{i\lambda_n t}\}_{\lambda_n\in\Lambda}$ is a Riesz basis in $L^2[0,1]$ if and only if the Toeplitz operator $T_{\bar{\Theta}S} : \mathcal{H}^2 \to \mathcal{H}^2$ is invertible;*
- (iv) *the sequence of complex exponentials $\{e^{i\lambda_n t}\}_{\lambda_n\in\Lambda}$ is l^2-independent in $L^2[0,1]$ if and only if the Toeplitz operator $T_{\bar{\Theta}S} : \mathcal{H}^2 \to \mathcal{H}^2$ is injective;*
- (v) *the sequence of complex exponentials $\{e^{i\lambda_n t}\}_{\lambda_n\in\Lambda}$ is complete in $L^2[0,1]$ if and only if the Toeplitz operator $T_{\bar{\Theta}S} : \mathcal{H}^2 \to \mathcal{H}^2$ has dense range.*

Remark 1. Notice that there the set of inner functions that satisfy the requirements of Theorem 1.1 may be empty. Still, if in addition to separation we also impose the stronger condition $\sup_n |\lambda_{n+1} - \lambda_n| < \infty$, then there always exists a meromorphic inner function Θ such that $|\Theta'| \simeq 1$ on $\mathbb{R}$ (see Lemma 6.1 in [8]). This stronger condition is necessary for Riesz bases and hence it is not a big restriction in this case.

Remark 2. Part (v) has been proved in [7] with no additional assumptions on Θ. We include our more restricted version here for completeness reasons only. It should be also mentioned that for part (iv) we only need $|\Theta'| \lesssim 1$ on $\mathbb{R}$.

2. Preliminaries

A bounded analytic function $\Theta : \mathbb{C}_+ \to \mathbb{C}$ in the upper half-plane $\mathbb{C}_+$ is said to be inner if its modulus coincides with 1 a.e. on the real line. We say that the inner function Θ is meromorphic inner function if it allows a meromorphic extension to the whole complex plane.

Each inner function Θ defines a model space

$$\mathcal{K}_\Theta := \mathcal{H}^2 \ominus \Theta\mathcal{H}^2.$$

This is a reproducing kernel space with reproducing kernels given by

$$K_w^\Theta(z) := \frac{1}{2\pi i}\frac{1-\Theta(z)\overline{\Theta(w)}}{\bar{w}-z}, \qquad w \in \mathbb{C}_+.$$

If Θ is meromorphic inner function then every function $f \in \mathcal{K}_\Theta$ can be extended analytically exactly at those points at which Θ can be extended. Moreover, point evaluations at those points are again bounded and consequently for each $\lambda \in \mathbb{R}$ there exists a reproducing kernel

$$K_\lambda^\Theta(z) = \frac{1}{2\pi i}\frac{1-\Theta(z)\overline{\Theta(\lambda)}}{\lambda - z} \in \mathcal{K}_\Theta.$$

A well known result of D. Clark [2] (see also [15]) says that the system of normalized reproducing kernels

$$k_{\lambda_n}^\Theta(z) = \frac{1}{i\,|\Theta'(\lambda_n)|}\frac{1-\Theta(z)}{\lambda_n - z} \in \mathcal{K}_\Theta$$

indexed by the points $\lambda_n \in \{\Theta = 1\}$ forms an orthonormal basis for $\mathcal{K}_\Theta$ which is sometimes called Clark basis.

The following (easy to check) equality between normalized reproducing kernels will be repeatedly use in our proof.

Lemma 2.1. *Let $\Theta(z)$ be an inner function with a level set $\{\Theta = 1\} = \{\lambda_n\}$ such that $\sup_n |\Theta'(\lambda_n)| < \infty$. Then for every sequence $\{a_n\} \in l^2$ the following equality between normalized reproducing kernels holds:*

$$(1-\Theta)\sum a_n k_{\lambda_n}^S = \sum a_n \left|\Theta'(\lambda_n)\right| k_{\lambda_n}^\Theta - S\sum a_n \left|\Theta'(\lambda_n)\right| \overline{S(\lambda_n)} k_{\lambda_n}^\Theta.$$

We will also use the following lemma. The first part of it is essentially Lemma 5.4 from [1]. For the sake of completeness we state it here in the form that will be used below.

