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Boundedness of Littlewood-Paley g-functions on non-homogeneous metric measure spaces

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Abstract. In this paper, we establish the boundedness of Littlewood-Paley g-functions on Lebesgue spaces, BMO-type spaces, and Hardy spaces over non-homogeneous metric measure spaces satisfying the weak reverse doubling condition.

Contents

1. Introduction

It is well known that the Littlewood-Paley theory plays an important role in harmonic analysis. It was first introduced by Littlewood and Paley [LP31, LP37II, LP37III] just for the one-dimensional case. In 1958, using real variable methods, Stein [S58] extended the theory to high-dimensional cases. From then on, the Littlewood-Paley theory drew wide concern in the field of analysis.

Many results, including the Littlewood-Paley theory, on the classical Euclidean space can be extended to the space of homogeneous type, which is generally regarded as a natural setting for singular integrals and function

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spaces. We call $(\mathcal{X}, d, \mu)$ a *space of homogeneous type* in the sense of Coifman and Weiss [CW71], if $(\mathcal{X}, d)$ is a metric space and μ is a non-negative Borel measure satisfying the *measure doubling condition*: there exists a positive constant $C_{(\mu)}$ such that, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, 2r)) \le C_{(\mu)}\mu(B(x, r)), \tag{1.1}$$

where $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$. For spaces of homogenous type with the additional property that a reverse doubling property holds, Han, Müller and Yang [HMY06] developed a Littlewood-Paley theory for atomic Hardy spaces, where a continuous version of the Littlewood-Paley g-function was used.

On the other hand, many results were proved to remain valid in other settings as well, for instance, $(\mathbb{R}^n, |\cdot|, \mu)$, the Euclid space with non-doubling measure. Recall that a non-negative Radon measure μ on $\mathbb{R}^n$ is called a *non-doubling measure*, if μ satisfies the *polynomial growth condition*: there exist some positive constants C_0 and $\kappa \in (0, n]$ such that, for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le C_0 r^\kappa, \tag{1.2}$$

where $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$. The measure as in (1.2) may not satisfy the doubling condition (1.1). The analysis on such non-doubling context plays a striking role in solving several long-standing problems related to the analytic capacity, like Vitushkin's conjecture or Painlevé's problem; see [T03]. Moreover, Tolsa [T01am] developed some Littlewood-Paley theory in this setting.

Recently, in [Hy10], Hytönen pointed out that the measure μ satisfying the polynomial growth condition is different from, not general than, the doubling measure. In other words, there exists no inevitable inclusion relation between the spaces of homogeneous type and the metric measure spaces with non-doubling measure. To unify these two spaces, Hytönen [Hy10] introduced the so-called non-homogeneous metric measure spaces satisfying both the upper doubling and the geometrically doubling condition (see, respectively, Definitions 1.1 and 1.2 below). We mention that several equivalent characterizations for the upper doubling condition were recently established by Tan and Li [TL15, TL17] and the so-called Bergman-type operator appearing in [VW12] can be seen as the Calderón-Zygmund operator in this new setting; see also [HM12] for an explanation. Furthermore, plenty of theoretical achievements, including some Littlewood-Paley theory, in this new context sprang up soon after 2010; see [LY11, HYY12, FYY12, BD13, LY14, FYY14, FLYY15, TL15] for more information. Very recently, Fu and Zhao [FZ16] obtained some endpoint estimates for the discrete version of Littlewood-Paley g-function. We refer the reader to the survey [YYF13] and the monograph [YYH13] for more developments on harmonic analysis in this setting.

The main purpose of this article is to establish the boundedness of the continuous version of Littlewood-Paley g-function on several function spaces over non-homogeneous metric measure spaces.

Definition 1.1. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling*, if μ is a Borel measure on $\mathcal{X}$ and there exist a *dominating function* $\lambda : \mathcal{X} \times (0, \infty) \to (0, \infty)$ and a positive constant $C_{(\lambda)}$, depending on λ, such that, for each $x \in \mathcal{X}$, $r \to \lambda(x, r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0, \infty)$,

$$\mu(B(x, r)) \le \lambda(x, r) \le C_{(\lambda)}\lambda(x, r/2). \tag{1.3}$$

Remark 1.1.

(i) Evidently, if a measure μ satisfies the measure doubling condition (1.1) or the polynomial growth condition (1.2), then it has the upper doubling property (1.3). In the former case, we take the dominating function $\lambda(x, r) := \mu(B(x, r))$ for all $x \in \mathcal{X}$ and $r \in (0, \infty)$; in the latter one, we take $\lambda(x, r) := C_0 r^\kappa$ for all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$.

(ii) For $(\mathcal{X}, d, \mu)$ and λ as in Definition 1.1, it was proved in [Hy10] that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \le \lambda$, $C_{(\widetilde{\lambda})} \le C_{(\lambda)}$ and, for all $x, y \in \mathcal{X}$ with $d(x, y) \le r$,

$$\widetilde{\lambda}(x, r) \le C_{(\widetilde{\lambda})}\widetilde{\lambda}(y, r). \tag{1.4}$$

The following notion of geometrically doubling can be found in [CW71, pp.66-67] and is also known as *metrically doubling* (see [He01, p.81]).

Definition 1.2. A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling*, if there exists some $N_0 \in \mathbb{N}^+ := \{1, 2, \ldots\}$ such that, for any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

What might also be noted is that spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss in [CW71, pp.66-68].

A metric measure space $(\mathcal{X}, d, \mu)$ is called a *non-homogeneous metric measure space*, if it is upper doubling and $(\mathcal{X}, d)$ is geometrically doubling. Based on Remark 1.1(ii), through the whole article, we *always assume* that $(\mathcal{X}, d, \mu)$ is a non-homogeneous metric measure space with the dominating function λ satisfying (1.4).

Now, we introduce the continuous version of Littlewood-Paley g-function on $(\mathcal{X}, d, \mu)$.

Definition 1.3. Let $\epsilon_1 \in (0, 1]$, $\epsilon_2 \in (0, \infty)$ and λ be a dominating function. The *kernel* $D_t(x, y)$ with $t \in (0, \infty)$ is a measurable function from $\mathcal{X} \times \mathcal{X}$ to $\mathbb{C}$ that satisfies the following estimates: there exists a positive constant C such that, for all $t \in (0, \infty)$ and $x, x', y \in \mathcal{X}$ with $d(x, x') \le (t + d(x, y))/2$, (A1) $|D_t(x,y)| \le C\frac{1}{\lambda(x,t)+\lambda(y,t)+\lambda(x,d(x,y))}\left[\frac{t}{t+d(x,y)}\right]^{\epsilon_2}$;

(A2)

$$\left|D_t(x,y)-D_t(x',y)\right| \le C\frac{1}{\lambda(x,t)+\lambda(y,t)+\lambda(x,d(x,y))}\left[\frac{t}{t+d(x,y)}\right]^{\epsilon_2} \times \left[\frac{d(x,x')}{t+d(x,y)}\right]^{\epsilon_1};$$

(A3) Property (A2) also holds with the roles of x and y interchanged;

(A4) $\int_{\mathcal{X}} D_t(x,y)d\mu(x) = 0 = \int_{\mathcal{X}} D_t(x,y)d\mu(y)$.

The *Littlewood-Paley g-function* $g(f)$ associated with $D_t(x,y)$ is defined by setting, for all suitable f and $x\in\mathcal{X}$,

$$g(f)(x) := \left\{\int_0^\infty \left|\int_{\mathcal{X}} D_t(x,y)f(y)d\mu(y)\right|^2 \frac{dt}{t}\right\}^{1/2}. \tag{1.5}$$

In the space of homogeneous type, if we take $\lambda(x,t)=\mu(B(x,t))$, then $g(f)$ as in (1.5) is just the Littlewood-Paley g-function introduced by Han et al. [HMY06]. To establish the boundedness of the operator g, throughout this paper, we always assume that g is bounded on $L^2(\mu)$ and the dominating function λ as in Definition 1.1 satisfies the following *weak reverse doubling condition* introduced by Fu et al. [FYY14]. In what follows, let $\operatorname{diam}(\mathcal{X}) := \sup_{x,y\in\mathcal{X}} d(x,y)$.

Definition 1.4. The dominating function λ as in Definition 1.1 is said to satisfy the *weak reverse doubling condition* if, for all $r\in(0,2\operatorname{diam}(\mathcal{X}))$ and $a\in(1,2\operatorname{diam}(\mathcal{X})/r)$, there exists a constant $C(a)\in[1,\infty)$, depending only on a and $\mathcal{X}$, such that, for all $x\in\mathcal{X}$,

$$\lambda(x,ar)\ge C(a)\lambda(x,r), \tag{1.6}$$

$$\sum_{k=1}^\infty \frac{1}{C(a^k)} < \infty. \tag{1.7}$$

The organization of this paper is as follows. Section 2 is devoted to recalling the notions of the (α,β)-doubling ball and the discrete coefficient $\widetilde{K}_{B,S}^{(\rho),p}$. Moreover, we establish some estimates for the Littlewood-Paley g-function $g(f)$, which will be used in the next sections. In section 3, by using the Calderón-Zygmund decomposition, we prove the boundedness of g from $L^1(\mu)$ into $L^{1,\infty}(\mu)$ (see Theorem 3.1 below). In section 4, we show that g is bounded from the space $\widetilde{\mathrm{RBMO}}(\mu)$ into the space $\widetilde{\mathrm{RBLO}}(\mu)$ (see Theorem 4.1 below). To this end, we establish a new characterization of the space $\widetilde{\mathrm{RBLO}}(\mu)$ (see Lemma 4.5 below), which is of independent interest. In section 5, via the boundedness criteria proved in [LL18], we establish the boundedness of g on the Hardy spaces H^p with $p\in(0,1]$ (see Theorems 5.1

and 5.2 and Corollary 5.2 below). The proof of the case of $p = 1$ is standard, and we borrow some ideas from the proof of [FLYY15, Theorem 4.8] to deal with the case of $p \in (0, 1)$. As a corollary, we obtain the boundedness of g on $L^q(\mu)$ with $q \in (1, \infty)$.

For convenience, we make some conventions on notation. Throughout this paper, C stands for a *positive constant* independent of the main parameters, but they may vary with different contexts. Moreover, *constants with subscripts* also denote positive constants. Concretely, constant like $C_{(\alpha)}$ depends on the parameter α; constant like C_0 does not change in different occurrences. For two real-valued functions f and g, we write $f \lesssim g$, if $f \le Cg$; we write $f \sim g$, if $f \lesssim g \lesssim f$. Given any $q \in (0, \infty)$, $q' := q/(q - 1)$ means its *conjugate index*. For any subset $E \subset \mathcal{X}$, χ_E denotes its *characteristic function*. A ball $B := B(x_B, r_B) \subset \mathcal{X}$ has positive and finite measure, where $x_B \in \mathcal{X}$ and $r_B \in (0, \infty)$ denote its *center* and *radius*, respectively. Furthermore, for any $\tau \in (0, \infty)$, $\tau B := B(x_B, \tau r_B)$. Finally, we write $\mathbb{N}^+ := \{1, 2, 3...\}$, $\mathbb{N} := \mathbb{N}^+ \cup \{0\}$, $\nu := \log_2 C_{(\lambda)}$ with $C_{(\lambda)}$ as in Definition 1.1 and $n_0 := \log_2 N_0$ with N_0 as in Definition 1.2.

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2. Preliminaries

In this section, we first recall some necessary notions and notation. Although the assumption concerning the measure doubling condition (1.1) do not strictly suit all balls in the non-homogeneous metric measure space $(\mathcal{X}, d, \mu)$, there still exist lots of balls having the following (α, β)-doubling property introduced in [Hy10].

Definition 2.1. Let $\alpha, \beta \in (1, \infty)$. The ball $B \subset \mathcal{X}$ is said to be (α, β)-*doubling*, if $\mu(\alpha B) \le \beta \mu(B)$.

Remark 2.1. The following statements were proved by Hytönen in [Hy10, Lemma 3.3].

(i) Let $(\mathcal{X}, d, \mu)$ be upper doubling with $\beta > \alpha^\nu$. Then, for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{N}$ such that $\alpha^j B$ is (α, β)-doubling.

(ii) Let $(\mathcal{X}, d)$ be a geometrically doubling space equipped with a nonnegative Borel measure μ which is finite on all bounded sets. Let $\beta > \alpha^{n_0}$. Then, for μ-almost every $x \in \mathcal{X}$, there exist arbitrary small (α, β)-doubling balls centered at x. Furthermore, the radii of these balls may be chosen to be of the form $\alpha^{-j} r$ for $j \in \mathbb{N}^+$ and any preassigned number $r \in (0, \infty)$.

