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Non-unital ASH algebras arising as crossed products of graph algebras

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Abstract. We study the structure of crossed products of graph algebras by quasi-free actions and show that they can be written as inductive limits of one-dimensional NCCW complexes for at least some dense G_δ set of the action parameters. The K-theory of certain AF algebras used in the construction is computed.

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1. Introduction

Recently, there have been major advances in the Elliott program to classify C^*-algebras using K-theoretic invariants. In particular, all unital, separable, simple C^*-algebras satisfying the UCT and having finite nuclear dimension are classified. Furthermore, the finite algebras in this class can be expressed as ASH algebras [ElN16], [ElGLN15], [TWW17]. Attention has now shifted to the non-unital case, where there has also been a lot of progress [GL16], [GL17]. Because of these results, it has become very interesting to know when a crossed product will have finite nuclear dimension. Recently, it was shown that if X is a finite dimensional locally compact Hausdorff space, then the crossed product of $C_0(X)$ by any automorphism has finite nuclear dimension [HW17]. The analysis naturally led to actions of the reals and it was shown that crossed products of flows with finite Rokhlin dimension have finite nuclear dimension [HSWW17].

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The C^*-algebras that are considered in this paper are examples of non-unital *Approximately Subhomogeneous* C^*-algebras; that is, C^*-algebras that are inductive limits of C^*-subalgebras of C^*-algebras whose irreducible representations all have the same (finite) dimension. We call such C^*-algebras *ASH algebras*. In this paper, the ASH algebras arise as crossed products of graph algebras by quasi-free actions (see Definition 2.2). A large class of the crossed products are non-unital simple stably projectionless C^*-algebras but it is unclear if these crossed products have finite nuclear dimension.

Over the years, crossed products of graph algebras by quasi-free automorphisms have garnered significant attention and have been a useful construction in generating examples of interesting C^*-algebras. Consider the quasi-free action α^λ on O_n given by $\alpha^\lambda(S_i) = e^{i\lambda_i t}S_i$. Kishimoto [Kis80] showed that the crossed product $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ is simple if and only if one of the following two cases occur:

1. All the labels λ_k are of the same sign and $\{\lambda_1, \ldots, \lambda_n\}$ generate $\mathbb{R}$ as a closed group.

2. The closed subsemigroup generated by all λ_k is $\mathbb{R}$.

It was shown that in Case 1, the crossed product is stably projectionless [KisK96], while in Case 2, the crossed product is purely infinite [KisK97]. In [Kat03], Katsura completely described the ideal structure of crossed products of O_n and O_∞ by quasi-free actions, giving another proof of Kishimoto's simplicity result. As an extension to [Kat03], Elliott and Fang [ElF10] investigated the ideal structure and simplicity of crossed products of graph algebras by quasi-free actions, where the corresponding graph is row-finite and without sinks. In [Kat02], a sufficient condition was obtained for the AF-embeddability of a crossed product of O_n and O_∞ by quasi-free actions and along with being stably projectionless, $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ is AF embeddable in Case 1 [Kat03]. The AF-embeddability of crossed products of certain graph C^*-algebras by quasi-free actions in [Fan09] shows that the methods are not easily extended to general graph algebras using the methods of Katsura, as the graphs are quite restrictive.

In [Dea01, Theorem 5.1], it was shown that for at least a dense G_δ set of labels, the crossed product of a Cuntz algebra by a quasi-free action can be written as an inductive limit of one-dimensional noncommutative CW-complexes, abbreviated NCCW complexes (See Definition 2.6). The special case of O_2 was considered in order to simplify calculations and book-keeping, however, the general argument also extends to O_n. The crossed products were viewed as fibres in a continuous field of C^*-algebras and the rational ones reduced to studying the mapping torus of $O_2 \rtimes_\alpha \mathbb{T}$ by an automorphism generating the dual action of $\mathbb{Z}$ [Dea01, Theorem 3.1]. Dean showed that $O_2 \rtimes_\alpha \mathbb{R}$ is isomorphic to the mapping torus of a simple AF algebra

$A(p,q) \cong O_2 \rtimes_\alpha \mathbb{T}$ by $\hat{\alpha}$, where $A(p,q)$ was a universal C^*-algebra given by generators and relations. The mapping torus was then deconstructed as an inductive limit of one-dimensional NCCW complexes [Dea01, Corollary 3.5]. The rational fibres satisfy a local approximation property and by stable relations and a Baire category argument, it followed that a dense G_δ set of the fibres have this local approximation property. Since they satisfy a local approximation property, they can be written as inductive limits of one-dimensional NCCW complexes [Dea01, Lemma 4.6].

As in [Dea01], the basis of our construction is viewing these crossed products as fibres in a continuous field over $\mathbb{R}^{E^1}$, where E^1 the the set of edges associated to the graph. The main result of this article extends the results of [Dea01, Theorem 5.1] to row-finite graph algebras. The case for finite graph algebras is proved in Theorem 4.3 and then, as a consequence, the row-finite case is addressed in Theorem 4.8. As a consequence of the construction, the crossed products are AF-embeddable.

In §5, the K-Theory of the AF algebras used in the construction of the mapping torus is calculated for the case when the graph has no sinks and the labels of the edges are either all positive integers or all negative integers (Theorem 5.2). Also, the ordered K_0-group is calculated for the Cuntz algebra case (Theorem 5.6).

2. Notation and preliminaries

2.1. Graph algebras. The definitions and terminology for directed graphs given below can be found in [Tom06, p. 3] and [Rae05, pp. 5–6]. A *directed graph* $E = (E^0, E^1, r, s)$ consists of countable sets E^0 and E^1 of vertices and edges, respectively, with range and source maps $r, s : E^1 \longrightarrow E^0$. A directed graph $E = (E^0, E^1, r, s)$ is called *finite* if both E^0 and E^1 are finite, and it is called *row-finite* if $|s^{-1}(v)| < \infty$ for all $v \in E^0$. A *path* of length $n \geq 1$ is a finite sequence of edges $\mu := \mu_1\mu_2 \cdots \mu_n$ with $r(\mu_i) = s(\mu_{i+1})$ for $1 \leq i \leq n-1$. We regard vertices as paths of length 0. For $n \geq 0$, we let E^n denote the set of all paths of length n and define $E^* := \bigcup_{n\geq 0} E^n$. The range and source maps extend to E^* in a natural way. For vertices v and w, we define $vE^n w$ to be the set $\{\mu \in E^n : s(\mu) = v \text{ and } r(\mu) = w\}$. The vertex matrix is the matrix $A_E \in M_{E^0}(\mathbb{N})$ such that $A_E(v,w) = |vE^1 w|$. A *cycle* is a path with its range and source equal; namely, a path $\mu := \mu_1\mu_2 \cdots \mu_n$ is a cycle provided that $r(\mu_n) = s(\mu_1)$. A cycle $\mu := \mu_1\mu_2 \cdots \mu_n$ has an exit if there is an edge $f \in E^1$ with the property that $s(f) = s(\mu_i)$ but $\mu_i \neq f$ for some $i \in \{1, 2, \ldots, n\}$. A vertex that does not emit an edge is called a *sink* and we write E^0_{sinks} for the set of all sinks in E^0. A vertex that does not receive an edge is called a *source* and we write E^0_{sources} for the set of all sources in E^0. For $v, w \in E^0$ we write $v \geq w$ if $vE^*w \neq \emptyset$ and we can define an equivalence relation $\sim$ on E^0 by $v \sim w \Longleftrightarrow v \geq w$ and $w \geq v$. We write $E/\sim$ for the set of equivalence classes of E^0 and refer to the equivalence classes as the *strongly connected components* of E. We say that a graph E

is *cofinal* if for every $v \in E^0$ and every infinite path $\mu \in E^\infty$ there is an $i \in \mathbb{N}$ with $v \geq s(\mu_i)$.

