Finite volume hyperbolic complements of 2-tori and Klein bottles in closed smooth simply connected 4-manifolds

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ABSTRACT. We give necessary conditions, for a closed smooth simply connected 4-manifold X, to contain a collection of surfaces L such that $X - L$ admits a complete finite volume hyperbolic structure. We then show that examples of non-compact hyperbolic 4-manifolds constructed by Ivanšić, and Ivanšić, Ratcliffe and Tschantz, give rise to examples of such link complements in $\#_{2k} S^2 \times S^2$.

CONTENTS

1. Introduction.

The aim of this article is to understand when the homeomorphism type of a closed smooth simply connected 4-manifold can contain a link of surfaces, whose complement admits a finite volume complete hyperbolic structure. The finite volume condition forces the surfaces to be either a 2-torus or a Klein bottle.

Our main approach to this problem is to use the work of S. Donaldson and M. Freedman, which provides us with a very nice classification theorem, on the possible homeomorphism types of a closed smooth simply connected 4-manifold. It can be expressed in the following simple form (see [9]).

Theorem 1.1. *Every closed smooth simply connected 4-manifold is homeomorphic to either*

$$ S^4 \ or \ \#_n \mathbb{CP}^2 \#_m \overline{\mathbb{CP}}^2 \ or \ \#_{\pm m} \mathcal{M}E_8 \#_n S^2 \times S^2 $$

Here $\mathcal{M}E_8$ denotes the non-smoothable topological 4-manifold with the E_8 intersection form.

Using this classification theorem we are able to prove the following theorem, which gives necessary conditions on the homeomorphism type, for a closed smooth simply connected 4-manifold, to contain a collection of surfaces, whose complement admits a finite volume hyperbolic structure.

Theorem 1.2. *Let X be a closed smooth simply connected 4-manifold and L be a collection of 2-tori and Klein bottles embedded in X. Suppose that the complement $X - L$ admits a finite volume complete hyperbolic structure. Then the homeomorphism type of X falls into one of the following three categories:*

- *S^4*
- *$\#_k(S^2 \times S^2)$, $k > 0$.*
- *$\#_k\mathbb{CP}^2\#_k\overline{\mathbb{CP}^2}$, $k > 0$.*

We prove this theorem by showing that any such closed smooth 4-manifold must have vanishing signature, see Proposition 2.6.

This result motivates the question as to whether all the homeomorphism types, appearing in Theorem 1.2, actually do contain such a link of surfaces L.

The first piece of work that was done in investigating this sort of question was carried out by D. Ivanšić in [4]. In that paper, Ivanšić showed that there exists a closed smooth simply connected 4-manifold, homeomorphic to S^4, with a collection of five embedded 2-tori, such that the complement of these 2-tori admits a finite volume complete hyperbolic structure. Soon after Ivanšić, Ratcliffe, and Tschantz constructed several more examples of such hyperbolic link complements in 4-manifolds that were homeomorphic to S^4 (see [5]).

In the same paper [4], Ivanšić showed the existence of closed smooth simply connected 4-manifolds, with Euler characteristic $2n$ for $n > 0$, each containing a collection of 2-tori, whose complement admitted a finite volume complete hyperbolic structure. However, he did not determine the homeomorphism type of these manifolds. Using Theorem 1.2 and some analysis to do with spin structures, we are able to show that, in the case $n = 2k+1$, his examples are homeomorphic to $\#_{2k}S^2 \times S^2$. This establishes the existence of a hyperbolic link complement in $\#_{2k}S^2 \times S^2$.

Theorem 1.3. *For $k \geq 1$ there exists a collection of $8k + 5$ 2-tori, embedded in a smooth 4-manifold X_k, such that X_k is homeomorphic to $\#_{2k}(S^2 \times S^2)$ and $X_k - L$ admits a finite volume hyperbolic structure.*

2. Hyperbolic link complements in closed smooth simply connected 4-manifolds.

In this section we will show how to prove Theorem 1.2. As mentioned in the introduction, the key point is to show that a closed smooth simply connected 4-manifold, containing a collection of embedded 2-tori and Klein bottles, whose complement admits a finite volume complete hyperbolic structure, must have vanishing signature.