In what follows, for any $\alpha \in (1, \infty)$ and ball B, $\widetilde{B}^{\alpha}$ denotes the *smallest* (α, β_α)*-doubling ball of the form* $\alpha^j B$ *with* $j \in \mathbb{N}$, where

$$\beta_\alpha := \alpha^{3(\max\{n_0, \nu\})} + (\max\{5\alpha, 30\})^{n_0} + (\max\{3\alpha, 30\})^{\nu}. \tag{2.1}$$

In particularly, for any ball $B \subset \mathcal{X}$, we use $\widetilde{B}$ to denote the smallest $(6, \beta_6)$-doubling ball of the form $6^j B$ with $j \in \mathbb{N}$.

Now we recall the definition of the discrete coefficient $\widetilde{K}_{B,S}^{(\rho),p}$ introduced by Bui and Duong in [BD13] when $p = 1$ and by Fu et. al in [FLYY15] when $p \in (0, 1]$. Before this, we first give an assumption: when we speak of a ball B in $(\mathcal{X}, d, \mu)$, it is understood that it comes with a fixed center and radius, although these in general are not uniquely determined by B as a set; see [He01, pp.1-2]. In other words, for any two balls $B, S \subset \mathcal{X}$, if $B = S$, then $x_B = x_S$ and $r_B = r_S$. Thus, if $B \subset S \subset \mathcal{X}$, then $r_B \leq 2r_S$, which guarantees the definition of $\widetilde{K}_{B,S}^{(\rho),p}$ make sense (see [FLYY15] for more details).

Definition 2.2. For any $\rho \in (1, \infty)$, $p \in (0, 1]$ and any two balls $B \subset S \subset \mathcal{X}$, let

$$\widetilde{K}_{B,S}^{(\rho),p} := \left\{ 1 + \sum_{k=-\lfloor \log_\rho 2 \rfloor}^{N_{B,S}^{(\rho)}} \left[\frac{\mu(\rho^k B)}{\lambda(x_B, \rho^k r_B)} \right]^p \right\}^{1/p},$$

here and hereafter, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ represents the *biggest integer which is not bigger than* a, and $N_{B,S}^{(\rho)}$ is the *smallest integer* satisfying $\rho^{N_{B,S}^{(\rho)}} r_B \geq r_S$.

Remark 2.2.

(i) We simply denote $\widetilde{K}_{B,S}^{(\rho),1}$ by $\widetilde{K}_{B,S}^{(\rho)}$. It is easy to see that

$$\widetilde{K}_{B,S}^{(\rho)} \sim 1 + \sum_{k=1}^{N_{B,S}^{(\rho)} + \lfloor \log_\rho 2 \rfloor + 1} \frac{\mu(\rho^k B)}{\lambda(x_B, \rho^k r_B)}.$$

(ii) The following coefficient $K_{B,S}$, introduced by Hytönen in [Hy10], can be deemed to be the continuous version of the discrete coefficient $\widetilde{K}_{B,S}^{(\rho)}$.

$$K_{B,S} := 1 + \int_{(2S) \setminus B} \frac{d\mu(x)}{\lambda(x_B, d(x, x_B))}$$

Obviously, $K_{B,S} \lesssim \widetilde{K}_{B,S}^{(\rho)}$. However, it is unclear whether $K_{B,S} \sim \widetilde{K}_{B,S}^{(\rho)}$. In particular, for $(\mathbb{R}^n, |\cdot|, \mu)$ with μ as in (1.2), $K_{B,S} \sim \widetilde{K}_{B,S}^{(\rho)}$. Moreover, if the dominating function λ satisfies the weak reverse doubling condition, then $K_{B,S} \sim \widetilde{K}_{B,S}^{(\rho)}$; see [FYY14].

The following properties of $\widetilde{K}_{B,S}^{(\rho),p}$ were proved in [FLYY15].

Lemma 2.1. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space, $p \in (0, 1]$ and $\rho \in (1, \infty)$.*

(i) *For all balls $B \subset R \subset S$,*

$$[\widetilde{K}_{B,R}^{(\rho),p}]^p \leq C_{(\rho)}[\widetilde{K}_{B,S}^{(\rho),p}]^p, \quad [\widetilde{K}_{R,S}^{(\rho),p}]^p \leq \widetilde{c}_{(\rho,p,\nu)}[\widetilde{K}_{B,S}^{(\rho),p}]^p$$

and

$$[\widetilde{K}_{B,S}^{(\rho),p}]^p \leq [\widetilde{K}_{B,R}^{(\rho),p}]^p + c_{(\rho,p,\nu)}[\widetilde{K}_{R,S}^{(\rho),p}]^p,$$

where $C_{(\rho)}$ is a positive constant depending on ρ, $c_{(\rho,p,\nu)}$ and $\widetilde{c}_{(\rho,p,\nu)}$ are positive constants depending on ρ, p and ν.

(ii) *Let $\alpha \in [1, \infty)$. For all balls $B \subset S$ with $r_S \leq \alpha r_B$, $[\widetilde{K}_{B,S}^{(\rho),p}]^p \leq C_{(\alpha,\rho)}$, where $C_{(\alpha,\rho)}$ is a positive constant depending on α and ρ.*

(iii) *There exists a positive constant $C_{(\rho,\nu)}$, depending on ρ and ν, such that, for all balls B, $\widetilde{K}_{B,\widetilde{B}^\rho}^{(\rho),p} \leq C_{(\rho,\nu)}$. Moreover, letting $\alpha, \beta \in (1, \infty)$, $B \subset S$ be any two concentric balls such that there exists no (α, β)-doubling ball in the form of $\alpha^k B$ with $k \in \mathbb{N}$, satisfying $B \subset \alpha^k B \subset S$, then there exists a positive constant $C_{(\alpha,\beta,\nu)}$, depending on α, β and ν, such that $\widetilde{K}_{B,S}^{(\rho),p} \leq C_{(\alpha,\beta,\nu)}$.*

(iv) *For any $\rho_1, \rho_2 \in (1, \infty)$, there exist positive constants $c_{(\rho_1,\rho_2,\nu)}$ and $C_{(\rho_1,\rho_2,\nu)}$, depending on ρ_1, ρ_2 and ν, such that, for all balls $B \subset S$,*

$$c_{(\rho_1,\rho_2,\nu)}\widetilde{K}_{B,S}^{(\rho_1),p} \leq \widetilde{K}_{B,S}^{(\rho_2),p} \leq C_{(\rho_1,\rho_2,\nu)}\widetilde{K}_{B,S}^{(\rho_1),p}.$$

At the end of this section, we present the following lemma which will be used frequently in the rest of this paper.

Lemma 2.2. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space, and g be as in Definition 1.3. Assume that $f \in L^1_{\rm loc}(\mu)$ and there exists a ball $B \subset \mathcal{X}$ such that* $\operatorname{supp}(f) \subset B$. *For any $x \notin 2B$,*

(i) *if f has the vanishing moment, that is, $\int_{\mathcal{X}} f(y)d\mu(y) = 0$, then*

$$g(f)(x) \lesssim \int_B \frac{|f(y)|}{\lambda(x, d(x, y)}\left[\frac{r_B}{d(x, y)}\right]^{\epsilon_1} d\mu(y), \tag{2.2}$$

where $\epsilon_1 \in (0, 1]$ is as in Definition 1.3;

(ii) *if λ satisfies the weak reverse doubling condition, then*

$$g(f)(x) \lesssim \int_B \frac{|f(y)|}{\lambda(x, d(x, y))} d\mu(y). \tag{2.3}$$

Proof. To prove (i), we use the regular conditions of $D_t(x, y)$ in Definition 1.3 (A2) and (A3). This, together with the assumption that $\displaystyle\int_{\mathcal{X}} f(y)d\mu(y) = 0$ and the Minkowski inequality, shows that

$$g(f)(x) = \left\{\int_0^\infty \left|\int_B [D_t(x, y) - D_t(x, x_B)]f(y)d\mu(y)\right|^2 \frac{dt}{t}\right\}^{1/2}$$

$$\begin{aligned}
&\leq \int_B \left\{ \int_0^\infty |D_t(x,y) - D_t(x,x_B)|^2 \frac{dt}{t} \right\}^{1/2} |f(y)| d\mu(y) \\
&\lesssim \int_B \left\{ \int_0^{d(x,y)} \left| \frac{1}{\lambda(x,d(x,y))} \left[\frac{t}{d(x,y)} \right]^{\epsilon_2} \left[\frac{d(x_B,y)}{d(x,y)} \right]^{\epsilon_1} \right|^2 \frac{dt}{t} \right\}^{1/2} \\
&\quad \times |f(y)| d\mu(y) \\
&\quad + \int_B \left\{ \int_{d(x,y)}^\infty \left| \frac{1}{\lambda(x,d(x,y))} \left[\frac{d(x_B,y)}{t} \right]^{\epsilon_1} \right|^2 \frac{dt}{t} \right\}^{1/2} |f(y)| d\mu(y) \\
&\leq \int_B |f(y)| \frac{(r_B)^{\epsilon_1}}{\lambda(x,d(x,y))[d(x,y)]^{\epsilon_1+\epsilon_2}} \left[\int_0^{d(x,y)} t^{2\epsilon_2 - 1} dt \right]^{1/2} d\mu(y) \\
&\quad + \int_B |f(y)| \frac{(r_B)^{\epsilon_1}}{\lambda(x,d(x,y))} \left[\int_{d(x,y)}^\infty t^{-2\epsilon_1 - 1} dt \right]^{1/2} d\mu(y) \\
&\lesssim \int_B \frac{|f(y)|}{\lambda(x,d(x,y))} \left[\frac{r_B}{d(x,y)} \right]^{\epsilon_1} d\mu(y).
\end{aligned}$$

To prove (ii), we use the size condition of $D_t(x,y)$ in Definition 1.3 (A1). From this, the Minkowski inequality, (1.6) and (1.7), we deduce that

$$\begin{aligned}
g(f)(x) &= \left\{ \int_0^\infty \left| \int_B D_t(x,y) f(y) d\mu(y) \right|^2 \frac{dt}{t} \right\}^{1/2} \\
&\leq \int_B \left\{ \int_0^\infty |D_t(x,y)|^2 \frac{dt}{t} \right\}^{1/2} |f(y)| d\mu(y) \\
&\lesssim \int_B \left\{ \int_0^{d(x,y)} \left| \frac{1}{\lambda(x,d(x,y))} \left[\frac{t}{d(x,y)} \right]^{\epsilon_2} \right|^2 \frac{dt}{t} \right\}^{1/2} |f(y)| d\mu(y) \\
&\quad + \int_B \left\{ \int_{d(x,y)}^\infty \left| \frac{1}{\lambda(x,t)} \right|^2 \frac{dt}{t} \right\}^{1/2} |f(y)| d\mu(y) \\
&= \int_B \frac{1}{\lambda(x,d(x,y))(d(x,y))^{\epsilon_2}} \left[\int_0^{d(x,y)} t^{2\epsilon_2 - 1} \frac{dt}{t} \right]^{1/2} |f(y)| d\mu(y) \\
&\quad + \int_B \left\{ \int_{d(x,y)}^\infty \frac{dt}{[\lambda(x,t)]^2 t} \right\}^{1/2} |f(y)| d\mu(y) \\
&\lesssim \int_B \frac{|f(y)|}{\lambda(x,d(x,y))} d\mu(y) \\
&\quad + \int_B \left\{ \sum_{n=0}^\infty \int_{2^n d(x,y)}^{2^{n+1} d(x,y)} \frac{dt}{[\lambda(x,t)]^2 t} \right\}^{1/2} |f(y)| d\mu(y)
\end{aligned}$$

$$\lesssim \int_B \frac{|f(y)|}{\lambda(x,d(x,y))}d\mu(y) + \sum_{n=0}^{\infty} \frac{1}{C(2^n)} \int_B \frac{|f(y)|}{\lambda(x,d(x,y))}d\mu(y)$$
$$\lesssim \int_B \frac{|f(y)|}{\lambda(x,d(x,y))}d\mu(y),$$

where $C(1) = 1$. This completes the proof of Lemma 2.2. □

3. Boundedness of g from $L^1(\mu)$ into $L^{1,\infty}(\mu)$

Theorem 3.1. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space and g be as in Definition 1.3. Assume that the dominating function λ satisfies the weak reverse doubling condition. If g is bounded on $L^2(\mu)$, then g is bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$.*

In order to prove Theorem 3.1, we first present the Calderón-Zygmund decomposition from [BD13].