If E is a graph, a *Cuntz-Krieger E-family* in a C^* -algebra is a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges that satisfy the following Cuntz-Krieger relations:

(CK1) $s_e^* s_e = p_{r(e)}$
(CK2) $p_v = \sum_{\{e \in E^1 : s(e)=v\}} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$, and
(CK3) $s_e s_e^* \leq p_{s(e)}$.

The *graph C^*-algebra* (or, simply, the graph algebra) of E is the C^*-algebra generated by the universal Cuntz-Krieger E-family, and it is denoted $C^*(E)$.

Remark 2.1. In this paper, we use the convention that the partial isometries go in a direction opposite the edges as in [Tom06, p. 3]. A path $\mu_1\mu_2\cdots\mu_n$ traverses edges from left to right since $r(\mu_i) = s(\mu_{i+1})$ for $i = 1, \dots, n-1$. The other convention is to have the isometries go in the same direction of the edges as seen in [Rae05, pp. 5–6]. In any case, the convention will not change the final results.

2.2. Quasi-free actions. Given a group G, we call a map $c : E^1 \to G$ a labeling map on the set of edges E^1. We naturally extend c to E^* by $c(\mu) = c(\mu_1)c(\mu_2)\cdots c(\mu_n)$ for $\mu = \mu_1\cdots\mu_n \in E^*\backslash E^0$ and $c(v) = 1_G$ for $v \in E^0$.

When the group is abelian, we will write the labeling map additively. For the case when $G = \mathbb{R}$, we will use λ to denote a labeling map, where $\lambda : E^* \to \mathbb{R}$ is given by $\lambda_\mu = \lambda_{\mu_1} + \cdots + \lambda_{\mu_n}$ for $\mu = \mu_1\cdots\mu_n \in E^*\backslash E^0$ and $\lambda_v = 0$ for $v \in E^0$.

Definition 2.2. Given a labeling map $\lambda : E^1 \to \mathbb{R}$, there is a strongly continuous action $\alpha^\lambda : \mathbb{R} \to \operatorname{Aut} C^*(E)$ such that $\alpha_t(s_e) = e^{i\lambda_e t} s_e$ for all $e \in E^1$ and $\alpha_t(p_v) = p_v$ for all $v \in E^0$. Actions of this form are referred to as *quasi-free actions*.

2.3. Skew-product graph. General theory for skew-product graphs can be found in [KumP99] and [KalQR01]. The skew-product graph defined below is the same as the one described in [KalQR01]. The skew-product graphs used in [KumP99], although defined differently, are isomorphic to the ones used in this paper [KalQR01, Remark 2.2].

Let E be a graph and G be a countable group. Given a labeling map $c : E^1 \to G$, we define the skew-product graph, denoted $E \times_c G$, to be the graph having vertex set $E^0 \times G$, edge set $E^1 \times G$, and with range and source maps defined by $r(e, g) = (r(e), g)$ and $s(e, g) = (s(e), c(e)g)$ for $(e, g) \in E^1 \times G$.

Note that $E \times_c G$ is row-finite if and only if E is row-finite. Also, (v, g) is a sink if and only if v is a sink. The C^*-algebra of a skew-product graph is an AF algebra if and only if $c(\mu) \neq 1_G$ for every cycle $\mu \in E$.

The group G acts on the skew-product graph via right translation:

$$g \cdot (v, h) = (v, hg^{-1})$$
$$g \cdot (e, h) = (e, hg^{-1}).$$

This induces an action $\beta : G \curvearrowright C^*(E \times_c G)$ such that

$$\beta_g(s_{(e,h)}) = s_{(e,hg^{-1})}$$
$$\beta_g(p_{(v,h)}) = p_{(v,hg^{-1})}$$

(see [KalQR01]). Below, we will use $G = \mathbb{Z}$. Then, we have the skew-product graph $E \times_c \mathbb{Z}$, with range and source maps as follows:

$$s(e, n) = (s(e), n - c(e)) \text{ and } r(e, n) = (r(e), n).$$

The induced action $\beta : \mathbb{Z} \to \operatorname{Aut} C^*(E \times_c \mathbb{Z})$ satisfies $\beta_m(s_{(e,n)}) = s_{(e,n+m)}$ and $\beta_m(p_{(v,n)}) = p_{(v,n+m)}$.

The proposition below will be useful in analyzing the crossed products of graph algebras by periodic quasi-free actions.

Proposition 2.3. [Rae05, Lemma 7.10],[KalQR01, Theorem 2.4] *Let E be a row-finite directed graph. Then, there is an isomorphism Φ of $C^*(E \times_c \mathbb{Z})$ onto $C^*(E) \rtimes_\alpha \mathbb{T}$ such that $\Phi \circ \beta_m = \widehat{\alpha}_m \circ \Phi$.*

Definition 2.4 and Proposition 2.5 below will be useful for the proof of Proposition 3.1, where the skew-product graph algebra is written as an inductive limit of finite-dimensional C^*-algebras.

Definition 2.4. [MT04, Definition 3.6] Let $E = (E^0, E^1, r, s)$ be a graph and let $F = (F^0, F^1, r_F, s_F)$ be a subgraph of E. Define a graph $E_F = (E_F^0, E_F^1, r_{E_F}, s_{E_F})$ as follows.
Set $S := \{v \in F^0 : |s_F^{-1}(v)| < \infty, \emptyset \subsetneq s_F^{-1}(v) \subsetneq s_E^{-1}(v)\}$, and let

$$E_F^0 := F^0 \cup \{v' : v \in S\} \text{ and } E_F^1 := F^1 \cup \{e' : e \in F^1 \text{ and } r(e) \in S\},$$

with range and source maps given by

$$s_{E_F}(e) = s(e),\ s_{E_F}(e') = s(e),\ r_{E_F}(e) = r(e),\ r_{E_F}(e') = r(e)'.$$

Proposition 2.5 shows that the C^*-subalgebra of $C^*(E)$ generated by elements that come from a subgraph F is isomorphic to a graph algebra whose corresponding graph E_F is defined above.

Proposition 2.5. [MT04, Theorem 3.7, Example 3.8] *Let E be a graph, $\{s_e, p_v\}$ be a generating Cuntz-Krieger E-family in $C^*(E)$, and F be a subgraph of E. Then, the C^*-subalgebra of $C^*(E)$ generated by $\{s_e : e \in F^1\} \cup \{p_v : v \in F^0\}$, denoted by $C^*(\{s_e : e \in F^1\} \cup \{p_v : v \in F^0\})$, is isomorphic to $C^*(E_F)$. Furthermore, if we define*

$$q_w := \begin{cases} p_v & \textit{if } w \in F^0 \backslash S \\ \sum_{\{e \in F^1 : s(e) = w\}} s_e s_e^* & \textit{if } w \in S \\ p_v - \sum_{\{e \in F^1 : s(e) = v\}} s_e s_e^* & \textit{if } w = v' \textit{ for some } v \in S \end{cases}$$

$$t_f := \begin{cases} s_f q_{r(f)} & \textit{if } f \in F^1 \\ s_e q_{r(e)'} & \textit{if } f = e' \textit{ for some } e \in F^1, \end{cases}$$

then $\{t_f, q_w\}$ *will be a generating Cuntz-Krieger* E_F*-family in* $C^*(\{s_e : e \in F^1\} \cup \{p_v : v \in F^0\})$.