We will need the following theorem of D. Long and A. Reid (see [7] Theorem 2.1, p. 173-174).

Theorem 2.1. *Let M be a non-compact orientable finite volume hyperbolic 4-manifold. Then $\sigma(M) = \eta(\partial M)$, where σ denotes the signature and η is the eta invariant.*

In the above theorem, M is a manifold with boundary. By $\sigma(M)$ we mean the signature of the nondegenerate symmetric form on the image of $H^2(M, \partial M; \mathbb{Z})$ in $H^2(M; \mathbb{Z})$, induced via the cup product. As we are restricting to the image of $H^2(M, \partial M; \mathbb{Z})$ in $H^2(M; \mathbb{Z})$, Poincare-Lefshetz duality tells us that this is nondegenerate. Also, note that ∂M could have more than one component. In such a situation, $\eta(\partial M)$ is to be understood as the sum of η on each component.

The importance of this theorem is that it translates the computation of the signature into the computation of the eta invariant of the cusp cross-sections of the manifold. The cusp cross-sections of a non-compact orientable finite volume hyperbolic 4-manifold are always compact flat 3-manifolds. There are six isometry classes of orientable closed flat 3-manifolds. We denote these six classes by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$, as in [3] (these are also denoted by $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, $\mathcal{G}_4$, $\mathcal{G}_5$, $\mathcal{G}_6$ in Wolf's book [11]). Therefore, in order to understand the signature of a non-compact orientable finite volume hyperbolic 4-manifold, one needs to understand what the eta invariant of the above six classes of flat 3-manifolds are.

The computation of the eta invariant for these six classes of flat 3-manifolds can be found in [10], example 1, p. 128. The following proposition gives the values of the eta invariant for these six classes.

Proposition 2.2.
$\eta(\boldsymbol{A}) = 0$
$\eta(\boldsymbol{B}) = 0$
$\eta(\boldsymbol{C}) = \frac{-2}{3}$
$\eta(\boldsymbol{D}) = -1$
$\eta(\boldsymbol{E}) = \frac{-4}{3}$
$\eta(\boldsymbol{F}) = 0$

From the classification theorem of closed flat 3-manifolds (see [11] Thm. 3.5.5, p. 117), it is known that only $\mathbf{A}$ and $\mathbf{B}$ are S^1-fibre bundles over a compact surface, with $\mathbf{A}$ being a 3-torus fibering over a 2-torus and $\mathbf{B}$ fibering over a Klein bottle. Recall that we are focusing on simply connected 4-manifolds, that contain a collection of 2-tori or Klein bottles, whose complement admits a finite volume complete hyperbolic structure. Therefore, it follows that the complement, which is a non-compact hyperbolic 4-manifold, must have cusp cross-section, a compact 3-manifold, an S^1 fibre bundle over a 2-torus or Klein bottle. We thus see that the cusp cross-sections must be of type $\mathbf{A}$ or $\mathbf{B}$.

Using the above proposition we have the following corollary.

Corollary 2.3. *Let M be an orientable non-compact finite volume hyperbolic 4-manifold, with cusp cross-sections of type $\boldsymbol{A}$ or $\boldsymbol{B}$. Then $\sigma(M) = 0$, where σ is the signature invariant.*

The flat 3-manifolds $\mathbf{A}$ and $\mathbf{B}$, being circles bundles, have associated disc bundles, which we denote by $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ respectively. We are going to need to know the signature of these disc bundles.

Lemma 2.4. $\sigma(\overline{\boldsymbol{A}}) = 0$ *and* $\sigma(\overline{\boldsymbol{B}}) = 0$.