Lemma 3.1. *Let $f \in L^1(\mu)$ and $\ell \in (0,\infty)$ ($\ell > \ell_0 := \gamma_0[\mu(\mathcal{X})]^{-1}\|f\|_{L^1(\mu)}$ if $\mu(\mathcal{X}) < \infty$, where γ_0 is any fixed positive constant satisfying that $\gamma_0 > max\{C_{(\lambda)}^{3\log_2 6}, 6^{3n}\}$, $C_{(\lambda)}$ is as in (1.3)). Then*

(i) *there exists an almost disjoint family $\{6B_j\}_j$ of balls such that $\{B_j\}_j$ is pairwise disjoint,*

$$\frac{1}{\mu(6^2B_j)}\int_{B_j}|f(x)|d\mu(x) > \frac{\ell}{\gamma_0} \quad \text{for all } j,$$

$$\frac{1}{\mu(6^2\eta B_j)}\int_{\eta B_j}|f(x)|d\mu(x) \le \frac{\ell}{\gamma_0} \quad \text{for all } j \text{ and all } \eta\in(2,\infty),$$

and

$$|f(x)| \le \ell \quad \text{for } \mu - \text{almost every } x \in \mathcal{X}\setminus(\bigcup_j 6B_j);$$

(ii) *for each j, let S_j be a $(3\times 6^2, C_{(\lambda)}^{\log_2(3\times 6^2)+1})$-doubling ball of the family $\{(3\times 6^2)^k B_j\}_{k\in\mathbb{N}^+}$ and $\omega_j := \chi_{6B_j}/(\sum_k \chi_{6B_k})$. Then, there exists a family $\{\varphi_j\}_j$ of functions such that, for each j, $\operatorname{supp}(\varphi_j)\subset S_j$, φ_j has a constant sign on S_j,*

$$\int_{\mathcal{X}} \varphi_j(x)d\mu(x) = \int_{6B_j} f(x)\omega_j(x)d\mu(x),$$

$$\sum_j |\varphi_j(x)| \le \gamma\ell \quad \text{for } \mu - \text{almost every } x\in\mathcal{X},$$

where γ is some positive constant, depending only on $(\mathcal{X},\mu)$, and there exists a positive constant C, independent of f, ℓ and j, such that, it holds true that

$$\|\varphi_j\|_{L^\infty(\mu)}\mu(S_j) \le C\int_{\mathcal{X}} |f(x)\omega_j(x)|d\mu(x)$$

Proof of Theorem 3.1. Let $f \in L^1(\mu)$ and $\ell \in (0,\infty)$. To obtain the desired conclusion, we only need to prove that

$$\mu(\{x\in\mathcal{X} : g(f)(x) > 2\ell\}) \lesssim \frac{1}{\ell}\|f\|_{L^1(\mu)}. \tag{3.1}$$

Let γ_0 be a positive constant as in Lemma 3.1. Apparently (3.1) holds true when $\mu(\mathcal{X}) < \infty$ and $\ell \in (0, \gamma_0\|f\|_{L^1(\mu)}[\mu(\mathcal{X})]^{-1}]$.

For other cases, we apply Calderón-Zygmund decomposition to $|f|$ at the level ℓ with the same notation as in Lemma 3.1. Let $\mathcal{F} := \mathcal{X}\backslash(\bigcup_j 6^2B_j)$. Decompose f as $f = a + b$, where

$$a := \chi_{\mathcal{F}} f + \sum_j \varphi_j \quad \text{and} \quad b := \sum_j b_j := \sum_j (\omega_j f - \varphi_j).$$

Now, we can transform the problem of proving (3.1) into certifying that

$$\mu(\{x\in\mathcal{X} : g(a)(x) > \ell\}) \lesssim \frac{1}{\ell}\|f\|_{L^1(\mu)} \tag{3.2}$$

and

$$\mu(\{x\in\mathcal{X} : g(b)(x) > \ell\}) \lesssim \frac{1}{\ell}\|f\|_{L^1(\mu)}. \tag{3.3}$$

From Lemma 3.1, it is easy to see that $\|a\|_{L^\infty(\mu)} \lesssim \ell$ and $\|a\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)}$. This, together with the $L^2(\mu)$-boundedness of g, enables us to derive (3.2).

On the other hand, it follows from Lemma 3.1(i) that

$$\mu\left(\bigcup_j 6^2B_j\right) \lesssim \frac{1}{\ell}\|f\|_{L^1(\mu)}.$$

Thus, to prove (3.3), we are only required to prove that

$$\mu(\{x\in\mathcal{F} : g(b) > \ell\}) \lesssim \frac{1}{\ell}\|f\|_{L^1(\mu)}. \tag{3.4}$$

Since g is non-negative and sublinear, we have

$$\begin{aligned}
&\mu(\{x\in\mathcal{F} : g(b)(x) > \ell\}) \\
&\quad\le \frac{1}{\ell}\int_{\mathcal{F}} g\left(\sum_j b_j\right)(x)d\mu(x) \\
&\quad\le \frac{1}{\ell}\sum_j\left[\int_{\mathcal{X}\backslash(2S_j)} g(b_j)(x)d\mu(x) + \int_{(2S_j)\backslash(6^2Bj)} g(b_j)(x)d\mu(x)\right] \\
&\quad=: \frac{1}{\ell}\sum_j(\mathrm{H}_{j,1} + \mathrm{H}_{j,2}).
\end{aligned}$$

We first give the conclusion as below, which will be repeatedly used afterward. When $x \notin 2B$ and $y \in B$, $d(x,y) \sim d(x,x_B)$. This, together with (1.1) and Remark 1.1(ii), implies that, for any $x \notin 2B$ and $y \in \operatorname{supp}(f) \subset B$,

$$\lambda(x,d(x,y)) \sim \lambda(y,d(x,y)) \sim \lambda(y,d(x,x_B)) \sim \lambda(x_B,d(x,x_B)). \tag{3.5}$$

By Lemma 3.1, we see that $\int_{\mathcal{X}} b_j(y)\,d\mu(y) = 0$ and $\operatorname{supp}(b_j) \subset S_j$. From this, together with (2.2), (3.5), (1.3) and Lemma 3.1(ii), we deduce that

$$\begin{aligned}
\mathrm{H}_{j,1} &\lesssim \int_{S_j} |b_j(y)|\,d\mu(y) \int_{\mathcal{X}\setminus(2S_j)} \frac{1}{\lambda(x_{S_j}, d(x,x_{S_j}))}\left[\frac{r_{S_j}}{d(x,x_{S_j})}\right]^{\epsilon_1} d\mu(x)\\
&\le \int_{\mathcal{X}} |b_j(y)|\,d\mu(y)\\
&\quad\times\left\{\sum_{n=1}^{\infty}\int_{(2^{n+1}S_j)\setminus(2^nS_j)} \frac{1}{\lambda(x_{S_j}, d(x,x_{S_j}))}\left[\frac{r_{S_j}}{d(x,x_{S_j})}\right]^{\epsilon_1} d\mu(x)\right\}\\
&\lesssim \int_{\mathcal{X}} |b_j(y)|\,d\mu(y)\left[\sum_{n=1}^{\infty}\frac{1}{2^{n\epsilon_1}}\frac{\mu(2^{n+1}S_j)}{\lambda(x_{S_j}, 2^n r_{S_j})}\right]\\
&\le \int_{\mathcal{X}} |\omega_j(y)f(y)|\,d\mu(y) + \int_{\mathcal{X}} |\varphi_j(y)|\,d\mu(y)\\
&\le \int_{6B_j} |f(y)|\,d\mu(y) + \|\varphi_j\|_{L^\infty(\mu)}\mu(S_j) \lesssim \int_{6B_j} |f(y)|\,d\mu(y).
\end{aligned}$$

To deal with $\mathrm{H}_{j,2}$, write

$$\begin{aligned}
\mathrm{H}_{j,2} &\lesssim \int_{(2S_j)\setminus(6^2B_j)} g(\omega_j f)(x)\,d\mu(x) + \int_{(2S_j)\setminus(6^2B_j)} g(\varphi_j)(x)\,d\mu(x)\\
&=: \mathrm{H}_{j,2}^{(1)} + \mathrm{H}_{j,2}^{(2)}.
\end{aligned}$$

Considering that $x \in (2S_j)\setminus(6^2B_j)$ and $\operatorname{supp}(\omega_j f) \subset 6B_j$, then, by (2.3), (3.5), Remark 2.2(ii) and Lemma 2.1, we gain that

$$\begin{aligned}
\mathrm{H}_{j,2}^{(1)} &\lesssim \int_{6B_j} |\omega_j(y)f(y)|\,d\mu(y) \int_{(2S_j)\setminus(6^2B_j)} \frac{1}{\lambda(x_{B_j}, d(x,x_{B_j}))}d\mu(x)\\
&\le \int_{6B_j} |f(y)|\,d\mu(y)\widetilde{K}^{(6)}_{6^2B_j,S_j} \lesssim \int_{6B_j} |f(y)|\,d\mu(y).
\end{aligned}$$

Due to the assumption that S_j is a $(3\times 6^2, C_{(\lambda)}^{\log_2(3\times 6^2)+1})$-doubling ball, we have $\mu(2S_j) \le \mu(3\times 6^2 S_j) \lesssim \mu(S_j)$, which, together with the Hölder inequality, the $L^2(\mu)$-boundedness of g and Lemma 3.1(ii), shows that

$$\begin{aligned}
\mathrm{H}_{j,2}^{(2)} &\le \int_{2S_j} g(\varphi_j)(x)\,d\mu(x) \le \left\{\int_{2S_j} [g(\varphi_j)(x)]^2\,d\mu(x)\right\}^{1/2} [\mu(2S_j)]^{1/2}\\
&\lesssim \left\{\int_{S_j} |\varphi_j(x)|^2\,d\mu(x)\right\}^{1/2} [\mu(2S_j)]^{1/2} \le \|\varphi_j\|_{L^\infty(\mu)}\,[\mu(S_j)\mu(2S_j)]^{1/2}
\end{aligned}$$

$$\lesssim \|\varphi_j\|_{L^\infty(\mu)}\,\mu(S_j) \lesssim \int_{6B_j} |f(x)|d\mu(x).$$

Combining the estimates for $\mathrm{H}_{j,1}$ and $\mathrm{H}_{j,2}$ yields that

$$\mu(\{x \in \mathcal{F} : g(b)(x) > \ell\}) \lesssim \frac{1}{\ell} \sum_j \int_{6B_j} |f(y)|d\mu(y) \lesssim \frac{1}{\ell}\|f\|_{L^1(\mu)},$$

which implies that (3.4) holds true. Then we finish the proof of Theorem 3.1. □

4. Boundedness of g from $\widetilde{\mathrm{RBMO}}(\mu)$ into $\widetilde{\mathrm{RBLO}}(\mu)$

To state our result in this section, we first recall the definitions of the space $\widetilde{\mathrm{RBMO}}(\mu)$ and the space $\widetilde{\mathrm{RBLO}}(\mu)$; see [FYY14] and [YYF13], respectively.

Definition 4.1. Let $\rho \in (1, \infty)$ and $\gamma \in [1, \infty)$. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\widetilde{\mathrm{RBMO}}(\mu)$, if there exist a positive constant C and a number f_B for any ball B such that, for all balls B,

$$\frac{1}{\mu(\rho B)} \int_B |f(y) - f_B|d\mu(y) \le C$$

and, for all balls $B \subset S$,

$$|f_B - f_S| \le C[\widetilde{K}^{(\rho)}_{B,S}]^\gamma.$$

Moreover, the *norm* of f in $\widetilde{\mathrm{RBMO}}(\mu)$ is defined to be the minimal constant C as above and denoted by $\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}$.

Definition 4.2. Let $\eta, \rho \in (1, \infty)$, and β_ρ be as in (2.1). A real-valued function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the *space* $\widetilde{\mathrm{RBLO}}(\mu)$, if there exists a non-negative constant C such that, for all balls B,

$$\frac{1}{\mu(\eta B)} \int_B \left[f(y) - \operatorname*{ess\,inf}_{\widetilde{B}^\rho} f\right] d\mu(y) \le C$$

and, for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$\operatorname*{ess\,inf}_B f - \operatorname*{ess\,inf}_S f \le C\widetilde{K}^{(\rho)}_{B,S}.$$

Moreover, the $\widetilde{\mathrm{RBLO}}(\mu)$ *norm* of f is defined to be the minimal constant C as above and denoted by $\|f\|_{\widetilde{\mathrm{RBLO}}(\mu)}$.

Remark 4.1.