Lemma 2.2. *Let $\Theta(z)$ be a meromorphic inner function such that $|\Theta'(t)| \lesssim 1$ for $t \in \mathbb{R}$.*

(a) *If $f \in \mathcal{K}_S$ vanishes on the whole level set $\Lambda = \{\lambda_n\} = \{\Theta = 1\}$, then $f/(1-\Theta) \in \mathcal{K}_S$.*
(b) *If $\{f_k\}$ is a sequence of functions in $\mathcal{K}_S$ such that $\|(1-\Theta)f_k\|_2 \to 0$ then $\|f_k\|_2 \to 0$.*

Proof. (b) Seeking contradiction, assume that $\{f_k\}$ does not converge to 0. In this case, by passing to a subsequence if necessary, we may assume that $\|f_k\|_2 > d$ for some constant $d > 0$ and all k. For small enough $\epsilon > 0$ there exists a constant $c > 0$ such that

$$|1-\Theta(t)| \geq c > 0,$$

for all t satisfying $\mathrm{dist}(t,\Lambda) \geq \epsilon$. Notice that we may choose this ϵ as small as we wish. With no loss of generality choose $\epsilon > 0$ so that the separation constant for Λ is larger than 6ϵ. Denote by E_ϵ the set consisting of all real t such that $\mathrm{dist}(t,\Lambda) \leq \epsilon$. Then,

$$\int_{E_\epsilon^c} |1-\Theta(t)|^2 |f_k(t)|^2 dt \geq c^2 \int_{E_\epsilon^c} |f_k(t)|^2.$$

Now, since

$$\lim_{k\to\infty} \frac{\|(1-\Theta)f_k\|}{\|f_k\|} = 0,$$

there exists $k(\epsilon)$ such that

$$\|(1-\Theta)f_k\|_2^2 < c^2 \frac{\|f_k\|_2^2}{2},$$

for all $k \geq k(\epsilon)$. Therefore, we have

$$c^2 \int_{E_\epsilon^c} |f_k(t)|^2 dt < c^2 \frac{\|f_k\|^2}{2},$$

and consequently

$$\|f_k\|^2 = \int_{\mathbb{R}} |f_k(t)|^2 dt < 2 \int_{E_\epsilon} |f_k(t)|^2 dt,$$

for all $k \geq k(\epsilon)$. Now, let $t \in (-3\epsilon, 3\epsilon)$ and fix $k \geq k(\epsilon)$. Using the well known inequality

$$\int_{-\infty}^{\infty} |f_k(x+iy)|^2 dx \leq e^{2|y|} \|f_k\|^2,$$

holding for functions $f_k \in \mathcal{K}_S = \mathcal{H}^2 \cap S\overline{\mathcal{H}^2}$ we obtain that

$$\sum_n |f_k(\lambda_n+t)|^2 \leq \sum_n \frac{1}{9\pi\epsilon^2} \int_{B(\lambda_n+t,3\epsilon)} |f_k(z)|^2 dA(z) \leq \frac{1}{9\pi\epsilon^2} \int_{-3\epsilon}^{3\epsilon} \int_{-\infty}^{\infty} |f_k(x+iy)|^2 dx dy$$

$$\leq \frac{2\|f_k\|_2^2}{3\pi\epsilon} \frac{e^{6\epsilon}-1}{6\epsilon}.$$

Thus,

$$\frac{1}{2}\|f_k\|^2 < \int_{E_\epsilon} |f_k(t)|^2 = \int_{-\epsilon}^{\epsilon} \sum_n |f_k(\lambda_n+t)|^2 dt \leq \frac{4\|f_k\|_2^2}{3\pi} \frac{e^{6\epsilon}-1}{6\epsilon}.$$

Since $f_k \neq 0$ we therefore have

$$\frac{1}{2} < \frac{4}{3\pi} \frac{e^{6\epsilon}-1}{6\epsilon},$$

which fails for small enough $\epsilon > 0$. Contradiction!