Proof. The manifold $\overline{\mathbf{A}}$ is a disc bundle with boundary $\mathbf{A}$. We remind the reader that, by definition, the signature of $\overline{\mathbf{A}}$ is defined as the signature of the nondegenerate symmetric form, on the image of $H^2(\overline{\mathbf{A}}, \mathbf{A}; \mathbb{Z})$ in $H^2(\overline{\mathbf{A}}; \mathbb{Z})$, induced via the cup product. As $\overline{\mathbf{A}}$ has zero Euler number it follows that its signature must vanish.

A similar proof proves the vanishing of $\sigma(\overline{\mathbf{B}})$. $\qquad\square$

From now on we suppose that X is a closed smooth simply connected 4-manifold, that contains a collection L of 2-tori and Klein bottles, such that $X - L$ admits a finite volume complete hyperbolic structure. We denote $X - L$ by M.

If we take each cusp of M, and chop it off, we produce a 4-manifold $\overline{M}$ with boundary given by the flat 3-manifolds $\mathbf{A}$ and $\mathbf{B}$. For each surface in L we can take a normal neighbourhood and construct the associated disk bundle. We choose the disc fibre small enough so that each such disc bundle is disjoint from any other one. If $T \in L$ is a 2-torus, then the disc bundle will be homeomorphic to $S^1 \times S^1 \times D^2$. If $K \in L$ is a Klein bottle, then the disc bundle is just a copy of $\overline{\mathbf{B}}$. Let V denote the union of all these disc bundles.

Each element in V is a disc bundle, and it has an associated circle bundle, which is either a copy of **A** or **B**. The circle fibre of this disc bundle will be called a *meridian*. It is then easy to see that $X - int(V) \cong \overline{M}$.

This viewpoint of X, as being obtained by gluing in disc bundles to the boundary of $\overline{M}$, allows one to compute the signature of X using the following theorem of Novikov (see [6] Thm. 5.3, p. 27).

Theorem 2.5. *Given two oriented 4n-dimensional manifolds M and N such that $\partial M = \partial N$. Then $\sigma(M \cup_\partial N) = \sigma(M) + \sigma(N)$, where $M \cup_\partial N$ denotes M glued to N along the common boundary.*

A nice application of Novikov's theorem and the theorem of Long and Reid is the following proposition.

Proposition 2.6. *Let N be an orientable closed smooth 4-manifold. Let T be a collection of embedded 2-tori and Klein bottles in N. Suppose that the complement $N - T$ admits a hyperbolic structure. Then $\sigma(N) = 0$, where σ denotes the signature.*

Proof. Denote the hyperbolic manifold $N - T$ by N_0. We have that the cusp cross sections of N_0 are all of type **A** or **B**. Furthermore, we have that N is obtained from N_0 by gluing in disc bundles associated to **A** and **B**. By Corollary 2.3 we have that $\sigma(N_0) = 0$. Furthermore, by Lemma 2.4 we have that the signature of these disc bundles vanishes. Applying Theorem 2.5 finishes the proof. $\square$

Remark. Note that in the above proposition there is no restriction on the fundamental group of N. In particular, it need not be simply connected.

We can now give the proof of Theorem 1.2.

Proof of Theorem 1.2. From Corollary 2.3, Proposition 2.4, and Theorem 2.5 we have that $\sigma(X) = 0$. Appealing to the classification theorem 1.1 finishes the proof. $\square$

3. Hyperbolic link complements in $\#_{2k}(S^2 \times S^2)$.

3.1. Preliminaries on Ivanšić's work. In [4] Ivanšić shows that the manifold numbered 1011 in the Ratcliffe-Tschantz census (see p. 123 in [8]) has an orientable double cover that is contained in a smooth manifold homeomorphic to S^4. More precisely, he shows that there exists a closed smooth 4-manifold W_1, homeomorphic to S^4, and a collection of five 2-tori L in W_1 such that the complement $W_1 - L$ is precisely the orientable double cover of the hyperbolic manifold numbered 1011 in the Ratcliffe-Tschantz census (see Theorem. 4.3, p. 18 in [4]).