(i) If we replace $\widetilde{K}^{(\rho)}_{B,S}$ by $K_{B,S}$ in Definitions 4.1 and 4.2, we then give the spaces $\mathrm{RBMO}(\mu)$ and $\mathrm{RBLO}(\mu)$, which were introduced by [Hy10] and [LY11], respectively.

(ii) It is a straightforward consequence of the definitions that $\widetilde{\mathrm{RBLO}}(\mu) \subset \widetilde{\mathrm{RBMO}}(\mu)$.

(iii) It is pointed out in [FYY14] that the space $\widetilde{\mathrm{RBMO}}(\mu)$ is independent of the choices of $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$. Moreover, the space $\widetilde{\mathrm{RBLO}}(\mu)$ is independent of the choices of $\eta, \rho \in (1,\infty)$; see [YYF13].

Theorem 4.1. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space and g be as in Definition 1.3. Assume that the dominating function λ satisfies the weak reverse doubling condition. If g is bounded on $L^2(\mu)$, then for all $f \in \widetilde{\mathrm{RBMO}}(\mu)$, $g(f)$ is either infinite everywhere or finite μ-almost everywhere. More precisely, if $g(f)$ is finite at some point $x_0 \in \mathcal{X}$, then $g(f)$ is finite μ-almost everywhere, and $\|g(f)\|_{\widetilde{\mathrm{RBLO}}(\mu)} \le C\|f\|_{\widetilde{\mathrm{RBMO}}}(\mu)$, where C is a positive constant independent of f.*

To prove Theorem 4.1, we first recall some useful lemmas related to the space $\widetilde{\mathrm{RBMO}}(\mu)$ as below. Lemmas 4.1 and 4.2 are showed in [LWY17], and the former one provides an equivalent characterization of the space $\widetilde{\mathrm{RBMO}}(\mu)$. Lemma 4.3 was proved in [CL17, Lemma 2.6].

Lemma 4.1. *Let $\eta, \rho \in (1,\infty)$ and β_ρ be as in (2.1). The following statements are equivalent:*

(i) *$f \in \widetilde{\mathrm{RBMO}}(\mu)$;*

(ii) *there exists a positive constant C such that, for all balls B,*

$$\frac{1}{\mu(\eta B)}\int_B \left|f(y) - m_{\widetilde{B}^\rho}(f)\right| d\mu(y) \le C \tag{4.1}$$

and, for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$|m_B(f) - m_S(f)| \le C\widetilde{K}^{(\rho)}_{B,S},$$

where above and in what follows, $m_B(f)$ denotes the mean of f over B, namely,

$$m_B(f) := \frac{1}{\mu(B)}\int_B f(y)d\mu(y).$$

Moreover, the infimum constant C is equivalent to $\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}$.

Lemma 4.2. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space, $f \in \widetilde{\mathrm{RBMO}}(\mu)$, $\eta \in (1,\infty)$ and $p \in [1,\infty)$. There exists a positive constant C such that, for any ball $B \subset \mathcal{X}$,*

$$\left[\frac{1}{\mu(\eta B)}\int_B |f(x) - f_B|^p d\mu(x)\right]^{1/p} \le C\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)},$$

where f_B is as in Definition 4.1.

Corollary 4.1. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space, $f \in \widetilde{\mathrm{RBMO}}(\mu)$, $\eta \in (1,\infty)$ and $p \in [1,\infty)$. Then there exists a positive constant C such* *that, for any ball $B \subset \mathcal{X}$,*

$$\left[\frac{1}{\mu(\eta B)}\int_B |f(x) - m_B(f)|^p d\mu(x)\right]^{1/p} \le C\,\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

Proof. Let f_B be as in Definition 4.1. It then follows from the Minkowski inequality, the Hölder inequality with $p \in (1, \infty)$ and Lemma 4.2 that

$$\begin{aligned}
&\left[\frac{1}{\mu(\eta B)}\int_B |f(x) - m_B(f)|^p d\mu(x)\right]^{1/p}\\
&\le \left[\frac{1}{\mu(\eta B)}\int_B |f(x) - f_B|^p d\mu(x)\right]^{1/p}\\
&\quad + \left[\frac{1}{\mu(\eta B)}\int_B |f_B - m_B(f)|^p d\mu(x)\right]^{1/p}\\
&\lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} + \left[\frac{1}{\mu(\eta B)}\int_B |f_B - m_B(f)|^p d\mu(x)\right]^{1/p}\\
&\le \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} + \left\{\frac{1}{\mu(\eta B)}\int_B \left[\frac{1}{\mu(B)}\int_B |f(y) - f_B| d\mu(y)\right]^p d\mu(x)\right\}^{1/p}\\
&\le \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} + \left\{\frac{1}{\mu(B)}\int_B \frac{1}{\mu(\eta B)}\int_B |f(y) - f_B|^p d\mu(y) d\mu(x)\right\}^{1/p}\\
&\lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} + \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} \lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)},
\end{aligned}$$

which completes the proof of Corollary 4.1. □

Lemma 4.3. *Let $f \in \widetilde{\mathrm{RBMO}}(\mu)$ and $\rho \in (1, \infty)$. Then, for all two balls $B \subset S \subset \mathcal{X}$, we have*

$$|m_{\widetilde{B}^\rho}(f) - m_{\widetilde{S}^\rho}(f)| \lesssim \widetilde{K}^{(\rho)}_{B,S}\,\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}$$

Now we show a new equivalent characterization of the space $\widetilde{\mathrm{RBLO}}(\mu)$. To this end, we need the following technical lemma (see also [FYY12, Lemma 3.13]), whose proof is parallel to that of [T01ma, Lemma 9.3] with a slight modification. We omit the details here.

Lemma 4.4. *Let $\rho \in (1, \infty)$. Assume that there exists a positive constant P_0 (big enough), depending on $C_{(\lambda)}$ from (1.3) and β_ρ as in (2.1), such that, if $x_0 \in \mathcal{X}$ is some fixed point and $\{f_B\}_{B \ni x_0}$ is a collection of numbers, for all (ρ, β_ρ)-doubling balls $B \subset S$ with $x_0 \in B$ such that $\widetilde{K}^{(\rho)}_{B,S} \le P_0$, which satisfies*

$$|f_B - f_S| \le C_{(x_0)},$$

then there exists a positive constant C, depending only on $C_{(\lambda)}$, β_ρ and P_0 such that, for all doubling balls $B \subset S$ with $x_0 \in B$,

$$|f_B - f_S| \le C\widetilde{K}^{(\rho)}_{B,S} C_{(x_0)}.$$

Lemma 4.5. *Let $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$ and β_ρ be as in (2.1). The following statements are equivalent:*

(i) $f \in \widetilde{\mathrm{RBLO}}(\mu)$;

(ii) *there exists a non-negative constant C_1 satisfying that, for all (ρ, β_ρ)-doubling balls B,*

$$\frac{1}{\mu(B)} \int_B \left[f(y) - \operatorname*{ess\,inf}_B f \right] d\mu(y) \le C_1 \tag{4.2}$$

and, for all (ρ, β_ρ)-doubling balls $B \subset S$,

$$m_B(f) - m_S(f) \le C_1 \widetilde{K}^{(\rho)}_{B,S}. \tag{4.3}$$

(iii) *there exists a non-negative constant C_2 satisfying (4.2) such that, for all (ρ, β_ρ)-doubling balls $B \subset S$,*

$$|m_B(f) - m_S(f)| \le C_2 [\widetilde{K}^{(\rho)}_{B,S}]^\gamma. \tag{4.4}$$

Moreover, the minimal constants C_1 and C_2 as above are equivalent to $\|f\|_{\widetilde{\mathrm{RBLO}}(\mu)}$.

Proof. The equivalence of (i) and (ii) can be proved by an argument similar to that used in [LY11, Proposition 2.3]. Thus, we only need to verify the equivalence of (ii) and (iii).

We first claim that (ii) is equivalent to (iii) with $\gamma = 1$. In fact, if (iii) holds true with $\gamma = 1$, then from the fact that $m_B(f) - m_S(f) \le |m_B(f) - m_S(f)|$, it is easy to see that (ii) holds true. To prove (ii) implies (iii) with $\gamma = 1$, notice that

$$m_B(f) \ge \operatorname*{ess\,inf}_B f \text{ for any } B \text{ and } \operatorname*{ess\,inf}_B f \ge \operatorname*{ess\,inf}_S f \text{ for any } B \subset S,$$

which, together with (4.2) and (4.3), show that

$$\begin{aligned}
&|m_B(f) - m_S(f)| \\
&\le \left| m_B(f) - \operatorname*{ess\,inf}_B f \right| + \left| \operatorname*{ess\,inf}_B f - \operatorname*{ess\,inf}_S f \right| + \left| \operatorname*{ess\,inf}_S f - m_S(f) \right| \\
&= \left[m_B(f) - \operatorname*{ess\,inf}_B f \right] + \left[\operatorname*{ess\,inf}_B f - \operatorname*{ess\,inf}_S f \right] + \left[m_S(f) - \operatorname*{ess\,inf}_S f \right] \\
&\le 2C_1 + \left[\operatorname*{ess\,inf}_B f - \operatorname*{ess\,inf}_S f \right] \\
&\le 2C_1 + \left[\operatorname*{ess\,inf}_B f - m_B(f) \right] + [m_B(f) - m_S(f)] \\
&\quad + \left[m_S(f) - \operatorname*{ess\,inf}_S f \right] \\
&\le 2C_1 + C_1 \widetilde{K}^{(\rho)}_{B,S} + C_1 \lesssim \widetilde{K}^{(\rho)}_{B,S}.
\end{aligned} \tag{4.5}$$

Hence, (iii) with $\gamma = 1$ holds true, which implies that our claim is valid.

Now we show that (iii) is independent of $\gamma \in [1, \infty)$. In fact, if (4.4) holds true for $\gamma = 1$, then it holds true for $\gamma \in (1, \infty)$. Assume that (4.4) holds true for $\gamma \in (1, \infty)$. Let $x \in \mathcal{X}$, and let $B \subset S$ be any two (ρ, β_ρ)-doubling balls with $x \in B$ such that $\widetilde{K}^{(\rho)}_{B,S} \le P_0$, where P_0 is as in Lemma 4.4. Then

$$|m_B(f) - m_S(f)| \le C[\widetilde{K}^{(\rho)}_{B,S}]^\gamma \le CP_0^\gamma := C_{(x)},$$

which, together with Lemma 4.4, implies that, for all (ρ, β_ρ)-doubling balls $B \subset S$ with $x \in B$,

$$|m_B(f) - m_S(f)| \le CC_{(x)}\widetilde{K}^{(\rho)}_{B,S}.$$

This yields that (4.4) holds true for $\gamma = 1$. Combining the above estimates, we conclude that (iii) is independent of $\gamma \in [1, \infty)$, which, together with our claim, completes the proof of Lemma 4.5. □

Proof of Theorem 4.1. Let $f \in \widetilde{\mathrm{RBMO}}(\mu)$ and $B \subset S$ be two (ρ, β_ρ)-doubling balls. According to Remark 4.1(iii), without loss of generality, we chose $\rho = 6$. To prove Theorem 4.1, we first claim that there exists a positive constant C such that

$$\frac{1}{\mu(B)} \int_B g(f)(x) d\mu(x) \le \inf_{y \in B} g(f)(y) + C \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}. \tag{4.6}$$

To prove (4.6), we decompose f as

$$\begin{aligned} f &= [f - m_{5B}(f)]\chi_{5B} + [f - m_{5B}(f)]\chi_{\mathcal{X}\setminus(5B)} + m_{5B}(f) \\ &=: f_1 + f_2 + m_{5B}(f). \end{aligned}$$

The vanishing condition of D_t implies that, for any $x, y \in B$,

$$\begin{aligned} g(f)(x) &\le g(f_1)(x) + g(f_2)(x) + g(m_{5B}(f))(x) \\ &= g(f_1)(x) + g(f_2)(x) \\ &= g(f_1)(x) + [g(f_2)(x) - g(f_2)(y)] + g(f_2)(y). \end{aligned}$$

Notice that B is $(6, \beta_6)$-doubling. By the Hölder inequality, the $L^2(\mu)$-boundedness of g and Corollary 4.1, we have

$$\begin{aligned} &\frac{1}{\mu(B)} \int_B g(f_1)(x) d\mu(x) \\ &\le \frac{1}{[\mu(B)]^{1/2}} \left\{ \int_{\mathcal{X}} [g(f_1)(x)]^2 \, d\mu(x) \right\}^{1/2} \\ &\lesssim \frac{1}{[\mu(6B)]^{1/2}} \left\{ \int_{5B} [f_1(x)]^2 \, d\mu(x) \right\}^{1/2} \\ &= \frac{1}{[\mu(6B)]^{1/2}} \left\{ \int_{5B} |f(x) - m_{5B}(f)|^2 \, d\mu(x) \right\}^{1/2} \lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}. \end{aligned} \tag{4.7}$$