The main result of this chapter deals with writing crossed products as inductive limits of NCCW-complexes. The definition is provided below.

Definition 2.6. [Ped99, Definition 11.2] A zero dimensional NCCW-complex is any finite dimensional C^*-algebra A_0. An n-dimensional NCCW-complex is defined as any C^*-algebra A_n, arising a pull-back of a diagram of the form:

$$\begin{tikzcd} & A_{n-1} \arrow[d, "\phi_n"] \\ C([0,1]^n, F_n) \arrow[r, "\delta"] & C(S^{n-1}, F_n) \end{tikzcd}$$

where A_{n-1} is an $(n-1)$-dimensional-NCCW complex, F_n is a finite dimensional C^* algebra, δ is the boundary restriction map, and ϕ_n is an arbitrary morphism called the connecting morphism. An NCCW complex A_n is called unital if if A_{n-1} is unital and the connecting morphism ϕ_n is also unital.

In this paper, we construct a one-dimensional NCCW complex in the following way. Let F_0 and F_1 be two finite dimensional C^*-algebras with maps $\alpha_1, \alpha_2 : F_0 \to F_1$. Let $\mathrm{ev}(0)$, $\mathrm{ev}(1)$ denote the maps from $F_1 \otimes C[0,1]$ to F_1, given by evaluation at zero and one, respectively. Then, we get a *one-dimensional NCCW-complex* as the pull-back of the following diagram:

$$\begin{tikzcd} & F_0 \arrow[d, "\alpha_1 \oplus \alpha_2"] \\ F_1 \otimes C[0,1] \arrow[r, "\mathrm{ev}(0) \oplus \mathrm{ev}(1)"] & F_1 \oplus F_1 \end{tikzcd}$$

3. The rational fibres

Let E be a graph and $c : E^1 \to \mathbb{R}$ a labeling map. A scaling of the parameters produces an isomorphic crossed product, and an action is periodic if and only if by a scaling we may assume labels are all integers. From now on, when α^c is referred to as a periodic action, we will assume the labeling map is $c : E^1 \to \mathbb{Z}$.

The 'rational' (periodic) fibres are viewed as mapping tori over skew-product graph algebras. In Proposition 3.1 below, the 'rational' (periodic) fibres $C^*(E) \rtimes_{\alpha^c} \mathbb{R}$ are shown to be inductive limits of one-dimensional NCCW complexes, extending [Dea01, Corollary 3.5] to finite-graph algebras.

Proposition 3.1. *Let E be a finite graph and let α^c be a periodic action on $C^*(E)$ with corresponding labeling map c. If $c(\mu) \neq 0$ for any cycle $\mu \in E^*$, then $C^*(E) \rtimes_{\alpha^c} \mathbb{R}$ is an inductive limit of one-dimensional NCCW-complexes.*

Proof. Since α^c is a periodic action, by rescaling we may assume $c : E^1 \to \mathbb{Z}$. Let $E \times_c \mathbb{Z} := (E^0 \times_c \mathbb{Z}, E^1 \times_c \mathbb{Z}, r, s)$ denote the skew-product graph. By Proposition 2.3, we have that there is an isomorphism Φ of $C^*(E \times_c \mathbb{Z})$ onto $C^*(E) \rtimes_\alpha \mathbb{T}$ with $\Phi \circ \beta_m = \widehat{\alpha}_m \circ \Phi$.

Let $E = (E^0, E^1, r, s)$ be a finite graph and let K be the set $K := \{v \in E^0 : r^{-1}(v) = \emptyset,\ s^{-1}(v) = \emptyset\}$. For each n, define a subgraph F_n of $E \times_c \mathbb{Z}$ by $F_n^0 = \{r(e,k) : e \in E^1,\ -n \leq k \leq n\} \cup \{s(e,k) : e \in E^1,\ -n \leq k \leq n\} \cup \{(v,k) : v \in K,\ -n \leq k \leq n\}$ and $F_n^1 = \{(e,k) : e \in E^1,\ -n \leq k \leq n\}$. Then, $F_n \subset F_{n+1}$ and $E \times_c \mathbb{Z} = \bigcup_n F_n$.

Let $\{s_{(e,k)}, p_{(v,k)} : e \in E^1, v \in E^0, k \in \mathbb{Z}\}$ be the canonical Cuntz-Krieger family generating $C^*(E \times_c \mathbb{Z})$. Define A_n to be C^*-subalgebra of $C^*(E \times_c \mathbb{Z})$ generated by $\{s_{(e,k)} : (e,k) \in F_n^1\} \cup \{p_{(v,k)} : (v,k) \in F_n^0\}$. Then,
$A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of C^*-subalgebras of $C^*(E \times_c \mathbb{Z})$ with

$$C^*(E \times_c \mathbb{Z}) = \overline{\bigcup_{n \geq 1} A_n}.$$

Since $c(\mu) \neq 0$ for any cycle $\mu \in E^*$, $E \times_c \mathbb{Z}$ has no cycles. Thus, each A_n is isomorphic to a finite graph algebra (see Proposition 2.5), in which the graph has no cycles. So, A_n is finite dimensional.

Lastly, since A_n and $\beta(A_n)$ are both included into A_{n+1}, we can now define the NCCW-complex B_n, as in [Dea01]. That is,

$$B_n = \{f \in C([0,1], A_{n+1}) : f(0) \in A_n, \beta(f(0)) = f(1)\}.$$

Then, $B_n \subseteq B_{n+1}$ for all n and

$$C^*(E) \rtimes_\alpha \mathbb{R} \cong M_{\widehat{\alpha}}(C^*(E) \rtimes_\alpha \mathbb{T}) \cong M_\beta(C^*(E \times_c \mathbb{Z})) = \overline{\bigcup_n B_n},$$

where $M_{\widehat{\alpha}}(C^*(E) \rtimes_\alpha \mathbb{T})$ denotes the mapping torus of $\widehat{\alpha}$ on $C^*(E) \rtimes_\alpha \mathbb{T}$ and $M_\beta(C^*(E \times_c \mathbb{Z}))$ denotes the mapping torus of β on $C^*(E \times_c \mathbb{Z})$. $\square$

4. The structure of the generic crossed product

Below we introduce the local approximation property and show in Proposition 4.2, the periodic fibres satisfy a local approximation property.

Definition 4.1. We say that a C^*-algebra A has the local approximation property with respect to the class of C^*-algebras $\mathfrak{C}$ if, for every finite set F of elements of A and every $\epsilon > 0$, there is a $C \in \mathfrak{C}$ and a $*$-homomorphism $\phi : C \to A$ such that each element of F lies within ϵ of the image of ϕ.