In the same paper Ivanšić constructs certain degree n cyclic covers of the manifold numbered 1011 in the Ratcliffe-Tschantz census, and shows that they are complements of 2-tori in closed smooth simply connected 4-manifolds with Euler characteristic $2n$. Let M denote the orientable double cover of the manifold numbered 1011 in the Ratcliffe-Tschantz census. This is a finite volume complete hyperbolic 4-manifold with five cusps, each cusp cross-section a 3-torus. His theorem can be expressed in the following way (see Theorem 4.3, p. 18 in [4]).

Theorem 3.1. *The hyperbolic 4-manifold M admits degree n cyclic covers M_n that are complements of $4n + 1$ 2-tori in some closed smooth simply connected 4-manifolds W_n, with Euler characteristic $2n$. In the case that $n = 1$, we have that W_1 is homeomorphic to S^4.*

Our goal is to classify the homeomorphism type of these closed smooth 4-manifolds, in the case that the covering degree is odd (and greater than one). Note that Proposition 2.6 implies that each of the W_n must have vanishing signature. Then Theorem 1.2 implies

that the manifolds W_n must be homeomorphic to $\#_{n-1}S^2 \times S^2$ or $\#_{n-1}(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2)$, for $n \geq 2$. This is because these are the only two groups that have vanishing signature and Euler characteristic greater than two. One key difference between these two groups of manifolds is that those in the first group all admit a spin structure, while those in the second group do not. It is this extra structure that will allow us to show that, in the case that we have an odd covering degree, of degree $2n + 1$ for $n \geq 1$, the manifolds W_{2n+1} must be homeomorphic to $\#_{2n}S^2 \times S^2$.

As we will see, the key idea of the argument for proving that the Ivanšić manifolds are spin is that every branched covering over S^4, above any collection of 3-tori, with odd degree is spin.

3.2. Spin structures on Ivanšić's cyclic covers and the proof of Theorem 1.3.
We start by setting up some of the notation we will be using throughout. As in the previous section we let M denote the orientable double cover of the manifold numbered 1011 in the Ratcliffe-Tschantz census. Let $\overline{M}$ denote the manifold obtained by removing each cusp of M. $\overline{M}$ is a compact 4-manifold with five boundary components, each one given by a 3-torus. We denote these 3-tori boundary components by T_i^3, $1 \leq i \leq 5$. Ivanšić proves the second part of Theorem 3.1 by making a choice of meridian in each boundary component T_i^3. He then glues a solid 3-torus, $S^1 \times S^1 \times D^2$, by a diffeomorphism $f_i : S^1 \times S^1 \times \partial D^2 \to T_i^3$, that sends $\{pt\} \times \{pt\} \times \partial D^2$ to his choice of meridian in T_i^3. The result produces a closed smooth simply connected 4-manifold, W_1, which he then proves is homeomorphic to S^4 (for the details we recommend the reader consult [4]). As we are going to need these meridians, we will denote them by m_i.

Using the notation of Ivanšić, see p. 18 in [4], we have that the meridians m_i correspond to certain parabolic generators in the fundamental group of M, which Ivanšić denotes by $a, k, i, e^{-1}g, c$. The notation for our meridians will correspond to these parabolic generators in the following way, $m_1 = a$, $m_2 = k$, $m_3 = i$, $m_4 = e^{-1}g$, $m_5 = c$.