□

3. Proof of Theorem 1.1

Notice first that if there exists an inner function Θ with $\Lambda = \{\Theta = 1\}$ such that $|\Theta'| \simeq 1$ on $\mathbb{R}$, then Λ must be separated (uniformly discrete). So we only consider such frequency sequences. It is well-known that a sequence of complex exponentials $\{e^{2\pi i\lambda_n t}\}_{\lambda_n\in\Lambda}$ (see e.g. [17]) is a Bessel sequence if and only if its frequency sequence Λ can be represented as a finite union of separated (uniformly discrete) sequences. Therefore, the Bessel condition is always fulfilled for sequences $\{e^{2\pi i\lambda_n t}\}_{\lambda_n\in\Lambda}$ considered in our theorem. Thus, we only need to concentrate to lower inequalities for frames and Riesz sequences.

Recall also (as already mentioned in the introduction) that a sequence of complex exponentials $\{e^{2\pi i\lambda_n t}\}_{\lambda_n\in\Lambda}$ is a Riesz sequence (frame, Riesz basis, l^2-independent, complete) in $L^2[0,1]$ if and only if the corresponding sequence of normalized reproducing kernels $\{k^S_{\lambda_n}\}_{\lambda_n\in\Lambda}$ is a Riesz sequence (frame, Riesz basis, l^2-independent, complete) in the model space $\mathcal{K}_S$.

3.1. Proof of (i) - Riesz sequences.

Proof. Assume that $\{k^S_{\lambda_n}\}_{\lambda_n\in\Lambda}$ is not a Riesz sequence in $L^2[0,1]$. Then there exists a sequence $\{a^k\} \in l^2$ in the unit sphere of l^2 ($\sum_n |a^k_n|^2 = 1$ for all k) such that $g_k := \sum_n a^k_n k^S_{\lambda_n} \to 0$ in $\mathcal{K}_S$ as $k \to \infty$. Therefore, by Lemma 2.1 we have that

$$S\sum a^k_n \left|\Theta'(\lambda_n)\right| \overline{S(\lambda_n)} k^\Theta_{\lambda_n} - \sum a^k_n \left|\Theta'(\lambda_n)\right| k^\Theta_{\lambda_n} \to 0,$$

as $k \to \infty$. Thus, if we take

$$f_k := \sum a^k_n \left|\Theta'(\lambda_n)\right| \overline{S(\lambda_n)} k^\Theta_{\lambda_n} \in \mathcal{H}^2,$$

we obtain $T_{\bar\Theta S} f_k \to 0$ and $\|f_k\|^2 = \sum_n \left|\Theta'(\lambda_n)\right|^2 |a^k_n|^2 \simeq 1$. Thus, $T_{\bar\Theta S}$ is not bounded below.

Conversely, assume that $T_{\bar\Theta S}$ is not bounded below. Then there exists a sequence $\{f_k\}$ of elements in $\mathcal{H}^2$ all with norm one such that $T_{\bar\Theta S} f_k \to 0$ as $k \to \infty$. Let $h_k := T_{\bar\Theta S} f_k \in \mathcal{H}^2$ and let $g_k := \bar\Theta S f_k - h_k \in \overline{\mathcal{H}^2}$. Notice that since $h_k \to 0$ and $\|f_k\| = 1$ we have that $\|g_k\| \simeq 1$.