Given a finite graph $E = (E^0, E^1, r, s)$ with edges $E^1 = \{e_1, e_2, \ldots, e_m\}$ we write $L^m := \{(\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_m}) \in \mathbb{Q}^m : \lambda$ is a labeling map on E^* and $\lambda_\mu \neq 0$ for any cycle $\mu \in E^*\}$.

Proposition 4.2. *Let E be a finite graph with edges $E^1 = \{e_1, e_2, \ldots, e_m\}$ and $\lambda_0 := (\lambda_{e_1}, \lambda_{e_2}, \ldots, \lambda_{e_m}) \in L^m$ so that α^{λ_0} is a periodic action of $\mathbb{R}$ on $C^*(E)$. Suppose further that $\epsilon > 0$ and $f_1, \ldots, f_n \in \mathrm{C}_c(\mathbb{R}, C^*(E))$. Then, there exists a neighbourhood U of λ_0 in $\mathbb{R}^m$, a non-unital one-dimensional NCCW-complex A, and for every $s \in U$, a $*$-homomorphism $\psi_s : A \to C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ such that $\{\varphi_s(f_1), \ldots, \varphi_s(f_n)\} \subseteq_\epsilon \psi_s(A)$, where φ_s denotes the canonical inclusion of $\mathrm{C}_c(\mathbb{R}, C^*(E))$ into $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$.*

Proof. Suppose $\lambda_0 \in \mathbb{R}^m$, $\epsilon > 0$ and $f_1, \ldots, f_n \in \mathrm{C}_c(\mathbb{R}, C^*(E))$. Let φ_{λ_0} denote the canonical inclusion of $\mathrm{C}_c(\mathbb{R}, C^*(E))$ into $C^*(E) \rtimes_{\alpha^{\lambda_0}} \mathbb{R}$. By Proposition 3.1, $C^*(E) \rtimes_{\lambda_0} \mathbb{R} \cong \overline{\bigcup_n B_n}$, where B_n are non-unital one-dimensional NCCW complexes. Thus, by choosing n large enough, there exists a $*$-homomorphism $\psi : B_n \to C^*(E) \rtimes_{\lambda_0} \mathbb{R}$ such that

$$\{\varphi_{\lambda_0}(f_1), \varphi_{\lambda_0}(f_2), \ldots, \varphi_{\lambda_0}(f_n)\} \subseteq_{\epsilon/2} C^*(E) \rtimes_{\lambda_0} \mathbb{R}.$$

The rest follows from [Dea01, Lemma 4.8]. □

As in [Dea01], we use stable relations and a Baire category argument to show that for a dense G_δ set of labels, the associated crossed products satisfy the local approximation property. Since they satisfy a local approximation property, they can be written as inductive limits of one-dimensional NCCW complexes by [San15, Proposition 6 (xiii)].

Theorem 4.3. (Finite Graph Case) *Let E be a finite graph with edges $E^1 = \{e_1, e_2, \ldots, e_m\}$. Then, the set of points $\lambda \in \overline{L^m}$, for which $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of one-dimensional NCCW-complexes contains a dense G_δ set in $\overline{L^m}$. For such λ, the crossed product $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is AF embeddable.*

Proof. The first statement follows from the same argument as in Theorem 4.10 of [Dea01]. As in the proof of [Dea01, Theorem 4.10], we need to show that the local approximation property with respect to the class of one-dimensional NCCW-complexes holds for such a set. Pick a countable dense subset of $C_c(\mathbb{R}, C^*(E))$ and call it $\mathfrak{D}$. Let φ_s be the canonical inclusion of $C_c(\mathbb{R}, C^*(E))$ into $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$. From Proposition 4.2 above, for each finite subset $F \subset \mathfrak{D}$, $\epsilon > 0$ and $\lambda \in L^m$, there is a neighbourhood $V(\lambda, F, \epsilon)$ of λ, a one-dimensional NCCW-complex $B(\lambda, F, \epsilon)$, and for every $s \in V(\lambda, F, \epsilon)$, a $*$-homomorphism $\psi(\lambda, s, F, \epsilon) : B(\lambda, F, \epsilon) \to C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ such that $\varphi_s(F) \subseteq_\epsilon \psi(\lambda, s, F, \epsilon) B(\lambda, F, \epsilon)$. Let $G(\epsilon, F) = \bigcup_{\lambda \in L^m} V(\lambda, F, \epsilon)$. Then, for every s in $G(\epsilon, F)$, $\varphi_s(F)$ is approximately contained to within

ϵ by the image of a one-dimensional NCCW complex. Also, $G(\epsilon, F)$ contains a dense open set in $\overline{L^m}$. Let ϵ_n be a sequence of positive numbers converging to zero and let $\mathfrak{F}(\mathfrak{D})$ denote the set of finite subsets of $\mathfrak{D}$. Then the set $G = \bigcap_{F\in\mathfrak{F}(\mathfrak{C})} \bigcap_{\epsilon_n} G(\epsilon_n, F)$ is contained in the set of points s in $\overline{L^m}$ for which $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ has the local approximation property and G clearly contains a dense G_δ set in $\overline{L^m}$. Since one-dimensional NCCW complexes are subhomogenous algebras, we have that $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ is an approximately subhomogenous (ASH) algebra for all $s \in G$. By Proposition 8.5.1 in [BO08], we have that $C^*(E) \rtimes_{\alpha^s} \mathbb{R}$ is AF embeddable for all $s \in G$. □

Remark 4.4. The condition on the labels comes from the fact that the skew product graph is AF. In [Dea01, Theorem 4.10], Dean scaled the action and made $\lambda_1 = 1$. In this case, the labels are all positive since the skew-product $O_n \times_c \mathbb{Z}$ is AF if and only if $c(E^1) \subseteq \mathbb{Z}^+$ or $c(E^1) \subseteq \mathbb{Z}^-$.

We can now recover Theorem 5.1 in [Dea01].

Corollary 4.5. *The set of points*

$$(1, \lambda_2, \ldots, \lambda_n) \in (0, \infty)^{n-1},$$

for which $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of non-unital one-dimensional NCCW-complexes, contains a dense G_δ set.

Proof. Since $\lambda_1 = 1$ we have that $L^{n-1} = (\mathbb{Q}^+)^{n-1}$ and thus $\overline{L^{n-1}} = [0, \infty)^{n-1}$. The rest follows from Theorem 4.3. □

Remark 4.6. Theorem 5.1 of [Dea01] assumes $\lambda_1 = 1$ by rescaling. In this case, we get a continuous field over $\mathbb{R}^{n-1}$ with fibres $O_n \rtimes_{\alpha^\lambda} \mathbb{R}$, where $\lambda = (1, \lambda_2, \ldots, \lambda_n)$. It was not necessary for us to rescale, as in our case, we can obtain a continuous field over $\mathbb{R}^n$ instead. Then, L^n will be a larger set than $(\mathbb{Q}^+)^n$.

The next proposition shows that a large family of crossed products are stably projectionless.