To estimate $g(f_2)(y)$, for any $y \in B$, write

$$g(f_2)(y) \le \left\{\int_0^{r_B} \left|\int_{\mathcal{X}} D_t(y,z) f_2(z)\,d\mu(z)\right|^2 \frac{dt}{t}\right\}^{1/2} + \left\{\int_{r_B}^{\infty} \cdots\right\}^{1/2}$$
$$=: \mathrm{I}_1(y) + \mathrm{I}_2(y).$$

For $\mathrm{I}_1(y)$, observe that $y \in B$ and $\mathrm{supp}(f_2) \subset \mathcal{X}\backslash(5B)$. From the Minkowski inequality, (A1) of D_t, (3.5), (4.1) and (1.3), we deduce that

$$\begin{aligned}
\mathrm{I}_1(y) &\lesssim \int_{\mathcal{X}\backslash(5B)} \left\{\int_0^{r_B} \left|\frac{1}{\lambda(z, d(y,z))}\left[\frac{t}{d(y,z)}\right]^{\epsilon_2}\right|^2 \frac{dt}{t}\right\}^{1/2} |f_2(z)|\,d\mu(z)\\
&\lesssim \int_{\mathcal{X}\backslash(5B)} \frac{1}{\lambda(x_B, d(z,x_B))}\left[\frac{r_B}{d(z,x_B)}\right]^{\epsilon_2} |f_2(z)|\,d\mu(x)\\
&\lesssim \sum_{n=1}^{\infty} \frac{1}{5^{n\epsilon_2}\lambda(x_B, 5^n r_B)} \int_{(5^{n+1}B)\backslash(5^n B)} |f(z) - m_{5B}(f)|\,d\mu(z)\\
&\le \sum_{n=1}^{\infty} \frac{1}{5^{n\epsilon_2}\lambda(x_B, 5^n r_B)} \left\{\int_{5^{n+1}B} \left|f(z) - m_{\widetilde{5^{n+1}B}}(f)\right| d\mu(z)\right.\\
&\quad + \int_{5^{n+1}B} \left|m_{\widetilde{5^{n+1}B}}(f) - m_B(f)\right| d\mu(z)\\
&\quad \left. + \mu(5^{n+1}B)\,|m_B(f) - m_{5B}(f)|\right\}\\
&\lesssim \sum_{n=1}^{\infty} \frac{n}{5^{n\epsilon_2}} \frac{\mu(6\times 5^{n+1}B)}{\lambda(x_B, 5^n r_B)} \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} \lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)},
\end{aligned}$$

where in the second to the last inequality, we use the facts that

$$\begin{aligned}
|m_B(f) - m_{5B}(f)| &\le \frac{1}{\mu(B)} \int_B |f(x) - m_{5B}(f)|\,d\mu(x)\\
&\lesssim \frac{1}{\mu(6B)} \int_{5B} |f(x) - m_{5B}(f)|\,d\mu(x)\\
&\lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)},
\end{aligned} \tag{4.8}$$

and

$$\left|m_{\widetilde{5^{n+1}B}}(f) - m_B(f)\right| \lesssim \widetilde{K}^{(6)}_{B,5^{n+1}B}\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} \lesssim n\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)},$$

which can be inferred from Lemmas 4.3 and 2.1.

On the other hand, for $\mathrm{I}_2(y)$, through the vanishing moment of D_t and the Minkowski inequality, it is easy to see that

$$\mathrm{I}_2(y) = \left\{\int_{r_B}^{\infty} \left|\int_{\mathcal{X}} D_t(y,z)[f(z) - f_1(z) - m_{5B}(f)]\,d\mu(z)\right|^2 \frac{dt}{t}\right\}^{1/2}$$

$$\leq \left\{\int_{r_B}^{\infty}\left|\int_{\mathcal{X}} D_t(y,z)f(z)d\mu(z)\right|^2\frac{dt}{t}\right\}^{1/2}$$
$$+\left\{\int_{r_B}^{\infty}\left|\int_{\mathcal{X}} D_t(y,z)f_1(z)d\mu(z)\right|^2\frac{dt}{t}\right\}^{1/2} =: \mathrm{I}_{2,1}(y)+\mathrm{I}_{2,2}(y).$$

Clearly, $\mathrm{I}_{2,1}(y)\leq g(f)(y)$. Besides this, an argument analogous to that used in (2.3), together with (1.4), (1.3) and Corollary 4.1, shows that, for $y\in B$,

$$\mathrm{I}_{2,2}(y)\lesssim \int_{5B}\frac{|f_1(z)|}{\lambda(y,r_B)}d\mu(z)\lesssim \int_{5B}\frac{|f(z)-m_{5B}(f)|}{\lambda(x_B,r_B)}d\mu(z)\lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

Combining the estimates for $\mathrm{I}_1(y)$ and $\mathrm{I}_2(y)$, we conclude that there exists a positive constant C_1 such that, for any $y\in B$,

$$g(f_2)(y)\leq g(f)(y)+C_1\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}. \tag{4.9}$$

By the Minkowski inequality, some arguments parallel to those used in (2.2) and the estimate for $\mathrm{I}_1(y)$, we have that, for any $x,y\in B$,

$$\begin{aligned}
&g(f_2)(x)-g(f_2)(y) \\
&=\left\{\int_0^{\infty}\left|\int_{\mathcal{X}} D_t(x,z)f_2(z)d\mu(z)\right|^2\frac{dt}{t}\right\}^{1/2} \\
&\quad-\left\{\int_0^{\infty}\left|\int_{\mathcal{X}} D_t(y,z)f_2(z)d\mu(z)\right|^2\frac{dt}{t}\right\}^{1/2} \\
&\leq\left\{\int_0^{\infty}\left|\int_{\mathcal{X}}[D_t(x,z)-D_t(y,z)]f_2(z)d\mu(z)\right|^2\frac{dt}{t}\right\}^{1/2} \\
&\lesssim \int_{\mathcal{X}\setminus(5B)}\frac{1}{\lambda(x_B,d(z,x_B))}\left[\frac{r_B}{d(z,x_B)}\right]^{\epsilon_1}|f_2(z)|d\mu(z) \\
&\leq \sum_{n=1}^{\infty}\frac{1}{5^{n\epsilon_1}\lambda(x_B,5^{n+1}r_B)}\int_{(5^{n+1}B)\setminus(5^nB)}|f(z)-m_{5B}(f)|\,d\mu(z) \\
&\lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}.
\end{aligned}\tag{4.10}$$

Now, combining the estimates for (4.7), (4.9) and (4.10) yields that there exists a positive constant C_2 such that, for any $y\in B$,

$$\frac{1}{\mu(B)}\int_B g(f)(x)d\mu(x)\leq g(f)(y)+C_2\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)},$$

which implies that (4.6) holds true. Based on (4.6), if there exists some $x_0\in\mathcal{X}$ satisfying $g(f)(x_0)<\infty$, then, for any $f\in\widetilde{\mathrm{RBMO}}(\mu)$ and any $(6,\beta_6)$-doubling ball $B\subset\mathcal{X}$ with $x_0\in B$,

$$\frac{1}{\mu(B)}\int_B g(f)(x)d\mu(x)\leq g(f)(x_0)+C\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}<\infty.$$

That is to say, $g(f)$ is finite μ-almost everywhere, furthermore,

$$\frac{1}{\mu(B)}\int_B [g(f)(x) - \operatorname*{essinf}_B g(f)]d\mu(x) \le C\,\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}. \tag{4.11}$$

In this case, by Lemma 4.5, to prove Theorem 4.1, we also need to prove that, for all $(6,\beta_6)$-doubling balls $B\subset S$,

$$|m_B(g(f)) - m_S(g(f))| \lesssim \left[\widetilde{K}^{(6)}_{B,S}\right]^2 \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}. \tag{4.12}$$

Write

$$\begin{aligned} f &= [f - m_{5B}(f)]\chi_{5B} + [f - m_{5B}(f)]\chi_{(5S)\setminus(5B)} \\ &\quad + [f - m_{5B}(f)]\chi_{\mathcal{X}\setminus(5S)} + m_{5B}(f) \\ &:= f_1 + f_3 + f_4 + m_{5B}(f). \end{aligned}$$

By the vanishing condition of D_t, we know that, for any $x\in B$ and $y\in S$,

$$\begin{aligned} g(f)(x) &\le g(f_1)(x) + g(f_3)(x) + g(f_4)(x) + g(m_{5B}(f))(x) \\ &= g(f_1)(x) + g(f_3)(x) + [g(f_4)(x) - g(f_4)(y)] + g(f_4)(y). \end{aligned}$$

Let $N_1 := N^{(6)}_{5B,5S} + \lfloor \log_6 2\rfloor + 1$ with $N^{(6)}_{5B,5S}$ as in Definition 2.2. Notice that $x\in B$ and $\operatorname{supp}(f_3)\subset (5S)\setminus(5B)$. An argument similar to that used in proof of (2.3), together with (3.5), (4.1), Lemma 4.3, (1.3), Lemma 2.1, (4.8) and Remark 2.2(i), gives us that

$$\begin{aligned} &g(f_3)(x) \\ &\quad\lesssim \int_{(5S)\setminus(5B)} \frac{|f_3(z)|}{\lambda(x,d(x,z))}d\mu(z) = \int_{(5S)\setminus(5B)} \frac{|f(z) - m_{5B}(f)|}{\lambda(x_B,d(z,x_B))}d\mu(z) \\ &\quad\le \sum_{n=1}^{N_1} \int_{(5^{n+1}B)\setminus(5^nB)} \frac{|f(z) - m_{5B}(f)|}{\lambda(x_B,d(z,x_B))}d\mu(z) \\ &\quad\le \sum_{n=1}^{N_1} \frac{1}{\lambda(x_B,5^nr_B)}\left[\int_{(5^{n+1}B)\setminus(5^nB)} |f(z) - m_{\widetilde{5^{n+1}B}}(f)| \right. \\ &\qquad\quad + |m_{\widetilde{5^{n+1}B}}(f) - m_B(f)|d\mu(z) \\ &\qquad \left. + \mu(5^{n+1}B)|m_B(f) - m_{5B}(f)|\right] \\ &\quad\lesssim \sum_{n=1}^{N_1}\left[\frac{\mu(2\times 5^{n+1}B)}{\lambda(x_B,5^nr_B)} + \widetilde{K}^{(6)}_{B,5^{n+1}B}\frac{\mu(5^{n+1}B)}{\lambda(x_B,5^nr_B)}\right]\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} \\ &\quad\lesssim \left[\widetilde{K}^{(6)}_{B,S}\right]^2 \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}. \end{aligned} \tag{4.13}$$

We now deal with $g(f_4)(y)$. For any $y\in S$, write

$$g(f_4)(y) \le \left\{\int_0^{r_S}\left|\int_{\mathcal{X}} D_t(y,z)f_4(z)d\mu(z)\right|^2\frac{dt}{t}\right\}^{1/2} + \left\{\int_{r_S}^\infty \ldots\right\}^{1/2}$$

$$=: \mathrm{J}_1(y) + \mathrm{J}_2(y).$$

Similar to the estimate for $\mathrm{I}_1(y)$, we have

$$\begin{aligned}
\mathrm{J}_1(y) &\lesssim \int_{\mathcal{X}\setminus(5S)} \frac{1}{\lambda(x_S, d(z,x_S))}\left[\frac{r_S}{d(z,x_S)}\right]^{\epsilon_2} |f_4(z)|d\mu(z)\\
&\lesssim \sum_{n=1}^{\infty} \frac{1}{5^{n\epsilon_2}\lambda(x_S, 5^n r_S)} \int_{(5^{n+1}S)\setminus(5^nS)} |f(z) - m_{5B}(f)|\,d\mu(z)\\
&\le \sum_{n=1}^{\infty} \frac{1}{5^{n\epsilon_2}\lambda(x_S, 5^n r_S)} \left\{\int_{5^{n+1}S} \left|f(z) - m_{\widetilde{5^{n+1}S}}(f)\right| d\mu(z)\right.\\
&\quad + \int_{5^{n+1}S} \left|m_{\widetilde{5^{n+1}S}}(f) - m_{\widetilde{5^{n+1}B}}(f)\right| d\mu(z)\\
&\quad + \int_{5^{n+1}S} \left|m_{\widetilde{5^{n+1}B}}(f) - m_B(f)\right| d\mu(z)\\
&\quad \left. + \mu(5^{n+1}S)\,|m_B(f) - m_{5B}(f)|\,\right\}\\
&\lesssim \sum_{n=1}^{\infty} \frac{\mu\left(6\times 5^{n+1}S\right)}{5^{n\epsilon_2}\lambda(x_S, 5^n r_S)}\left[1 + \widetilde{K}^{(6)}_{5^{n+1}B,5^{n+1}S} + \widetilde{K}^{(6)}_{B,5^{n+1}B}\right] \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}\\
&\lesssim \widetilde{K}^{(6)}_{5B,5S}\,\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)} \sum_{n=1}^{\infty} \frac{n}{5^{n\epsilon_2}} \lesssim \widetilde{K}^{(6)}_{B,S}\,\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}\,.
\end{aligned}$$