Since $\Theta g_k \in K_\Theta$ we have $\Theta g_k = \sum_n g^k_n k^\Theta_{\lambda_n}$ for some $\{g^k_n\} \in l^2$. Therefore, by Lemma 2.1 we have

$$\Theta h_k = S f_k - \sum_n g^k_n k^\Theta_{\lambda_n} = S f_k - S \sum_n g^k_n \overline{S(\lambda_n)} k^\Theta_{\lambda_n} - (1-\Theta) \sum_n \frac{g^k_n}{|\Theta'(\lambda_n)|} k^S_{\lambda_n}.$$

Thus,

$$S f_k - S \sum_n g^k_n \overline{S(\lambda_n)} k^\Theta_{\lambda_n} - \sum_n \frac{g^k_n}{|\Theta'(\lambda_n)|} k^S_{\lambda_n} = \Theta h_k - \Theta \sum_n \frac{g^k_n}{|\Theta'(\lambda_n)|} k^S_{\lambda_n}.$$

The difference of the first two terms on the left side is in $S\mathcal{H}^2$ while the last term is in $\mathcal{K}_S$. Using orthogonality and the triangle inequality, we obtain

$$\left\|Sf_k - S\sum_n g_n^k\overline{S(\lambda_n)}k_{\lambda_n}^\Theta\right\|^2 + \left\|\sum_n \frac{g_n^k}{|\Theta'(\lambda_n)|}k_{\lambda_n}^S\right\|^2 \leq \left(\left\|\Theta\sum_n \frac{g_n^k}{|\Theta'(\lambda_n)|}k_{\lambda_n}^S\right\| + \|\Theta h_k\|\right)^2.$$

Since $h_k \to 0$, the last inequality implies $Sf_k - S\sum_n g_n^k\overline{S(\lambda_n)}k_{\lambda_n}^\Theta \to 0$. Using this and 3.1 we obtain

$$(1-\Theta)\sum_n \frac{g_n^k}{|\Theta'(\lambda_n)|}k_{\lambda_n}^S \to 0.$$

Finally, using Lemma 2.2 (b) we derive

$$\sum_n \frac{g_n^k}{|\Theta'(\lambda_n)|}k_{\lambda_n}^S \to 0,$$

which together with

$$\sum_n \left|\frac{g_n^k}{\Theta'(\lambda_n)}\right|^2 \simeq \sum_n \left|g_n^k\right|^2 = \|\Theta g_k\|^2 = \|g_k\|^2 \simeq 1,$$

implies that $\{k_{\lambda_n}^S\}$ cannot be a Riesz sequence in $\mathcal{K}_S$.

□

3.2. Proof of (ii) - frames.

Proof. Assume that $\{k_{\lambda_n}^S\}_{\lambda_n\in\Lambda}$ is not a frame in $\mathcal{K}_S$. Then there exists a sequence of elements $f_k \in \mathcal{K}_S$ all with norm 1 such that

$$\sum_n \left|\left\langle f_k, k_{\lambda_n}^S\right\rangle\right|^2 \to 0$$

as $k \to \infty$.

Let $h_k \in K_\Theta$ be defined by $h_k = \sum_n \left\langle f_k, k_{\lambda_n}^S\right\rangle k_{\lambda_n}^\Theta$. Then

$$\|h_k\|_2^2 = \sum_n |f_k(\lambda_n)|^2 \to 0$$

as $k \to \infty$. Finally, define

$$g_k := \frac{f_k - h_k}{1-\Theta}.$$

By Lemma 2.2 $g_k \in \mathcal{H}^2$. Moreover, since clearly $f_k - h_k \in \mathcal{K}_{\Theta S}$ we also have that $g_k \in \mathcal{K}_S$. Now,

$$\bar{S}f_k = \bar{S}g_k - \bar{S}\Theta g_k + \bar{S}h_k.$$

By projecting to $\mathcal{H}^2$ we obtain

$$T_{\bar{S}\Theta}g_k = P_+\bar{S}h_k.$$

Since $\|h_k\|_2 \to 0$ as $k \to \infty$ this implies that $\|T_{\bar{S}\Theta}g_k\|_2 \to 0$. Finally, since $\|f_k - h_k\|_2 \leq 2\|g_k\|$ we have that the Toeplitz operator $T_{\bar{S}\Theta}$ is not bounded from below and consequently $T_{\bar{\Theta}S}$ is not surjective.