Proposition 4.7. *Let E be a finite graph that is cofinal and in which every cycle has an exit. If E contains a strongly connected component that is not a single cycle and $\lambda : E^1 \to \mathbb{R}$ is a labeling map with $\lambda_e > 0$ for all $e \in E^1$, then $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is stably projectionless.*

Proof. We note that $C^*(E)$ is a simple unital C^*-algebra. Since E is a graph having a strongly connected component that is not a single cycle, there exists a $\beta > 0$, with $\rho(C_\beta) = 1$ [Chl16, Proposition 4.3, 4.4]. By [Chl16, Corollary 4.5] , there exists a KMS_β state with $\beta \neq 0$. By [KisK96, Corollary 3.4], $C^*(E) \rtimes_{\alpha^\omega} \mathbb{R}$ is stably projectionless. □

Let $E = (E^0, E^1, r, s)$ be an infinite graph with edges $E^1 = \{e_1, e_2, \ldots\}$. Write $L^\infty := \{(\lambda_{e_1}, \lambda_{e_2}, \ldots) \in \mathbb{Q}^\infty : \lambda$ is a labeling map on E^* and $\lambda_\mu \neq 0$ for any cycle $\mu \in E^*\}$ equipped with the product topology on $\mathbb{R}^\infty$.

Theorem 4.8. (Infinite Graph Case) *Let E be an row-finite graph. Then, the set of points $\lambda \in \overline{L^\infty}$ for which $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of one-dimensional NCCW-complexes contains a dense G_δ set. For such λ, the crossed product $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is AF embeddable.*

Proof. Using Proposition 2.4 , we can construct a sequence of finite subgraphs $F_1 \subseteq F_2 \subseteq \cdots \subseteq E$ with $E = \cup_{n=1}^\infty F_n$, so that $C^*(E)$ is an inductive limit of finite graph algebras $C^*(F_n)$ that are invariant under α. Hence, $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R} = \varinjlim C^*(F_n) \rtimes_{\alpha^{\lambda(n)}} \mathbb{R}$. Let $\{s_k\}_{k=1}^\infty$ be a strictly increasing sequence with $|F_n^1| = s_n$. By Theorem 4.3, we have that $C^*(F_n) \rtimes_{\alpha^{\lambda(n)}} \mathbb{R}$ is an inductive limit of one-dimensional NCCW complexes for all $\lambda(n) \in G(n)$, where $G(n)$ contains a dense G_δ set in $\overline{L^{s_n}}$. Let $L_{s_n}^\infty := \{(\lambda_{e_{s_{n+1}}}, \lambda_{e_{s_{n+2}}}, \dots) \in \mathbb{Q}^\infty : \lambda$ is a labeling map and $\lambda_\mu \neq 0$for any cycle $\mu \in E^*\}$. Then, $G(n) \times \overline{L_{s_n}^\infty}$ contains a dense G_δ set in $\overline{L^\infty}$ and so does $G := \bigcap_n G(n) \times \overline{L_{s_n}^\infty}$. We have that $C^*(E) \rtimes_{\alpha^\lambda} \mathbb{R}$ is an inductive limit of one-dimensional NCCW complexes for all $\lambda \in G$. $\square$

5. *K*-theory: rational fibres

In [Rob12], a classification result was obtained for C^*-algebras that are stably isomorphic to inductive limits of one-dimensional NCCW complexes with trivial K_1 group. There are many examples of crossed products of graph algebras by quasi-free actions that are not classified under the results of [Rob12].

Based on the proof of Proposition 3.1, the rational fibres are mapping tori over skew-graph algebras that are inductive limits of one-dimensional NCCW complexes. In this section, the K-theory of these skew-graph algebras is computed as well as the ordered K_0 group of the skew-graph algebras $C^*(\tilde{O}_n \times_c \mathbb{Z})$.

Lemma 5.1. *Suppose G is a countable abelian group. Then, $E \times_c G \cong E \times_{-c} G$.*

Proof. Let $\phi^0 : (E \times_c G)^0 \to (E \times_{-c} G)^0$ be defined by $\phi^0(v, n) = (v, -n)$ and $\phi^1 : (E \times_c G)^1 \to (E \times_{-c} G)^1$ be defined by $\phi^0(e, n) = (e, -n)$. Then, ϕ^0 and ϕ^1 are bijective maps that satisfy $r_{E\times_{-c}G} \circ \phi^1 = \phi^0 \circ r_{E\times_c G}$ and $s_{E\times_{-c}G} \circ \phi^1 = \phi^0 \circ s^0_{E\times_c G}$. $\square$

Theorem 5.2. *Let E be a finite graph without sinks and $\alpha : \mathbb{T} \curvearrowright C^*(E)$ be an action such that $\alpha_z(s_e) = z^{c(e)} s_e$, where $c : E^1 \to \mathbb{Z}$ is a labelling map with $c(E^1) \subseteq \mathbb{Z}^+$ or $c(E^1) \subseteq \mathbb{Z}^-$. If v is not a source, define $M_v := \max\{|c(e)| : r(e) = v\}$. Let $M := |E^0_{\text{sources}}| + \sum_{\{v\in E^0 : r^{-1}(v) \neq \emptyset\}} M_v$. Then,*

$$K_0(C^*(E) \rtimes_\alpha \mathbb{T}) = \varinjlim(\mathbb{Z}^M, B)$$

for some $M \times M$ matrix B and $K_1(C^*(E) \rtimes_\alpha \mathbb{T}) = 0$.

Proof. By Proposition 2.3, we know that $C^*(E) \rtimes_\alpha \mathbb{T}$ is isomorphic to the graph algebra $C^*(E \times_c \mathbb{Z})$, where $E \times_c \mathbb{Z} = ((E \times_c \mathbb{Z})^0, (E \times_c \mathbb{Z})^1, r, s)$. By Lemma 5.1, we may assume $c(E^1) \subseteq \mathbb{Z}^+$.

Let $\{s_{(e,k)}, p_{(v,k)} : k \in \mathbb{Z}, e \in E^1, v \in E^0\}$ be the canonical Cuntz-Krieger family generating $C^*(E \times_c \mathbb{Z})$. Since the skew product has no cycles, $C^*(E \times_c \mathbb{Z})$ is AF.

Let $V_m = \{(v, k) : v \in E^0, k \in \mathbb{Z} \text{ and } -\infty \leq k \leq m\}$ for $m \geq 0$. For $m \geq 1$, define F_m to be the subgraph of $E \times_c \mathbb{Z}$ with vertices

$$F_m^0 = V_m \cup \{(v, m+1), \ldots, (v, m-1+M_v) : v \in E^0, r^{-1}(v) \neq \emptyset\}$$

and edges

$$F_m^1 := s^{-1}(V_{m-1}).$$

We have that each F_m is a graph without loops, where $F_m \subseteq F_{m+1}$ for $m \geq 1$ and $E \times_c \mathbb{Z} = \bigcup_{n=1}^\infty F_m$. Let A_m denote the C^*-subalgebra of $C^*(E \times_c \mathbb{Z})$ generated by $\{s_{(e,\ell)} : (e, \ell) \in F_m^1\} \cup \{p_{(v,n)} : (v, n) \in F_m^0\}$. The generating set for A_m is a Cuntz-Krieger F_m family in $C^*(E \times_c \mathbb{Z})$ with all projections nonzero. Hence, by the Cuntz-Krieger uniqueness theorem, there is an injection of $C^*(F_m)$ into $C^*(E \times_c \mathbb{Z})$ and this map gives $C^*(F_m) \cong A_m$. Thus, $C^*(F_m) \subseteq C^*(F_{m+1})$ and $C^*(E \times_c \mathbb{Z}) = \overline{\bigcup_{m=1}^\infty C^*(F_m)}$.