For $\mathrm{J}_2(y)$, notice that $f_4 = f - f_1 - f_3 - m_{5B}(f)$. Thus, through the vanishing moment of D_t, it is easy to see that, for $y \in S$,

$$\begin{aligned}
\mathrm{J}_2(y) &= \left\{\int_{r_S}^{\infty} \left|\int_{\mathcal{X}} D_t(y,z)[f(z) - f_1(z) - f_3(z) - m_{5B}(f)]d\mu(z)\right|^2 \frac{dt}{t}\right\}^{1/2}\\
&\le \left\{\int_{r_S}^{\infty} \left|\int_{\mathcal{X}} D_t(y,z)f(z)d\mu(z)\right|^2 \frac{dt}{t}\right\}^{1/2}\\
&\quad + \left\{\int_{r_S}^{\infty} \left|\int_{\mathcal{X}} D_t(y,z)f_1(z)d\mu(z)\right|^2 \frac{dt}{t}\right\}^{1/2}\\
&\quad + \left\{\int_{r_S}^{\infty} \left|\int_{\mathcal{X}} D_t(y,z)f_3(z)d\mu(z)\right|^2 \frac{dt}{t}\right\}^{1/2}\\
&=: \mathrm{J}_{2,1}(y) + \mathrm{J}_{2,2}(y) + \mathrm{J}_{2,3}(y).
\end{aligned}$$

Obviously, $\mathrm{J}_{2,1}(y) \le g(f)(y)$ and $\mathrm{J}_{2,2}(y) \le g(f_1)(y)$. By some argument similar to that used in $\mathrm{J}_1(y)$, we conclude that

$$\begin{aligned}
&\mathrm{J}_{2,3}(y)\\
&\lesssim \int_{(5S)\setminus(5B)} \frac{|f(z) - m_{5B}(f)|}{\lambda(x_S, r_S)} d\mu(z)
\end{aligned}$$

$$\le \int_{5S} \frac{|f(z)-m_{\widetilde{5S}}(f)|+|m_{\widetilde{5S}}(f)-m_B(f)|+|m_B(f)-m_{5B}(f)|}{\lambda(x_S, r_S)} d\mu(z)$$
$$\lesssim \widetilde{K}^{(6)}_{B,S}\, \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

It then follows that there exists a positive constant C_3 such that, for any $y \in S$,

$$\mathrm{J}_2(y) \le g(f)(y) + g(f_1)(y) + C_3 \widetilde{K}^{(6)}_{B,S}\, \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)},$$

which, together with the estimate for $\mathrm{J}_1(y)$, implies that there exists a positive constant C_4 such that, for any $y \in S$,

$$g(f_4)(y) \le g(f)(y) + g(f_1)(y) + C_4 \widetilde{K}^{(6)}_{B,S} \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}. \tag{4.14}$$

Similar to the estimates for (4.10) and $\mathrm{J}_1(y)$, we see that, for any $x \in B$ and $y \in S$,

$$\begin{aligned}
(4.15)\qquad & g(f_4)(x) - g(f_4)(y) \\
&\le \left\{ \int_0^\infty \left| \int_{\mathcal{X}} [D_t(x,z) - D_t(y,z)] f_4(z)\, d\mu(z) \right|^2 \frac{dt}{t} \right\}^{1/2} \\
&\lesssim \int_{\mathcal{X}\setminus(5S)} \frac{1}{\lambda(x_S, d(z,x_S))} \left[\frac{r_S}{d(z,x_S)}\right]^{\epsilon_1} |f_4(z)|\, d\mu(z) \\
&\le \sum_{n=1}^{\infty} \frac{1}{5^{n\epsilon_1} \lambda(x_S, 5^{n+1} r_S)} \int_{(5^{n+1}S)\setminus(5^n S)} |f(z) - m_{5B}(f)|\, d\mu(z) \\
&\lesssim \widetilde{K}^{(6)}_{B,S}\, \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}.
\end{aligned}$$

From (4.13), (4.14) and (4.15), we deduce that there exists a positive constant C_5 such that, for any $x \in B$ and $y \in S$,

$$g(f)(x) - g(f)(y) \le g(f_1)(x) + g(f_1)(y) + C_5 \left[\widetilde{K}^{(6)}_{B,S}\right]^2 \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

On the other hand, an argument analogous to (4.7) shows that

$$\frac{1}{\mu(S)} \int_S g(f_1)(y)\, d\mu(y) \lesssim \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

It then follows from the above two estimates and (4.7) that

$$\begin{aligned}
m_B(g(f)) - m_S(g(f)) &= \frac{1}{\mu(S)} \frac{1}{\mu(B)} \int_S \int_B [g(f)(x) - g(f)(y)]\, d\mu(x)\, d\mu(y) \\
&\lesssim \left[\widetilde{K}^{(6)}_{B,S}\right]^2 \|f\|_{\widetilde{\mathrm{RBMO}}(\mu)},
\end{aligned}$$

which, together with an argument similar to that used in (4.5) and (4.11), yields that (4.12) holds true, and then completes the proof of Theorem 4.1. ∎

5. Boundedness of g on the Hardy space $H^p(\mu)$ with $p \in (0,1]$

We begin with the definitions and some boundedness criteria of the atomic Hardy space $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and the molecular Hardy space $\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$; see [FLYY15] and [LL18], respectively.

Definition 5.1. Let $\rho \in (1,\infty)$, $0 < p \le 1 \le q \le \infty$, $p \ne q$, and $\gamma \in [1,\infty)$. A function b in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$ is called a $(p,q,\gamma,\rho)_\lambda$-*atomic* block if

- (i) there exists a ball B such that $\mathrm{supp}(b) \subset B$;
- (ii) $\displaystyle\int_{\mathcal{X}} b(x)d\mu(x) = 0$;
- (iii) for any $j \in \{1,2\}$, there exist a function a_j supported on a ball $B_j \subset B$ and a number $\lambda_j \in \mathbb{C}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$ and

$$\|a_j\|_{L^q(\mu)} \le [\mu(\rho B_j)]^{1/q-1}[\lambda(x_B, r_B)]^{1-1/p}[\widetilde{K}^{(\rho),p}_{B_j,B}]^{-\gamma}. \tag{5.1}$$

Moreover, let $|b|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} := |\lambda_1| + |\lambda_2|$.

A function f is said to belong to the *space* $\widetilde{\mathbb{H}}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ if there exists a sequence of $(p,q,\gamma,\rho)_\lambda$-atomic blocks, $\{b_i\}_{i=1}^\infty$, such that $f = \sum_{i=1}^\infty b_i$ in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$, and

$$\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} < \infty.$$

Moreover, define

$$\|f\|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)} := \inf\left\{\left[\sum_{i=1}^\infty |b_i|^p_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}\right]^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

The *atomic Hardy space* $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ is then defined as the completion of $\widetilde{\mathbb{H}}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ with respect to the p-quasi-norm $\|\cdot\|^p_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}$.

Definition 5.2. Let $\rho \in (1,\infty)$, $0 < p \le 1 \le q \le \infty$, $p \ne q$, $\gamma \in [1,\infty)$ and $\epsilon \in (0,\infty)$. A function b in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$ is called a $(p,q,\gamma,\epsilon,\rho)_\lambda$-*molecular* block if

- (i) $\displaystyle\int_{\mathcal{X}} b(x)d\mu(x) = 0$;
- (ii) there exist some ball $B := B(x_B, r_B)$, with $x_B \in \mathcal{X}$ and $r_B \in (0,\infty)$, and some constants $\widetilde{M}, M \in \mathbb{N}^+$ such that, for all $k \in \mathbb{N}$ and $j \in \{1,\cdots,M_k\}$ with $M_k := \widetilde{M}$ if $k = 0$ and $M_k := M$ if $k \in \mathbb{N}^+$, there exist functions $m_{k,j}$ supported on some balls $B_{k,j} \subset U_k(B)$ for all $k \in \mathbb{N}$, where $U_0(B) := \rho^2 B$ and $U_k(B) := \rho^{k+2}B \setminus \rho^{k-2}B$ with

$k \in \mathbb{N}^+$, and $\lambda_{k,j} \in \mathbb{C}$ such that $b = \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j}$ in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$,

$$\|m_{k,j}\|_{L^q(\mu)} \le \rho^{-k\epsilon}[\mu(\rho B_{k,j})]^{1/q-1}[\lambda(x_B, \rho^{k+2} r_B)]^{1-1/p} \times [\widetilde{K}^{(\rho),p}_{B_{k,j},\rho^{k+2}B}]^{-\gamma} \tag{5.2}$$

and

$$|b|^p_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} := \sum_{k=0}^{\infty} \sum_{j=1}^{M_k} |\lambda_{k,j}|^p < \infty.$$

A function f is said to belong to the *space* $\widetilde{\mathbb{H}}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ if there exists a sequence of $(p,q,\gamma,\epsilon,\rho)_\lambda$-molecular blocks, $\{b_i\}_{i=1}^{\infty}$, such that $f = \sum_{i=1}^{\infty} b_i$ in $L^2(\mu)$ when $p \in (0,1)$ and in $L^1(\mu)$ when $p = 1$, and

$$\sum_{i=1}^{\infty} |b_i|^p_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} < \infty.$$

Moreover, define

$$\|f\|_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} := \inf\left\{\left[\sum_{i=1}^{\infty} |b_i|^p_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}\right]^{1/p}\right\},$$

where the infimum is taken over all possible decompositions of f as above.

The *molecular Hardy space* $\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ is then defined as the completion of $\widetilde{\mathbb{H}}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ with respect to the p-quasi-norm $\|\cdot\|^p_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}$.

Remark 5.1.

(i) It was pointed out in [FLYY15] that $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu) \subset \widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ in the sense that there exists a map T from $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ to $\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ such that, for any $f \in \widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$, there is a unique element $\widetilde{f} \in \widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ satisfying $T(f) = \widetilde{f}$ and $\|\widetilde{f}\|_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} \lesssim \|f\|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)}$.

(ii) When $p = 1$, it was proved in [FYY14] that $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu) = \widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ and they are independent of the choices of ρ, q, γ and ϵ. Thus, in what follows, we denote $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ simply by $\widetilde{H}^1(\mu)$.

(iii) When $p \in (0,1)$, it is unclear whether the similar properties of $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ as in (b) still hold true.

The following two boundedness criteria, respectively for the sublinear operator and the non-negative sublinear operator, on the Hardy spaces were proved in [LL18].

Lemma 5.1. *Let $\rho, q \in (1,\infty)$, $\gamma \in [1,\infty)$ and T be a sublinear operator bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$. If there exists a positive constant C such* *that, for all* $(1, q, \gamma, \rho)_\lambda$*-atomic blocks* b*,*

$$\|Tb\|_{L^1(\mu)} \le C|b|_{\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)},$$

then T *is extended to be a bounded sublinear operator from* $\widetilde{H}^1(\mu)$ *to* $L^1(\mu)$*.*

Lemma 5.2. *Let* $\rho \in (1, \infty)$*,* $0 < p < 1 \le q \le \infty$*,* $\gamma \in [1, \infty)$ *and* $\epsilon \in (0, \infty)$*. Let* T *be a non-negative sublinear operator. Assume that* T *is bounded on* $L^2(\mu)$*.*

(i) *If there exists a positive constant* C *such that, for all* $(p, q, \gamma, \epsilon, \rho)_\lambda$*-molecular blocks* b*,*

$$\|T(b)\|_{L^p(\mu)} \le C|b|_{\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)},$$

then T *is extended to be a bounded operator from* $\widetilde{H}^{p,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ *to* $L^p(\mu)$*.*

(ii) *If there exists a positive constant* C *such that, for all* $(p, q, \gamma, \rho)_\lambda$*-atomic blocks* b*,*

$$\|T(b)\|_{L^p(\mu)} \le C|b|_{\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)},$$

then T *is extended to be a bounded operator from* $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ *to* $L^p(\mu)$*.*

We first consider the boundedness of g on the Hardy space $\widetilde{H}^1(\mu)$.