Conversely, suppose that $T_{\bar{\Theta}S}$ is not surjective. Then there exists a sequence $\{g_k\}$ in $\mathcal{H}^2$ with $\|g_k\|_2 = 1$ for all k such that $\|T_{\bar{S}\Theta} g_k\|_2 \to 0$. Decompose $g_k = f_k + Sh_k$ in orthogonal components such that $f_k \in \mathcal{K}_S$ and $h_k \in \mathcal{H}^2$. Then $T_{\bar{S}\Theta} f_k \in \mathcal{K}_\Theta$ and $T_{\bar{S}\Theta} Sh_k = \Theta h_k \in \Theta\mathcal{H}^2$. Consequently, they are orthogonal and hence each of them tends to 0 as $k \to \infty$. Therefore, for all but finitely many k we have that $\|f_k\|_2^2 = \|g_k\|_2^2 - \|Sh_k\|_2^2 \geq c_1 > 0$. Now, decompose $\bar{S}\Theta f_k = p_k + \bar{n}_k$ into orthogonal components $p_k \in \mathcal{K}_\Theta$ and $n_k \in \mathcal{H}^2$. Notice that since $\bar{S} n_k = \bar{\Theta}\bar{f}_k - \overline{Sp_k} \in \overline{\mathcal{H}^2}$ we also have $n_k \in \mathcal{K}_S$. Finally, define $q_k := n_k - S\bar{f}_k$. It easy to check that $q_k \in \mathcal{K}_S$. Moreover,

$$q_k(\lambda_n) = S(\lambda_n)\bar{\Theta}(\lambda_n)\bar{f}_k(\lambda_n) - \bar{p}_k(\lambda_n) - S(\lambda_n)\bar{f}_k(\lambda_n) = -\bar{p}_k(\lambda_n).$$

Therefore,

$$\sum_n \left|\left\langle q_k, k_{\lambda_n}^S \right\rangle\right|^2 = \sum_n |q_k(\lambda_n)|^2 = \sum_n |p_k(\lambda_n)|^2 \simeq \|p_k\|_2^2 = \|T_{\bar{S}\Theta} f_k\| \to 0.$$

So, we would be done if we show that $\limsup \|q_k\|_2 > 0$. To see this, first notice that $q_k = S\bar{\Theta}\bar{f}_k - S\bar{f}_k - \bar{p}_k$. Using orthogonality we then obtain $\|q_k\|^2 = \|(1-\Theta)f_k\|_2 + \|p_k\|_2^2$. Thus, it is enough to exclude the possibility that $\lim_{k\to\infty} \|(1-\Theta)f_k\| = 0$. Indeed, otherwise we would be able to find a convergent subsequence with a positive limit which would easily give $\limsup \|q_k\|_2 > 0$. However, if $\lim_{k\to\infty} \|(1-\Theta)f_k\| = 0$ then, by Lemma 2.2, we would have $\|f_k\| \to 0$ which contradicts our assumption. So, we are done. □

3.3. Proof of (iii) - Riesz bases. This is a direct consequence of (i) and (ii).

3.4. Proof of (iv) - l^2-independence.

Proof. Assume that $\{k_{\lambda_n}^S\}$ is not l^2-independent in $\mathcal{K}_S$. Then there exists a non zero sequence $\{a_n\} \in l^2$ such that $\sum_n a_n k_{\lambda_n}^S = 0$ in $\mathcal{K}_S$. Therefore, by Lemma 2.1 we have that

$$S \sum a_n \left|\Theta'(\lambda_n)\right| \overline{S(\lambda_n)} k_{\lambda_n}^\Theta = \sum a_n \left|\Theta'(\lambda_n)\right| k_{\lambda_n}^\Theta \in \mathcal{K}_\Theta.$$

Thus, if we take

$$f := \sum a_n \left|\Theta'(\lambda_n)\right| \overline{S(\lambda_n)} k_{\lambda_n}^\Theta \in \mathcal{H}^2,$$

we have that $T_{\bar{\Theta}S} f = 0$ and $f \neq 0$, i. e., $T_{\bar{\Theta}S}$ is not injective.