A typical element in the spanning set for $C^*(F_m)$ is $s_\mu s_\nu^*$ with $r(\mu) = r(\nu)$. Suppose $r(\mu) = r(\nu) = (w, k)$. If (w, k) is not a sink, we can apply the Cuntz-Krieger relations, so that $s_\mu s_\nu^*$ can be written as a finite sum of terms of the form $s_\alpha s_\beta^*$, where $r(\alpha) = r(\beta)$ is a sink. The set of all sinks in the graph F_m, denoted by S_{F_m}, is the set

$$\{(v, m) : v \in E^0\} \cup \{(v, m+1), \ldots, (v, m-1+M_v) : v \in E^0, r^{-1}(v) \neq \emptyset\}.$$

Therefore,

$$C^*(F_m) = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) \in S_{F_m}\}.$$

Fix an element $(v, k) \in S_{F_m}$. Let

$$F_m^{(v,k)} = \{\alpha \in F_m^* : r(\alpha) = (v, k)\}$$

and

$$A_{(v,k)} = \overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in F_m^{(v,k)}\}.$$

The elements $s_\alpha s_\beta^*$ with $\alpha, \beta \in F_m^{(v,k)}$ form a family of matrix units, and thus, $A_{(v,k)}$ is isomorphic to the algebra of compact operators on $\ell^2(F_m^{(v,k)})$. For any two elements $(v, k), (w, n) \in S_{F_m}$, $A_{(v,k)}$ is orthogonal to $A_{(w,n)}$ when $(v, k) \neq (w, n)$. Hence,

$$C^*(F_m) = \bigoplus_{(v,k) \in S_{F_m}} A_{(v,k)}.$$

Since $p_{(v,k)}$ is a rank one projection in $A_{(v,k)}$, we have that $K_0(A_{(v,k)})$ is a free abelian group generated by $[p_{(v,k)}]$. Therefore,

$$\begin{aligned} K_0(C^*(F_m)) &= K_0\left(\bigoplus_{(v,k)\in S_{F_m}} A_{(v,k)}\right) \\ &= \bigoplus_{(v,k)\in S_{F_m}} K_0(A_{(v,k)}) \\ &= \bigoplus_{(v,k)\in S_{F_m}} \mathbb{Z}[p_{(v,k)}]. \end{aligned}$$

Continuity of K_0 gives $K_0(C^*(E\times_c\mathbb{Z})) = \varinjlim K_0(C^*(F_m))$.

To calculate the bonding maps $\phi_{m,m+1} : K_0(C^*(F_m)) \longrightarrow K_0(C^*(F_{m+1}))$, we will see how the projections $[p_{(v,k)}]$, with $(v,k)\in S_{F_m}$, decompose in $K_0(C^*(F_{m+1}))$. If $v\in E^0$, then

$$\begin{aligned} [p_{(v,m)}] &= \sum_{s(e,n)=(v,m)} [s_{(e,n)}s^*_{(e,n)}] \\ &= \sum_{s(e,n)=(v,m)} [p_{r(e,n)}] \\ &= \sum_{s(e)=v} [p_{(r(e),m+c(e))}], \end{aligned}$$

where $m < m + c(e) \le m + M_{r(e)}$.

Since $S_{F_m}\cap S_{F_{m+1}} = S_{F_m}\backslash\{(v,m) : v\in E^0\}$, we have that

$$[p_{(v,k)}] \xmapsto{\phi_{m,m+1}} \begin{cases} [p_{(v,k)}] & \text{if } (v,k)\in S_{F_m}\backslash\{(v,m) : v\in E^0\} \\ \sum_{s(e)=v}[p_{(r(e),m+c(e))}] & \text{otherwise.} \end{cases}$$

We note that

$$\begin{aligned} |S_{F_m}| &= |E^0| + |F_m^0\backslash V_m| \\ &= |E^0| + \sum_{\{v\in E^0 : r^{-1}(v)\neq\emptyset\}} (M_v - 1) \\ &= |E^0| + \left(\sum_{\{v\in E^0 : r^{-1}(v)\neq\emptyset\}} M_v\right) - |E^0\backslash E^0_{\text{sources}}| \\ &= |E^0_{\text{sources}}| + \sum_{\{v\in E^0 : r^{-1}(v)\neq\emptyset\}} M_v \\ &= M. \end{aligned}$$

The matrix representations of the bonding maps $\phi_{m,m+1}$ are all the same and we will denote them by B. Hence, $K_0(C^*(E\times_c\mathbb{Z})) \cong \varinjlim(\mathbb{Z}^M, B)$. □

As a consequence of Theorem 5.2, we can now recover the result for the gauge action (see Corollary 7.14 of [Rae05]).

Corollary 5.3. *Let $C^*(E)$ be a row-finite graph without sinks and let $\gamma : \mathbb{T} \curvearrowright C^*(E)$ be the standard gauge action. Then, $K_0(C^*(E) \rtimes_\gamma \mathbb{T}) = \varinjlim(\mathbb{Z}^{E^0}, A_E^t)$, where A_E is the vertex matrix.*

Proof. For the standard gauge action, we have $c(e) = 1$ for all edges $e \in E^1$. Thus, $C^*(E) \rtimes_\gamma \mathbb{T} \cong C^*(E \times_1 \mathbb{Z})$. Using Theorem 5.2, we see that $F_m^0 = V_m$, $F_m^1 := s^{-1}(V_{m-1})$ and $S_{F_m} := \{(v, m) : v \in E^0\}$. We note $M = |E^0|$ and for all $v \in E^0$, we have that

$$\begin{aligned}\phi_{m,m+1}([p_{(v,m)}]) &= \sum_{s(e)=v} [p_{(r(e),m+1))}] \\ &= \sum_{w \in E^0} A_E(v, w)[p_{(w,m+1))}].\end{aligned}$$

Hence, in this case, the bonding map is multiplication by the transpose of the vertex matrix, as required. □

The Cuntz Algebra Case

Let O_n be the Cuntz algebra with corresponding graph $\tilde{O}_n$ having vertex v and edges $\{e_i\}_{i=1}^n$. The skew-product graph algebra $C^*(\tilde{O}_n \times_c \mathbb{Z})$ is AF if and only if $c(E^1) \subseteq \mathbb{Z}^+$ or $c(E^1) \subseteq \mathbb{Z}^-$. Without loss of generality, we assume $c(E^1) \subseteq \mathbb{Z}^+$. Suppose we have $s \leq n$ distinct labels; namely, $k_1 < k_2 < \cdots < k_s$.

Here, we note that $k_s = M$. For all $j = 1, 2, \ldots, k_s$, define

$$c_j := \begin{cases} |\{e \in E^1 : c(e) = j\}| & \text{if } j \in \{k_1, k_2, \ldots, k_s\} \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem 5.2, we have $S_{F_m} = \{(v, m), (v, m+1), \ldots, (v, m-1+M_v)\}$ and the bonding maps $\phi_{m,m+1}$ send

$$\begin{aligned}[p_{(v,m)}] \longmapsto & \sum_{s(e)=v} [p_{(r(e),m+c(e))}] \\ &= \sum_{i=1}^n [p_{(v,m+c(e_i))}] \\ &= \sum_{i=1}^s c_{k_i} [p_{(v,m+k_i)}],\end{aligned}$$

while the remaining elements remain fixed under $\phi_{m,m+1}$. Hence,

$$B = \begin{pmatrix} c_1 & 1 & 0 & \cdots & 0 \\ c_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{k_s-1} & 0 & 0 & \cdots & 1 \\ c_{k_s} & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{1}$$

is the matrix representation of the bonding maps $\phi_{m,m+1}$. Therefore, we have that $K_0(C^*(\tilde{O}_n \times_c \mathbb{Z})) \cong \varinjlim(\mathbb{Z}^{M_v}, B)$.