Theorem 5.1. *Let* $(\mathcal{X}, d, \mu)$ *be a non-homogeneous space and* g *be as in Definition 1.3. Assume that the dominating function* λ *satisfies the weak reverse doubling condition. If* g *is bounded on* $L^2(\mu)$*, then* g *is bounded from* $\widetilde{H}^1(\mu)$ *into* $L^1(\mu)$*.*

Proof. By Remark 5.1(ii), without loss of generality, we choose $\rho = 2$, $q = 2$ and $\gamma = 1$. Let $b = \sum_{j=1}^{2} \lambda_j a_j$ be an arbitrary $(1, 2, 1, 2)_\lambda$-atomic block. For any $j \in \{1, 2\}$, $\mathrm{supp}(a_j) \subset B_j \subset B$, where B_j, B are as in Definition 5.1. Applying Lemma 5.1, we only need to prove that

$$\|g(b)\|_{L^1(\mu)} \lesssim |b|_{\widetilde{H}^{1,2,1}_{atb,2}(\mu)}.$$

Write

$$\begin{aligned}
\int_{\mathcal{X}} g(b)(x)d\mu(x) &= \int_{\mathcal{X}\setminus(2B)} g(b)(x)d\mu(x) + \int_{2B} g\left(\lambda_1 a_1 + \lambda_2 a_2\right)(x)d\mu(x) \\
&\le \int_{\mathcal{X}\setminus(2B)} g(b)(x)d\mu(x) \\
&\quad + \sum_{j=1}^{2} |\lambda_j| \left\{ \int_{(2B)\setminus(2B_j)} g(a_j)(x)d\mu(x) \right. \\
&\quad \left. + \int_{2B_j} g(a_j)(x)d\mu(x) \right\}
\end{aligned}$$

$$=: \mathrm{K}_0 + \sum_{j=1}^{2} |\lambda_j| \, (\mathrm{K}_1^j + \mathrm{K}_2^j).$$

For K_0, notice that $x \in \mathcal{X}\backslash(2B)$, $\mathrm{supp}(b) \subset B$, and $\int_{\mathcal{X}} b(y)d\mu(y) = 0$. This, together with (2.2), (3.5), (1.3), the Hölder inequality and (5.1), implies that

$$\begin{aligned}(5.3)\quad \mathrm{K}_0 &\lesssim \int_B |b(y)|d\mu(y) \int_{\mathcal{X}\backslash(2B)} \frac{1}{\lambda(x_B, d(x, x_B))} \left[\frac{r_B}{d(x, x_B)}\right]^{\epsilon_1} d\mu(x) \\ &\leq \int_B |b(y)|d\mu(y) \\ &\quad \times \left\{\sum_{n=1}^{\infty} \int_{(2^{n+1}B)\backslash(2^nB)} \frac{1}{\lambda(x_B, d(x, x_B))} \left[\frac{r_B}{d(x, x_B)}\right]^{\epsilon_1} d\mu(x)\right\} \\ &\lesssim \int_B |b(y)|d\mu(y) \left[\sum_{n=1}^{\infty} \frac{1}{2^{n\epsilon_1}} \frac{\mu(2^{n+1}B)}{\lambda(x_B, 2^n r_B)}\right] \leq \|b\|_{L^1(\mu)} \lesssim |b|_{\widetilde{H}^{1,2,1}_{atb,2}(\mu)}.\end{aligned}$$

To estimate K_1^j, notice that, for any fixed $j \in \{1, 2\}$, $x \in (2B)\backslash(2B_j)$, $\mathrm{supp}(a_j) \subset B_j$, and λ satisfies the weak reverse doubling condition. It then follows from (2.3), (3.5), Remark 2.2(ii), the Hölder inequality and (5.1) that

$$\begin{aligned}(5.4)\qquad \mathrm{K}_1^j &\lesssim \int_{(2B)\backslash(2B_j)} \frac{d\mu(x)}{\lambda(x_{B_j}, d(x, x_{B_j}))} \int_{\mathcal{X}} |a_j(y)| \, d\mu(y) \\ &\lesssim \widetilde{K}^{(2)}_{B_j,B} \, \|a_j\|_{L^1(\mu)} \leq 1.\end{aligned}$$

On the other hand, for K_2^j with any fixed $j \in \{1, 2\}$, from the Hölder inequality, the $L^2(\mu)$-boundedness of g and (5.1), we deduce that

$$\begin{aligned}(5.5)\qquad \mathrm{K}_2^j &\leq \left\{\int_{2B_j} [g(a_j)(x)]^2 \, d\mu(x)\right\}^{1/2} [\mu(2B_j)]^{1/2} \\ &\lesssim \left\{\int_{\mathcal{X}} [a_j(x)]^2 \, d\mu(x)\right\}^{1/2} [\mu(2B_j)]^{1/2} \\ &= \|a_j\|_{L^2(\mu)} \, [\mu(2B_j)]^{1/2} \leq 1.\end{aligned}$$

Combining the estimates for (5.3), (5.4) and (5.5) yields that

$$\int_{\mathcal{X}} g(b)(x)d\mu(x) \lesssim |b|_{\widetilde{H}^{1,2,1}_{atb,2}(\mu)} + \sum_{j=1}^{2} |\lambda_j| \lesssim |b|_{\widetilde{H}^{1,2,1}_{atb,2}(\mu)},$$

which completes the proof of Theorem 5.1. □

Theorem 4.1 tells us that g is bounded from the Lebesgue space $L^\infty(\mu)$ into the space $\widetilde{\mathrm{RBMO}}(\mu)$. From this, together with Theorem 5.1 and the following interpolation lemma for sublinear operator, we obtain the boundedness of g on the Lebesgue spaces $L^q(\mu)$ with $q \in (1, \infty)$. We mention that the interpolation lemma can be proved by using some arguments similar to those used in the proof of [LY12, Theorem 1.1]. We omit the details here.

Lemma 5.3. *Suppose that T is a sublinear operator bounded from $L^\infty(\mu)$ into the space $\widetilde{\mathrm{RBMO}}(\mu)$ and from $\widetilde{H}^1(\mu)$ into $L^{1,\infty}(\mu)$. Then T is extended to be a bounded operator on $L^q(\mu)$ for every $q \in (1, \infty)$.*

Corollary 5.1. *Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space and g be as in Definition 1.3. Assume that the dominating function λ satisfies the weak reverse doubling condition. If g is bounded on $L^2(\mu)$, then g is extended to be a bounded operator on $L^q(\mu)$ for every $q \in (1, \infty)$.*

At the end of this section, we establish the boundedness of g on the Hardy spaces H^p with $p \in (0, 1)$. Recall that $\nu := \log_2 C_{(\lambda)}$, where $C_{(\lambda)}$ is as in Definition 1.1.

Theorem 5.2. *Let $\rho \in (1, \infty)$, $\gamma \in [1, \infty)$, $\frac{\nu}{\nu + \epsilon_1} < p < 1 < q < \infty$, where $\epsilon_1 \in (0, 1]$ is as in Definitions 1.3. Let $(\mathcal{X}, d, \mu)$ be a non-homogeneous space and g be as in Definition 1.3. Assume that the dominating function λ satisfies the weak reverse doubling condition. If g is bounded on $L^2(\mu)$, then g is bounded from the molecular Hardy space $\widetilde{H}^{p,q,\gamma,\epsilon_1}_{\mathrm{mb},\rho}(\mu)$ into $L^p(\mu)$.*

With some proof completely analogous to that in [FLYY15, Corollary 4.9], we establish the following corollary and omit the details here.

Corollary 5.2. *Under the same assumption as in Theorem 5.2, g is bounded from the atomic Hardy space $\widetilde{H}^{p,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ into $L^p(\mu)$.*

Proof of Theorem 5.2. For the sake of simplicity, we choose $\rho = 2$, $\gamma = 1$, and assume that $\widetilde{M} = M$ in Definition 5.2. Our argument can be expanded to general cases via some slight modifications. Let $b = \sum_{k=0}^{\infty} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j}$ be an arbitrary $\widetilde{H}^{p,q,1,\epsilon_1}_{\mathrm{mb},2}(\mu)$-molecular block, where, for any $k \in \mathbb{N}$ and $j \in \{1, 2, ..., M\}$, $\mathrm{supp}(m_{k,j}) \subset B_{k,j} \subset U_k(B)$. According to Lemma 5.2(i), it is sufficient to prove that

$$\|g(b)\|_{L^p(\mu)} \lesssim |b|_{\widetilde{H}^{p,q,\gamma,\epsilon_1}_{\mathrm{mb},\rho}(\mu)}.$$

Write

$$\begin{aligned}
\|g(b)\|^p_{L^p(\mu)} &\le \sum_{n=0}^{\infty} \int_{U_n(B)} \left| g\left(\sum_{k=0}^{\infty} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right)(x) \right|^p d\mu(x) \\
&\le \sum_{n=0}^{4} \int_{U_n(B)} \left| g\left(\sum_{k=0}^{\infty} \sum_{j=1}^{M} \lambda_{k,j} m_{k,j} \right)(x) \right|^p d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=5}^{\infty}\int_{U_n(B)}\left|g\left(\sum_{k=0}^{n-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}\right)(x)\right|^p d\mu(x)\\
&+\sum_{n=5}^{\infty}\int_{U_n(B)}\left|g\left(\sum_{k=n-4}^{n+4}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}\right)(x)\right|^p d\mu(x)\\
&+\sum_{n=5}^{\infty}\int_{U_n(B)}\left|g\left(\sum_{k=n+5}^{\infty}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}\right)(x)\right|^p d\mu(x)\\
=:&\ \mathrm{L}_0+\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3.
\end{aligned}
$$

Using the Hölder inequality and (5.2), we obtain the following estimate for $m_{k,j}$ with $k \in \mathbb{N}$ and $j \in \{1, \cdots, M\}$,

$$
\|m_{k,j}\|_{L^1(\mu)}^p \lesssim \left[2^{k\epsilon_1}\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}\right]^{-p}\left[\lambda(x_B, 2^{k+2}r_B)\right]^{p-1}. \tag{5.6}
$$

To estimate L_2, write

$$
\begin{aligned}
\mathrm{L}_2 &\lesssim \sum_{n=5}^{\infty}\sum_{k=n-4}^{n+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_n(B)}|g(m_{k,j})(x)|^p d\mu(x)\\
&\lesssim \sum_{n=5}^{\infty}\sum_{k=n-4}^{n+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left[\int_{U_n(B)\setminus(2B_{k,j})}|g(m_{k,j})(x)|^p d\mu(x)\right.\\
&\quad\left.+\int_{2B_{k,j}}|g(m_{k,j})(x)|^p d\mu(x)\right]\\
&= \sum_{n=5}^{\infty}\sum_{k=n-4}^{n+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_n(B)\setminus(2B_{k,j})}|g(m_{k,j})(x)|^p d\mu(x)\\
&\quad+\sum_{n=5}^{\infty}\sum_{k=n-4}^{n+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{2B_{k,j}}|g(m_{k,j})(x)|^p d\mu(x) := \mathrm{L}_{2,1}+\mathrm{L}_{2,2}.
\end{aligned}
$$