Also, as in the previous section, we let M_n denote the cyclic cover of M, constructed by Ivanšić. We then define $\overline{M_n}$, as we did for $\overline{M}$, so that $\overline{M_n}$ is an n-fold cover of $\overline{M}$. From Ivanšić's work (see p. 18 in [4]) we know that four of the boundary components in $\overline{M}$ each lift to n boundary components in $\overline{M_n}$, and one boundary component lifts to one boundary 3-torus in $\overline{M_n}$. We will order the boundary components of $\overline{M}$ so that the first four T_i^3, for $1 \leq i \leq 4$, lift to n boundary components, which we denote by T_{ij}^3 for $1 \leq j \leq n$. The fifth boundary component, T_5^3, will be the one that lifts to one boundary component in $\overline{M_n}$, which we denote by T_{5n}^3.

Ivanšić also tells us how the meridians lift (see p. 18 in [4]). For $1 \leq i \leq 4$, the meridian m_i in T_i^3 lifts to a meridian m_{ij} in each T_{ij}^3 for $1 \leq j \leq n$. For the case of T_5^3, the meridian m_5 lifts to one meridian m_{5n} in T_{5n}^3. Viewing T_{5n}^3 as a copy of $S^1 \times S^1 \times S^1$, with the meridian m_{5n} corresponding to the last S^1 factor, and similarly viewing T_5^3 as $S^1 \times S^1 \times S^1$, with the meridian m_5 corresponding to the last S^1 factor. Ivanšić shows that the induced covering $T_{51}^3 \to T_5^3$ is simply the covering $S^1 \times S^1 \times S^1 \to S^1 \times S^1 \times S^1$ induced by the standard degree n covering $S^1 \to S^1$ on the last S^1 factor. In this way we see that the induced covering $T_{5n}^3 \to T_5^3$ is such that the meridian m_{5n} is a degree n cover of m_5.

We also point out to the reader that the manifolds W_n, for $n \geq 2$, are obtained from $\overline{M_n}$ by gluing in solid 3-tori, using the meridians m_{ij} and m_{5n}, in exactly the same way W_1 was obtained from $\overline{M}$.

We are going to analyse spins structures on the manifolds $\overline{M_n}$, and their associated boundary components. By a spin structure, we mean a spin structure on the tangent bundle. We say that a spin manifold X^n spin bounds a spin manifold (of one dimension higher) Y^{n+1}, if X bounds Y, and the spin structure induced on X, via the spin structure on Y (from being a boundary of Y), is the original spin structure on X. We will also need

to know about spin structures on S^1. We remind the reader that S^1 has two distinct spin structures. One of them coming from viewing S^1 as a Lie group, and the other coming from viewing S^1 as the boundary of a disc. The reader who is unfamiliar with this material can consult the book [6].

Lemma 3.2. *There exists a unique spin structure on W_1. This spin structure induces a spin structure on $\overline{M}$ and on the boundary components T_i^3, $1 \le i \le 5$. Furthermore, this induced spin structure on the boundary T_i^3 is the one that spin bounds a solid 3-torus. In fact, the induced spin structure on each T_i^3 is such that the meridian factor m_i is given the spin structure that spin bounds a disc.*

Proof. We saw above that the manifold W_1 is obtained from $\overline{M}$ by gluing in a solid 3-torus to each boundary component, using the meridians m_i, $1 \le i \le 5$. Therefore, inside W_1 each T_i^3 bounds a solid 3-torus, which we denote by DT_i^3, so that the meridian m_i bounds a disc D_i in DT_i^3.

From Theorem 3.1, we know that W_1 is a smooth manifold homeomorphic to S^4. Therefore, it has vanishing second cohomology group. This then implies the second Stiefel-Whitney class of W_1 must vanish. As the set of distinct spin structures is parametrised by $H^1(W_1, \mathbb{Z}_2)$, which vanishes, it follows that W_1 admits a unique spin structure.