Conversely, assume that $T_{\bar{\Theta}S}$ is not injective. Let f be some non-zero element from $\ker T_{\bar{\Theta}S}$. Then $Sf \in K_\Theta$ and by the Clark formula $Sf = \sum b_n k_{\lambda_n}^\Theta$ for some non-zero l^2 sequence $\{b_n\}$. Define also $h := \sum b_n \overline{S(\lambda_n)} k_{\lambda_n}^\Theta \in \mathcal{K}_\Theta$. Clearly, by Lemma 2.1

$$(1-\Theta) \sum a_n k_{\lambda_n}^S = Sf - Sh \in S\mathcal{H}^2,$$

where $a_n := b_n \left|\Theta'(\lambda_n)\right|$.

Denote $g := \sum a_n k^S_{\lambda_n} \in \mathcal{K}_S$. We have $g \perp S(f-h)$ and $\Theta g = g - S(f-h)$. Therefore $\|\Theta g\|^2 = \|g\|^2 + \|S(f-h)\|^2$. This implies that $S(f-h) = 0$. Therefore $(1-\Theta)g = 0$ and hence $\sum a_n k^S_{\lambda_n} = g = 0$. Thus, $\{k^S_{\lambda_n}\}$ is not l^2-independent in $\mathcal{K}_S$. □

3.5. Proof of (v) - completeness.

Proof. Assume that $\{k^S_{\lambda_n}\}_{\lambda_n \in \Lambda}$ is not complete in $\mathcal{K}_S$. Then there exists a non-zero $f \in \mathcal{K}_S$ which vanishes on Λ. By the Lemma 2.2, $g := f/(1-\Theta) \in \mathcal{K}_S$, and hence

$$\bar{S}\Theta g = \bar{S}g - \bar{S}f \in \overline{\mathcal{H}^2}.$$

Consequently, the Toeplitz operator $T_{\bar{S}\Theta}$ is not injective, and hence $T_{\bar{\Theta}S}$ does not have a dense range.

Conversely, suppose $T_{\bar{\Theta}S}$ does not have a dense range. Then there is a non-zero $g \in \ker T_{\bar{S}\Theta}$. This implies that both g and Θg belong to $\mathcal{K}_S$. So, $f := (1-\Theta)g \in \mathcal{K}_S$ is a non-zero function which vanishes on Λ, and hence $\{k^S_{\lambda_n}\}_{\lambda_n \in \Lambda}$ is not complete in $\mathcal{K}_S$. □

Acknowledgment. A portion of this work was done while the author was a graduate student at Texas A&M. The author is grateful to A. Poltoratski for many enlightening discussions during this time. The author would also like to thank the anonymous referee for the comments and suggestions.

References

[1] Baranov, Anton. Completeness and Riesz bases of reproducing kernels in model subspaces. *Int. Math. Res. Not.* **2006**, Art. ID 81530, 34 pp. MR2264717, Zbl 1118.47007, doi: 10.1155/IMRN/2006/81530. ↑420, 422

[2] Clark, Douglas N. One dimensional perturbations of restricted shifts. *J. Analyse Math.* **25** (1972), 169–191. MR0301534, Zbl 0252.47010, doi: 10.1007/BF02790036. ↑419, 422

[3] Douglas, Ronald G.; Sarason, Donald. A class of Toeplitz operators. Proceedings of an International Symposium on Operator Theory (Indiana Univ., Bloomington, Ind., 1970). *Indiana Univ. Math. J.* **20** (1971), no. 10, 891–895. MR0407642, Zbl 0199.19103, doi: 10.1512/iumj.1971.20.20073. ↑419

[4] Higgins, John Rowland. Completeness and basis properties of sets of special functions. Cambridge Tracts in Mathematics, 72. *Cambridge University Press, Cambridge-New York-Melbourne*, 1977. x+134 pp. ISBN: 0-521-21376-2. MR0499341, Zbl 0351.42021, doi: 10.1017/CBO9780511566189. ↑419