For $\mathrm{L}_{2,1}$, notice that $x \in U_n(B)\setminus(2B_{k,j})$ and $\mathrm{supp}(m_{k,j}) \subset B_{k,j}$. By (2.3), (3.5), the Hölder inequality, (5.6), Remark 2.2(ii), Lemma 2.1 and (1.3), we see that

$$
\begin{aligned}
\mathrm{L}_{2,1} &\lesssim \sum_{n=5}^{\infty}\sum_{k=n-4}^{n+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\\
&\quad\times\int_{U_n(B)\setminus(2B_{k,j})}\left[\frac{1}{\lambda(x_{B_{k,j}}, d(x, x_{B_{k,j}}))}\int_{B_{k,j}}|m_{k,j}(y)|d\mu(y)\right]^p d\mu(x)\\
&= \sum_{n=5}^{\infty}\sum_{k=n-4}^{n+4}\sum_{j=1}^{M}|\lambda_{k,j}|^p\,\|m_{k,j}\|^p{}_{L^1(\mu)}
\end{aligned}
$$

$$\begin{aligned}
&\times \int_{U_n(B)\setminus(2B_{k,j})} \left[\frac{1}{\lambda(x_{B_{k,j}}, d(x, x_{B_{k,j}}))}\right]^p d\mu(x) \\
&\le \sum_{n=5}^{\infty} \sum_{k=n-4}^{n+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \, \|m_{k,j}\|^p{}_{L^1(\mu)} \\
&\times \left[\int_{(2^{n+2}B)\setminus(2B_{k,j})} \frac{d\mu(x)}{\lambda(x_{B_{k,j}}, d(x, x_{B_{k,j}}))}\right]^p \left[\mu(2^{n+2}B)\right]^{1-p} \\
&\le \sum_{n=5}^{\infty} \sum_{k=n-4}^{n+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[\frac{\widetilde{K}^{(2),p}_{B_{k,j},2^{n+2}B}}{2^{k\epsilon_1}\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}}\right]^p \left[\frac{\mu(2^{n+2}B)}{\lambda(x_B, 2^{k+2}B)}\right]^{1-p} \\
&\lesssim \sum_{n=5}^{\infty} \sum_{k=n-4}^{n+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \, 2^{-pk\epsilon_1} \lesssim \sum_{j=1}^{M} \sum_{k=1}^{\infty} \sum_{n=5}^{k+4} |\lambda_{k,j}|^p \, 2^{-pk\epsilon_1} \\
&\lesssim \sum_{j=1}^{M} \sum_{k=0}^{\infty} k 2^{-pk\epsilon_1} \, |\lambda_{k,j}|^p \lesssim \sum_{k=0}^{\infty} \sum_{j=1}^{M} |\lambda_{k,j}|^p = |b|^p_{\widetilde{H}^{p,q,\gamma,\epsilon_1}_{\mathrm{mb},\rho}(\mu)}.
\end{aligned}$$

On the other hand, the Hölder inequality, together with Corollary 5.1, (5.2) and (1.3), shows that

$$\begin{aligned}
\mathrm{L}_{2,2} &\le \sum_{n=5}^{\infty} \sum_{k=n-4}^{n+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left\{\int_{2B_{k,j}} [g(m_{k,j}(x))]^q \, d\mu(x)\right\}^{p/q} [\mu(2B_{k,j})]^{1-p/q} \\
&\lesssim \sum_{n=5}^{\infty} \sum_{k=n-4}^{n+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \, \|m_{k,j}\|^p_{L^q(\mu)} \, [\mu(2B_{k,j})]^{1-p/q} \\
&\lesssim \sum_{n=5}^{\infty} \sum_{k=n-4}^{n+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \left[\frac{1}{2^{k\epsilon_1}\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}}\right]^p \left[\frac{\mu(2^{k+3}B)}{\lambda(x_B, 2^{k+2}r_B)}\right]^{1-p} \\
&\lesssim \sum_{n=5}^{\infty} \sum_{k=n-4}^{n+4} \sum_{j=1}^{M} |\lambda_{k,j}|^p \, 2^{-pk\epsilon_1} \lesssim |b|^p_{\widetilde{H}^{p,q,\gamma,\epsilon_1}_{\mathrm{mb},\rho}(\mu)}.
\end{aligned}$$

Combining the estimates for $\mathrm{L}_{2,1}$ and $\mathrm{L}_{2,2}$, we have

$$\mathrm{L}_2 \lesssim |b|_{\widetilde{H}^{p,q,\gamma,\epsilon_1}_{\mathrm{mb},\rho}(\mu)}.$$

Now we deal with L_3. For any $x \in U_n(B)$ and $y \in B_{k,j} \subset U_k(B)$ with $k \ge n+5$, we have $d(x,y) \ge 2^{n+2}r_B \ge d(x,x_B) \ge 2^{n-2}r_B$, which, together with (1.3) and (3.5), implies that

$$\lambda(x, d(x,y)) \ge \lambda(x, d(x,x_B)) \sim \lambda(x_B, d(x,x_B)) \ge \lambda(x_B, 2^{n-2}r_B).$$

This, together with an argument similar to that used in the estimate for (2.3), (5.6) and (1.3), shows that

$$\begin{aligned}
\mathrm{L}_3 &\le \sum_{n=5}^{\infty}\int_{U_n(B)}\left|\sum_{k=n+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|g(m_{k,j})(x)\right|^p d\mu(x)\\
&\lesssim \sum_{n=5}^{\infty}\sum_{k=n+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\int_{U_n(B)}|g(m_{k,j})(x)|^p d\mu(x)\\
&\lesssim \sum_{n=5}^{\infty}\sum_{k=n+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\|m_{k,j}\|_{L^1(\mu)}^p\int_{U_n(B)}\left[\frac{1}{\lambda(x_B,2^{n-2}r_B)}\right]^p d\mu(x)\\
&\lesssim \sum_{n=5}^{\infty}\sum_{k=n+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left[\frac{1}{2^{k\epsilon_1}\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}}\right]^p\\
&\quad\times\frac{1}{[\lambda(x_B,2^{k+2}r_B)]^{1-p}}\frac{\mu(2^{n+2}B)}{[\lambda(x_B,2^{n-2}r_B)]^p}\\
&\lesssim \sum_{n=5}^{\infty}\sum_{k=n+5}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p 2^{-pk\epsilon_1}\frac{1}{[\lambda(x_B,2^{n+2}r_B)]^{1-p}}\frac{\mu(2^{n+2}B)}{[\lambda(x_B,2^{n+2}r_B)]^p}\\
&\lesssim \sum_{j=1}^{M}\sum_{k=10}^{\infty}\sum_{n=5}^{k-5}|\lambda_{k,j}|^p 2^{-pk\epsilon_1}\\
&\lesssim \sum_{j=1}^{M}\sum_{k=10}^{\infty}k2^{-pk\epsilon_1}|\lambda_{k,j}|^p\lesssim\sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p=|b|_{\widetilde{H}^{p,q,\gamma,\epsilon_1}_{\mathrm{mb},\rho}(\mu)}.
\end{aligned}$$

We now turn to estimate L_1. Write

$$\begin{aligned}
\mathrm{L}_1 &= \sum_{n=5}^{\infty}\int_{U_n(B)}\left|\int_0^{\infty}\left|\int_{\mathcal{X}}D_t(x,y)\right.\right.\\
&\qquad\left.\left.\times\left[\sum_{k=0}^{n-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)\right]d\mu(y)\right|^2\frac{dt}{t}\right|^{p/2}d\mu(x)\\
&\le \sum_{n=5}^{\infty}\int_{U_n(B)}\left|\int_0^{\infty}\left|\int_{\mathcal{X}}[D_t(x,y)-D_t(x,x_B)]\right.\right.\\
&\qquad\left.\left.\times\left[\sum_{k=0}^{n-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)\right]d\mu(y)\right|^2\frac{dt}{t}\right|^{p/2}d\mu(x)\\
&\quad+\sum_{n=5}^{\infty}\int_{U_n(B)}\left|\int_0^{\infty}\left|\int_{\mathcal{X}}D_t(x,x_B)\right.\right.
\end{aligned}$$

$$\times \left[\sum_{k=0}^{n-5}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)\right]d\mu(y)\Bigg|^2\frac{dt}{t}\Bigg|^{p/2}d\mu(x)$$
$$=: \mathrm{L}_{1,1}+\mathrm{L}_{1,2}.$$

Notice that, for any $x\in U_n(B)$ and $y\in B_{k,j}\subset U_k(B)$ with $k\le n-5$, we have $d(x,y)\ge 2^{n-3}r_B\ge 2^{-5}d(x,x_B)\ge 2^{-5}2^{n-2}r_B$, which, together with (1.3) and (3.5), implies that

$$\lambda(x,d(x,y))\gtrsim\lambda(x,d(x,x_B))\sim\lambda(x_B,d(x,x_B))\ge\lambda(x_B,2^{n-2}r_B).$$

From this, together with an argument similar to the estimate for (2.2), (5.6) and (1.3), we conclude that, for any $p\in(\frac{\nu}{\nu+\epsilon_1},1)$,

$$\begin{aligned}
\mathrm{L}_{1,1}&\lesssim\sum_{n=5}^{\infty}\sum_{k=0}^{n-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p\\
&\quad\times\int_{U_n(B)}\left\{\int_{B_{k,j}}\frac{|m_{k,j}(y)|}{\lambda(x,d(x,y))}\left[\frac{d(y,x_B)}{d(x,y)}\right]^{\epsilon_1}d\mu(y)\right\}^p d\mu(x)\\
&\lesssim\sum_{n=5}^{\infty}\sum_{k=0}^{n-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p\|m_{k,j}\|^p{}_{L^1(\mu)}\\
&\quad\times\int_{2^{n+2}B}\left\{\frac{1}{\lambda(x_B,2^{n-2}r_B)}\left[\frac{2^{k+2}r_B}{2^{n-2}r_B}\right]^{\epsilon_1}\right\}^p d\mu(x)\\
&\lesssim\sum_{n=5}^{\infty}\sum_{k=0}^{n-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left\{\frac{2^{-k\epsilon_1}}{\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}}\frac{\mu(2^{n+2}B)}{\lambda(x_B,2^{n-2}r_B)}\left[\frac{2^{k+2}}{2^{n-2}}\right]^{\epsilon_1}\right\}^p\\
&\quad\times\left\{\frac{\mu(2^{n+2}B)}{\lambda(x_B,2^{k+2}r_B)}\right\}^{1-p}\\
&\lesssim\sum_{n=5}^{\infty}\sum_{k=0}^{n-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p2^{-pn\epsilon_1}[C_{(\lambda)}]^{(n-k)(1-p)}\\
&=\sum_{n=5}^{\infty}\sum_{k=0}^{n-5}\sum_{j=1}^{M}|\lambda_{k,j}|^p2^{-(1-p)\nu k}2^{[(1-p)\nu-p\epsilon_1]n}\\
&\lesssim\sum_{k=0}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p=|b|_{\widetilde{H}^{p,q,\gamma,\epsilon_1}_{\mathrm{mb},\rho}(\mu)}.
\end{aligned}$$

Furthermore, by the vanishing moment of b, an argument similar to the estimate of (2.3), (3.5), (5.6) and (1.3), we have

$$\mathrm{L}_{1,2}=\sum_{n=5}^{\infty}\int_{U_n(B)}\left|\int_0^{\infty}\left|\int_{\mathcal{X}}D_t(x,x_B)\right.\right.$$

$$
\begin{aligned}
&\times \left[b(y) - \sum_{k=n-4}^{\infty}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)\right] d\mu(y)\Bigg|^2 \frac{dt}{t}\Bigg|^{p/2} d\mu(x)\\
&= \sum_{n=5}^{\infty}\int_{U_n(B)}\left|\int_0^{\infty}\left|\int_{\mathcal{X}} D_t(x,x_B)\right.\right.\\
&\quad\times \left[\sum_{k=n-4}^{\infty}\sum_{j=1}^{M}\lambda_{k,j}m_{k,j}(y)\right] d\mu(y)\Bigg|^2 \frac{dt}{t}\Bigg|^{p/2} d\mu(x)\\
&\lesssim \sum_{n=5}^{\infty}\sum_{k=n-4}^{\infty}\sum_{j=1}^{M}\|m_{k,j}\|^p{}_{L^1(\mu)}\int_{U_n(B)}\left[\frac{1}{\lambda(x_B,2^{n-2}r_B)}\right]^p d\mu(x)\\
&\le \sum_{n=5}^{\infty}\sum_{k=n-4}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\left[\frac{1}{2^{k\epsilon_1}\widetilde{K}^{(2),p}_{B_{k,j},2^{k+2}B}}\right]^p\left[\frac{\mu(2^{n+2}B)}{\lambda(x_B,2^{k+2}r_B)}\right]^{1-p}\\
&\lesssim \sum_{n=5}^{\infty}\sum_{k=n-4}^{\infty}\sum_{j=1}^{M}|\lambda_{k,j}|^p\,2^{-pk\epsilon_1}\\
&\lesssim \sum_{j=1}^{M}\sum_{k=1}^{\infty}\sum_{n=5}^{k+4}|\lambda_{k,j}|^p\,2^{-pk\epsilon_1}\lesssim |b|^p_{\widetilde{H}^{p,q,\gamma,\epsilon_1}_{\mathrm{mb},\rho}(\mu)}.
\end{aligned}
$$

It then follows from the estimates for $\mathrm{L}_{1,1}$ and $\mathrm{L}_{1,2}$ that

$$\mathrm{L}_1 \lesssim |b|_{\widetilde{H}^{p,q,1,\epsilon_1}_{\mathrm{mb},2}(\mu)}.$$

Using some argument used in the estimates for L_1 and L_2, we obtain that

$$\mathrm{L}_0 \lesssim |b|_{\widetilde{H}^{p,q,1,\epsilon_1}_{\mathrm{mb},2}(\mu)}.$$

Combining all the estimates for L_0 to L_3, we finish the proof of Theorem 5.2. □

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