As $\overline{M}$ sits inside W_1, it follows that the unique spin structure on W_1 induces a spin structure on $\overline{M}$. Furthermore, the unique spin structure on W_1 also induces a spin structure on each boundary T_i^3 and each solid 3-torus DT_i^3. As each T_i^3 bounds DT_i^3, in W_1, it follows that the induced spin structure on T_i^3 is one that spin bounds the induced spin structure on DT_i^3. In fact, since the meridians m_i bound the discs D_i in DT_i^3, we find that the induced spin structure on each of the m_i is the one that spin bounds a disc (viewing each m_i as a copy of S^1). $\qquad\square$

The spin structure induced on $\overline{M}$, via the unique spin structure on W_1, lifts to a spin structure on the cover $\overline{M}_n$. This spin structure on $\overline{M}_n$ then induces a spin structure on each of its boundary 3-tori T_{ij}^3 for $1 \le i \le 4$, $1 \le j \le n$, and T_{5n}^3. The following lemma examines these induced spin structures on the boundary 3-tori.

In the lemma to follow, by spin structure on $\overline{M}_n$ we mean the one lifted from $\overline{M}$, which in turn will always be the one induced from the unique spin structure on W_1. We will also need to know how spin structures on S^1 lift under the usual degree n covering maps of S^1 onto itself.

Let $\rho_n : S^1 \to S^1$ be the standard degree n covering map. If we give the base S^1 the Lie group spin structure, then the lifted spin structure on the total space S^1 will also be the Lie group spin structure. On the other hand, if we give the base S^1 the spin structure that spin bounds a disc. We find that the lifted spin structure on the total space S^1 will be the one that spins bounds a disc if n is odd, and will be the Lie group one if n is even. We will use this fact in the following lemma.

Lemma 3.3. *The spin structure on $\overline{M}_n$, for $n \ge 2$, induces a spin structure on each T_{ij}^3, for $1 \le i \le 4$ and $1 \le j \le n$, that spin bounds a solid 3-torus. In the case that $n = 2k + 1$ is odd, the spin structure induced on T_{5n} is also the one that spin bounds a solid 3-torus.*

In particular, the induced spin structure on the meridians m_{ij} of T_{ij}^3, for $1 \le i \le 4$ and $1 \le j \le n$, are all the ones that spin bound a disc. Furthermore, when $n = 2k + 1$ the induced spin structure on the meridian m_{5n} is also the one that spin bounds a disc. When $n = 2k$, the induced spin structure on the meridian m_{5n} is the Lie group spin structure.

Proof. Each boundary component T_i^3 of $\overline{M}$, for $1 \le i \le 4$, lifts to n distinct boundary components T_{ij}^3, $1 \le j \le n$, in $\overline{M}_n$. It is therefore clear that since the spin structure on each T_i^3 is the one that spin bounds a solid 3-torus (see lemma 3.2), the induced spin structure on each T_{ij}^3 is the one that also spin bounds a solid 3-torus. In particular, since

each meridian m_i of T_i^3, for $1 \leq i \leq 4$, has the spin structure that spin bounds a disc (see lemma 3.2), and each of these meridians lifts to n meridians m_{ij}, each a meridian of T_{ij}^3, $1 \leq j \leq n$, it follows that the lifted meridians m_{ij} must also inherit the spin structure that spin bounds a disc.

For the analysis of the last boundary component, we know that the induced covering $T_{5n}^3 \to T_5^3$ is given by the degree n covering of the meridian m_{5n} to m_5. We also know, from Lemma 3.2, that the spin structure on m_5 is the one that spin bounds a disc. It follows, from the discussion before the lemma, that for $n = 2k + 1$ the spin structure on m_{5n} spin bounds a disc. Hence, the spin structure on T_{5n}^3 is the one that spin bounds a solid 3-torus. On the other hand, in the case that $n = 2k$ it follows that the spin structure on m_{5n} will be the Lie group spin structure. $\qquad\square$

Remark. When the degree of the covering is even, $n = 2k$, the above lemma shows us that the induced spin structure on the meridian m_{5n}, in the boundary component T_{5n}^3 of $\overline{M}_n$, must be the Lie group spin structure. Therefore, it is natural to ask if the induced spin structure on the whole of T_{5n}^3 is the Lie group spin structure? I.e. viewing T_{5n}^3 as $S^1 \times S^1 \times S^1$, with the last S^1 factor corresponding to the meridian m_{5n}, does it follow that the first two S^1 factors also have the Lie group spin structure? The answer is no, at least one of the first two S^1 factors must have the spin structure that spin bounds a disc.