Also, the determinant of B is $c_{k_s}(-1)^{k_s+1} \neq 0$. So, if we suppose that $c_{k_s} = 1$, then the bonding maps are bijective and in this case, $K_0(C^*(\tilde{O}_n \times_c \mathbb{Z})) \cong \mathbb{Z}^{M_v}$.

Next, we consider the positive cones of the K_0 groups. A matrix A is called *unimodular* if the determinant of A is $+1$ or -1 and *primitive* if A is nonnegative and $A^m > 0$ for some positive integer m, where $B > 0$ means $b_{ij} > 0$ for all i, j. The bonding maps in Proposition 5.4 are nonnegative unimodular primitive matrices in $\mathrm{M}_k(\mathbb{Z})$.

Proposition 5.4. [She81, p. 464] *Suppose we are given a sequence*

$$\mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^k \xrightarrow{A} \cdots$$

where A is a nonnegative unimodular primitive matrix in $\mathrm{M}_k(\mathbb{Z})$. Then, the resulting stationary dimension group $\varinjlim(\mathbb{Z}^k, A)$ has a unique state, and we can express its positive cone as

$$P_{(1,\alpha_2,\ldots,\alpha_n)} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^k : x_1 + \alpha_2 x_2 + \cdots \alpha_n x_n > 0\} \cup \{(0, \ldots, 0)\},$$

where $(1, \alpha_2, \ldots, \alpha_n)$ is the eigenvector of the Perron-Frobenius eigenvalue of A^{tr}, at least one of α_i is irrational, and $\alpha_2, \ldots, \alpha_n > 0$.

For the rest of the section, we will suppose that $s = n$ and $\gcd(k_1, \ldots, k_n) = 1$. Then $c_{k_1} = c_{k_2} = \cdots c_{k_n} = 1$ and $c_j = 0$ otherwise. We will use the notation $O_n(k_1, k_2, \ldots, k_n)$ to represent $C^*(\tilde{O}_n \times_c \mathbb{Z})$, where c is a labeling map with distinct labels $k_n > k_{n-1} > \cdots > k_1 > 0$. We denote the transpose of the matrix B in (1) as

$$A_{(k_1,\ldots,k_n)} = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{k_n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

This matrix is known in the literature as the Leslie matrix [CFR05, p. 140]. The matrix $A_{(k_1,\ldots,k_n)}$ has characteristic polynomial

$$x^{k_n} - x^{k_n - k_1} - \cdots - x^{k_n - k_{n-1}} - 1.$$

In order to apply Proposition 5.4, a description of the Perron-Frobenius eigenvalue and its corresponding eigenvector for $A_{(k_1,k_2,\ldots,k_n)}$ is needed. This is described in Lemma 5.5, along with the fact that α is the limit of a ratio of terms from a difference equation. The description of α in this way was given in [Flo11], but with more cumbersome calculations as the size of the matrix increased (for example, [Flo11, p.26]). The proof of Lemma 5.5 makes use of some standard results from matrix theory.

Lemma 5.5. *The matrix $A_{(k_1,k_2,\ldots,k_n)}$ has eigenvector $(1,\alpha^{-1},\ldots,\alpha^{-k_n+1})^{tr}$, where α is the Perron-Frobenius eigenvalue of $A_{(k_1,k_2,\ldots,k_n)}$ and is irrational. It satisfies $\alpha = \lim_{m\to\infty}\dfrac{f_{m+k_n}}{f_{m+k_n-1}}$, where*

$$f_{m+k_n} = f_{m+k_n-k_1} + f_{m+k_n-k_2} + \cdots + f_m$$

is a difference equation with initial conditions $f_0 = f_1 = \cdots = f_{k_n-2} = 0$ and $f_{k_n-1} = 1$.

Proof. The matrix $A_{(k_1,k_2,\ldots,k_n)}$ yields a difference equation of the form

$$f_{m+1} = f_{m-k_1+1} + f_{m-k_2+1} + \cdots + f_{m-k_n+1},$$

with initial conditions $f_0 = f_1 = \cdots = f_{k_n-2} = 0$ and $f_{k_n-1} = 1$ or equivalently, in matrix form

$$\begin{pmatrix} f_{m+k_n} \\ \vdots \\ f_{m+2} \\ f_{m+1} \end{pmatrix} = A_{(k_1,k_2,\ldots,k_n)} \begin{pmatrix} f_{m+k_n-1} \\ \vdots \\ f_{m+1} \\ f_m \end{pmatrix}, \qquad \begin{pmatrix} f_{k_n-1} \\ f_{k_n-2} \\ \vdots \\ f_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

(see [Mey00, pp. 683–684]). If we let $\mathbf{g}(m) = \begin{pmatrix} f_{m+k_n-1} \\ \vdots \\ f_{m+1} \\ f_m \end{pmatrix}$, then it is not hard to see that $\mathbf{g}(m) = A^m_{(k_1,k_2,\ldots,k_n)}\mathbf{g}(0)$.

The matrix $A_{(k_1,k_2,\ldots,k_n)}$ is primitive if and only if the $\gcd(k_1,\ldots,k_n) = 1$ (see, for example, Theroem 6.11 in [CFR05]). If $r = \rho(A_{(k_1,k_2,\ldots,k_n)})$, then $\lim_{m\to\infty}\left(\dfrac{A_{(k_1,k_2,\ldots,k_n)}}{r}\right)^m = \dfrac{\mathbf{p}\mathbf{q}^{tr}}{\mathbf{q}^{tr}\mathbf{p}} > 0$, where $\mathbf{p}$ and $\mathbf{q}$ are the Perron-Frobenius eigenvectors of $A_{(k_1,k_2,\ldots,k_n)}$ and $A^{tr}_{(k_1,k_2,\ldots,k_n)}$, respectively [Mey00, p. 674]. From this, we get that

$$\lim_{m\to\infty}\frac{\mathbf{g}(m)}{||\mathbf{g}(m)||_1} = \mathbf{p},$$

where $\mathbf{p}$ is the Perron-Frobenius eigenvector of $A_{(k_1,k_2,\ldots,k_n)}$ (see [Mey00, p. 684]). For $0 \le q \le k_n - 1$, $\lim_{m\to\infty}\dfrac{f_{m+q}}{||\mathbf{g}(m)||_1}$ exists and is positive. Hence, so is $\lim_{m\to\infty} \dfrac{f_{m+q}}{f_{m+k_n-1}}$. Since

$$\frac{f_{m+k_n}}{f_{m+k_n-1}} = \frac{f_{m+k_n-k_1}}{f_{m+k_n-1}} + \frac{f_{m+k_n-k_2}}{f_{m+k_n-1}} + \cdots + \frac{f_m}{f_{m+k_n-1}}, \tag{2}$$

we have that $\lim_{m\to\infty} \dfrac{f_{m+k_n}}{f_{m+k_n-1}}$ exists and we will denote it by α.