[5] Hruščëv, Sergei V.; Nikol'skiĭ, Nikolaĭ K.; Pavlov, Boris S. Unconditional bases of exponentials and of reproducing kernels. *Complex analysis and spectral theory* (Leningrad, 1979/1980), 214–335. Lecture Notes in Math., 864. *Springer, Berlin-New York*, 1981. MR0643384, Zbl 0466.46018. ↑419

[6] Levinson, Norman. Gap and density theorems. American Mathematical Society Colloquium Publications, 26. *American Mathematical Society, New York*, 1940. viii+246 pp. MR0003208, Zbl 0145.08003. ↑419

[7] Makarov, Nikolai G.; Poltoratski, Alexei G. Meromorphic inner functions, Toeplitz kernels, and the uncertainty principle. *Perspectives in analysis*, 185–252. Math. Phys. Stud., 27. *Springer, Berlin*, 2005. MR2215727, Zbl 1118.47020, doi: 10.1007/3-540-30434-7_10. ↑420, 421

[8] Makarov, Nikolai G.; Poltoratski, Alexei G. Beurling–Malliavin theory for Toeplitz kernels. *Invent. Math.* **180** (2010), no. 3, 443–480. MR2609247, Zbl 1186.47025, arXiv:math/0702497, doi: 10.1007/s00222-010-0234-2. ↑420, 421

[9] Mitkovski, Mishko; Poltoratski, Alexei. Pólya sequences, Toeplitz kernels, and gap theorems. *Adv. Math.* **224** (2010), no. 3, 1057–1070. MR2628803, Zbl 1204.30018, arXiv:0903.4499, doi: 10.1016/j.aim.2009.12.014. ↑420

[10] Mitkovski, Mishko; Poltoratski, Alexei. On the determinacy problem for measures. *Invent. Math.* **202** (2015), no. 3, 1241–1267. MR3425390, Zbl 1346.30015, arXiv:1406.0025, doi: 10.1007/s00222-015-0588-6. ↑420

[11] Ortega-Cerdà, Joaquim; Seip, Kristian. Fourier frames. *Ann. of Math.* (2) **155** (2002), no. 3, 789–806. MR1923965, Zbl 1015.42023, doi: 10.2307/3062132. ↑420

[12] Paley, Raymond E.A.C.; Wiener, Norbert. Fourier transforms in the complex domain. American Mathematical Society Colloquium Publications, 19. *American Mathematical Society, Providence, RI*, 1934. x+184 pp. ISBN: 0-8218-1019-7. MR1451142, Zbl 0011.01601. ↑419

[13] Poltoratski, Alexei. Spectral gaps for sets and measures. *Acta Math.* **208** (2012), no. 1, 151–209. MR2910798, Zbl 1245.42008, arXiv:0908.2079, doi: 10.1007/s11511-012-0076-4. ↑420

[14] Poltoratski, Alexei. A problem on completeness of exponentials. *Ann. Math.* (2) **178** (2013), no. 3, 983–1016. MR3092474, Zbl 1285.42009, arXiv:1006.1840, doi: 10.4007/annals.2013.178.3.4. ↑420

[15] Poltoratski, Alexei; Sarason, Donald. Aleksandrov–Clark measures. *Recent advances in operator-related function theory*, 1–14. Contemp. Math., 393. *Amer. Math. Soc., Providence, RI*, 2006. MR2198367, Zbl 1102.30032, doi: 10.1090/conm/393/07366. ↑422

[16] Redheffer, Raymond M. Completeness of sets of complex exponentials. *Advances in Math.* **24** (1977), no. 1, 1–62. MR0447542, Zbl 0358.42007, doi: 10.1016/S0001-8708(77)80002-9. ↑419

[17] Young, Robert M. An introduction to nonharmonic Fourier series. revised first edition. *Academic Press, Inc., San Diego, CA*, 2001. xiv+234 pp. ISBN: 0-12-772955-0. MR1836633, Zbl 0981.42001. ↑419, 424

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This paper is available via http://nyjm.albany.edu/j/2018/24-23.html.