The following proposition gives details of the above remark, and may be of independent interest.

Proposition 3.4. *Let (X, s) be a non-compact finite volume hyperbolic 4-manifold with a fixed spin structure s. Assume the cusp cross sections of M are all 3-tori. Then the number of cusp cross sections with the induced Lie group spin structure from s must be even.*

Proof. By removing each cusp, we produce a manifold with boundary $\overline{X}$, each boundary component being a 3-torus. X deformation retracts onto $\overline{X}$, so we can view the spin structure s on $\overline{X}$. Now, the induced spin structures on the boundary 3-tori are either one of the seven spin structures that spin bounds a solid 3- torus, or the Lie group spin structure.

By Corollary 2.3 we know that $\overline{X}$ has vanishing signature. Also, by Corollary 2.4, we have that a solid 3-torus has vanishing signature. We now take those boundary components of $\overline{X}$ that have the induced spin structure that spin bounds a solid 3-torus. Then glue in a solid 3-torus to each of these components. We then have a spin manifold X_0 whose boundary components are all 3-tori, and such that the induced spin structure on these boundary components is the Lie group spin structure. Furthermore, by Novikov's theorem 2.5, X_0 has vanishing signature.

Let us suppose that the number of boundary components with the induced Lie group spin structure is odd. In other words, X_0 has an odd number of boundary components. Let Y denote the manifold obtained by cutting out the interior of a tubular neighbourhood of a generic fibre of the elliptic surface $\mathbb{CP}^2 \#_9 \overline{\mathbb{CP}}^2$. It is known that Y is a spin manifold, with one boundary component a 3-torus, such that the induced spin structure on this boundary 3-torus is the Lie group one. Furthermore, it is known that Y has signature 8 (see [9], chap. V).

We then glue a copy of Y to each component of X_0. This produces a closed smooth spin 4-manifold Z with signature an odd multiple of 8. However, no such manifold can exist by Rokhlin's theorem (see [6], chap. III). $\qquad\square$

Using Lemma 3.3 we can detect the homeomorphism type of the manifolds W_{2k+1}, for $k > 0$, leading to a proof of Theorem 1.3.

Proof of Theorem 1.3. By Proposition 2.6, we know that each W_{2k+1} has vanishing signature. Therefore, by Theorem 1.2, we have that the homeomorphism type of W_{2k+1} must be $\#_{2k}S^2 \times S^2$ or $\#_{2k}(\mathbb{CP}^2 \# \overline{CP}^2)$, as these are the only ones with vanishing signature.

By Lemma 3.3, we know that the manifolds $\overline{M}_{2k+1}$ are spin, and they admit a spin structure that induces the spin structure on the boundary 3-tori, which spin bounds a solid 3-torus. We also know that the meridians of each boundary 3-torus has the spin structure that spin bounds a disc.

When we glue in a solid 3-torus, $S^1 \times S^1 \times D^2$, we are using the diffeomorphism that identifies $\{pt\} \times \{pt\} \times \partial D^2$ with the meridian of the boundary 3-torus. Taking the spin structure on $S^1 \times S^1 \times D^2$ that agrees with the spin structure on the boundary 3-torus of $\overline{M}_{2k+1}$, we see that the spin structure on $\overline{M}_{2k+1}$ extends to each glued-in solid 3-torus. This implies that W_{2k+1} admits a spin structure. It then follows that W_{2k+1} must be homeomorphic to $\#_{2k}S^2 \times S^2$ for $k \geq 1$, as these are the only ones in Theorem 1.1 that have vanishing signature and are spin.