Taking the limit of both sides of equation (2), we get $\alpha = \alpha^{-(k_1-1)} + \alpha^{-(k_2-1)} + \cdots + \alpha^{-(k_n-1)}$, or equivalently,

$$\alpha^{k_n} - \alpha^{k_n-k_1} - \alpha^{k_n-k_2} - \cdots - 1 = 0.$$

Therefore, α satisfies the characteristic polynomial of $A_{(k_1,k_2,\ldots,k_n)}$. By Descartes' rule of signs, the characteristic polynomial has one positive root and since α is positive, it must be the Perron-Frobenius eigenvalue.

Lastly, we have that

$$\begin{aligned}\begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{k_n} \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \alpha^{-1} \\ \alpha^{-2} \\ \vdots \\ \alpha^{-k_n+1} \end{pmatrix} &= \begin{pmatrix} \alpha^{-k_1+1} + \alpha^{-k_2+1} + \cdots + \alpha^{-k_n+1} \\ 1 \\ \alpha^{-1} \\ \vdots \\ \alpha^{-k_n} \end{pmatrix} \\ &= \alpha \begin{pmatrix} 1 \\ \alpha^{-1} \\ \alpha^{-2} \\ \vdots \\ \alpha^{-k_n+1} \end{pmatrix}.\end{aligned}$$

The only possible rational root of the characteristic polynomial is -1, hence α must be irrational. ∎

Theorem 5.6. *Let $\mathbf{v} = (1, \alpha^{-1}, \ldots, \alpha^{-k_n+1})^{tr}$ be the eigenvector of $A_{(k_1,k_2,\ldots,k_n)}$, where α is the corresponding Perron-Frobenius eigenvalue. Then,*

$$K_0(O_n(k_1,\ldots,k_n)) = \mathbb{Z}^{k_n}$$

and

$$K_0^+(O_n(k_1,\ldots,k_n)) = P_{\mathbf{v}},$$

where

$$\begin{aligned} P_{\mathbf{v}} = \{&(x_1,\ldots,x_{k_n}) \in \mathbb{Z}^{k_n} : x_1 + \alpha^{-1}x_2 + \cdots + \alpha^{-k_n+1}x_{k_n} > 0\} \\ &\cup \{(0,0,\ldots,0)\}. \end{aligned}$$

Furthermore, if the characteristic polynomial of $A_{(k_1,k_2,\ldots,k_n)}$ is irreducible over the rationals, then

$$\Big(K_0(O_n(k_1,k_2,\ldots,k_n)), K_0^+(O_n(k_1,k_2,\ldots,k_n)), \begin{pmatrix}1 & 0 & 0 & \cdots & 0\end{pmatrix}^{tr}\Big)$$
$$\cong (\mathbb{Z}+\alpha^{-1}\mathbb{Z}+\cdots+\alpha^{-k_n+1}\mathbb{Z}, (\mathbb{Z}+\alpha^{-1}\mathbb{Z}+\cdots+\alpha^{-k_n+1}\mathbb{Z})\cap\mathbb{R}_+, 1).$$

Proof. By Theorem 5.2, Lemma 5.5 and Proposition 5.4, we have the K_0 group and its cone are described as above. The map $(1,\alpha^{-1},\ldots,\alpha^{-k_n+1}) : \mathbb{Z}^{k_n} \to \mathbb{Z}+\alpha^{-1}\mathbb{Z}+\cdots+\alpha^{-k_n+1}\mathbb{Z}$ is a positive surjective homomorphism that preserves the order unit and the image of the cone $K_0^+(O_n(k_1,k_2,\ldots,k_n))$ is exactly $(\mathbb{Z}+\alpha^{-1}\mathbb{Z}+\cdots+\alpha^{-k_n+1}\mathbb{Z})\cap\mathbb{R}_+$. Furthermore, if the characteristic polynomial is irreducible, then the map $(1,\alpha^{-1},\ldots,\alpha^{-k_n+1})$ is injective since the set $\{1,\alpha^{-1},\ldots,\alpha^{-k_n+1}\}$ is linearly independent. Indeed, the set $\{1,\alpha,\ldots,\alpha^{k_n-1}\}$ is linearly independent since the characteristic polynomial is irreducible and therefore, so is the set $\{1,\alpha^{-1},\ldots,\alpha^{-k_n+1}\}$. Hence, we have an order isomorphism, as required. $\square$

Remark 5.7. $K_0(O_n(k_1,k_2,\ldots,k_n))$ is not totally ordered when the number of nonzero even entries is one greater than the number of nonzero odd entries in the first row of the bonding maps $A_{(k_1,k_2\ldots,k_n)}$, since the characteristic polynomial will have -1 as a root (see [Han81, pp. 63–64]).

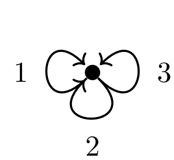
The K-theory of the standard Fibonacci algebra was calculated in [Dav96] and extended for the generalized Fibonacci algebras in [Flo11]. The standard embedding was given by the matrix $A_{(k_1,k_2,\ldots,k_n)}$, where $k_j = j$ for $j = 1,\ldots,n$. As a consequence of Theorem 5.6, we arrive at the same results, but in a more indirect way.

Corollary 5.8. *Suppose $k_j = j$ for $j = 1,2,\ldots,n$. Then,*

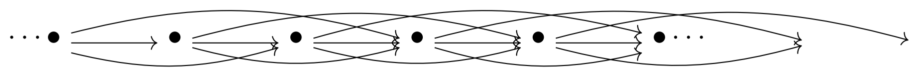
$$\Big(K_0(O_n(k_1,k_2,\ldots,k_n)), K_0^+(O_n(k_1,k_2,\ldots,k_n)), \begin{pmatrix}1 & 0 & 0 & \cdots & 0\end{pmatrix}^{tr}\Big)$$
$$\cong (\mathbb{Z}+\alpha^{-1}\mathbb{Z}+\cdots+\alpha^{-k_n+1}\mathbb{Z}, (\mathbb{Z}+\alpha^{-1}\mathbb{Z}+\cdots+\alpha^{-k_n+1}\mathbb{Z})\cap\mathbb{R}_+, 1).$$

Proof. By [Bra51, Theorem 2], the characteristic polynomial of $A_{(k_1,k_2\ldots,k_n)}$ is irreducible over the rationals. Then, the result follows from Theorem 5.6. $\square$

Example 5.9. Suppose we have the Cuntz-algebra O_3 with the following labels:



Then, we have the following skew-product graph:



Hence, $K_0(O_3(1,2,3)) = \varinjlim \left(\mathbb{Z}^3, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right) \cong \mathbb{Z}^3$. In this case, $\alpha \approx 1.83929$ is the Perron-Frobenus eigenvalue of the above bonding map and

$$\begin{aligned} &\Big(K_0(O_3(1,2,3)), K_0^+(O_3(1,2,3)), \begin{pmatrix}1 & 0 & 0 & \cdots & 0\end{pmatrix}^{tr}\Big) \\ &\cong (\mathbb{Z}+\alpha^{-1}\mathbb{Z}+\alpha^{-2}\mathbb{Z}, (\mathbb{Z}+\alpha^{-1}\mathbb{Z}+\alpha^{-2}\mathbb{Z})\cap\mathbb{R}_+, 1). \end{aligned}$$

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