Applying this result, with Ivanšić's theorem 3.1, completes the proof. $\qquad\square$

In [5], Ivanšić, Ratcliffe, and Tschantz construct several more examples of hyperbolic link complements, in closed smooth simply connected 4-manifolds with Euler characteristic $2n$, based on the approach of Ivanšić in [4]. Using the same techniques as above, one can show that, in the case that n is odd, their examples give rise to hyperbolic link complements in manifolds that are homeomorphic to $\#_{n-1}S^2 \times S^2$, $n > 1$.

References

[1] DONALDSON, SIMON K. An application of gauge theory to four-dimensional topology. *J. Differential Geom.* **18** (1983), no. 2, 279–315. MR0710056, Zbl 0507.57010, doi: 10.4310/jdg/1214437665.

[2] FREEDMAN, MICHAEL HARTLEY. The topology of four-dimensional manifolds. *J. Differential Geom.* **17** (1982), no. 3, 357–453. MR0679066, Zbl 0528.57011, doi: 10.4310/jdg/1214437136.

[3] HANTZSCHE, WALTER; WENDT, HILMAR. Dreidimensionale euklidische Raumformen. *Math. Ann.* **110** (1935), no. 1, 593–611. MR1512956, Zbl 0010.18003, doi: 10.1007/BF01448045. 445

[4] IVANŠIĆ, DUBRAVKO. Hyperbolic structure on a complement of tori in the 4-sphere. *Adv. Geom.* **4** (2004), no. 1, 119–139. MR2155369, Zbl 1039.57012, doi: 10.1515/advg.2004.002. 444, 446, 447, 450

[5] IVANŠIĆ, DUBRAVKO; RATCLIFFE, JOHN G.; TSCHANTZ, STEVEN T. Complements of tori and Klein bottles in the 4-sphere that have hyperbolic structure. *Algebr. Geom. Topol.* **5** (2005), 999–1026. MR2171801, Zbl 1085.57013, arXiv:math/0502293, doi: 10.2140/agt.2005.5.999. 444, 450

[6] KIRBY, ROBION C. The topology of 4-manifolds. Lecture Notes in Mathematics, 1374. *Springer-Verlag, Berlin*, 1989. vi+108 pp. ISBN: 3-540-51148-2. MR1001966, Zbl 0668.57001, doi: 10.1007/BFb0089031. 446, 448, 449

[7] LONG, DARREN D.; REID, ALAN W. On the geometric boundaries of hyperbolic 4-manifolds. *Geom. Topol.* **4** (2000), 171–178. MR1769269, Zbl 0961.57011, arXiv:math/0007197, doi: 10.2140/gt.2000.4.171. 444

[8] RATCLIFFE, JOHN G.; TSCHANTZ, STEVEN T. The volume spectrum of hyperbolic 4-manifolds. *Experiment. Math.* **9** (2000), no. 1, 101–125. MR1758804, Zbl 0963.57012, doi: 10.1080/10586458.2000.10504640. 446

[9] SCORPAN, ALEXANDRU. The wild world of 4-manifolds. *American Mathemtical Society, Providence, RI*, 2005. xx+609 pp. ISBN: 0-8218-3749-4. MR2136212, Zbl 1075.57001. 443, 449

[10] SZCZEPAŃSKI, ANDRZEJ. Eta invariants for flat manifolds. *Ann. Global Anal. Geom.* **41** (2012), 125–138. MR2876691, Zbl 1237.58026, arXiv:1001.1270, doi: 10.1007/s10455-011-9274-0. 445

[11] WOLF, JOSEPH A. Spaces of constant curvature. *McGraw-Hill Book Co., New York-London-Sydney*, 1967. xv+408 pp. MR0217740, Zbl 0162.53304. 445

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