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Connectedness and Lusternik-Schnirelmann categories of the spaces of persistence modules

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Abstract. The classes of various interval decomposable persistence modules in literature were analyzed, the sets were determined, and some topological characteristics that these sets gained through interleaving metric were studied. In this study, connectedness of these spaces and their Lusternik-Schnirelmann categories were considered.

Contents

1. Introduction

In topological data analysis, a persistence module is obtained with applying homology with coefficients in some fixed field to the increasing family of topological spaces or complexes [14]. The distance between two persistence modules can be measured with the interleaving metric [6]. The collection of persistence modules with the interleaving metric fails to be a topological space since it is not a set but a class.

For this, the collections of persistence modules were classified under specific characteristics in [5], and the collections, which constitute sets, were proven. Then the authors restrict themselves to the identified sets together with the interleaving metric in order to study their basic topological properties such as countability, separability, compactness, completeness and path connectedness.

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The majority of the spaces included in this study are not contractible or path-connected. Here, we will first discuss whether these spaces are connected or not. On the other hand, since a Lusternik-Schnirelmann category, or LS-category for short, shows how far a topological space from contractibility, we will calculate LS-categories of these spaces discussed in this study.

This paper is organised as follows. Since this study is sub-study of [5], we will only provide the bare minimum of necessary background in Section 2. We will provide brief definitions of persistence module, interleaving metric, and the persistence module sets included in this study. In Section 3, we will analyze the connectedness of the topologies of these sets, which were gained through interleaving metric, and then we will compute their LS-categories in Section 4.

2. Preliminaries

In this section, we give the concepts of persistence modules, the interleaving metrics and the sets of persistence modules on which we work on.

2.1. Persistence modules. We start with the definition of persistence modules [8, 14] Let $\mathbf{k}$ be a fixed field. A *persistence module* M is a set of $\mathbf{k}$-vector spaces $\{M(a) \mid a \in \mathbb{R}\}$ together with $\mathbf{k}$-linear maps $\{v_a^b : M(a) \to M(b) \mid a \leq b\}$ such that

i): for all a, $v_a^a : M(a) \to M(a)$ is the identity map, and
ii): if $a \leq b \leq c$ then $v_a^c = v_b^c \circ v_a^b$.

The definiton of a persistence module M can be given in terms of functors in a categorical way. Let $\underline{\mathbf{R}}$ be the category whose set of objects is $\mathbb{R}$ and whose morphisms are the inequalities $a \leq b$. Then a persistence module is a functor $M : \underline{\mathbf{R}} \to \underline{\mathrm{Vect}}_{\mathbf{k}}$, where $\underline{\mathrm{Vect}}_{\mathbf{k}}$ is the category of $\mathbf{k}$-vector spaces and $\mathbf{k}$-linear maps. For more details, we refer to [1, 2, 3, 4].

Example 2.1 (Zero persistence module)**.** *The zero persistence module, denoted by* 0*, is a persistence module such that* $0(a) = 0$ *for all* a*.*

Example 2.2. *Consider the interval* $[1, \infty)$ *in* $\mathbb{R}$*. We define the persistence module* $[1, \infty)$ *by*

$$[1,\infty)(a) = \begin{cases} \mathbf{k} & a \in [1,\infty) \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and} \quad [1,\infty)(a \leq b) = \begin{cases} 1 & a, b \in [1,\infty) \\ 0 & \textit{otherwise} \end{cases}$$

where 1 *is the identity map on* $\mathbf{k}$*.*

Generalization of the previous example to an arbitrary interval in $\mathbb{R}$ yields a concept of a persistence modules called *interval modules*. For an interval I in $\mathbb{R}$, we define the persistence module $\mathcal{I}$ given by

$$\mathcal{I}(a) = \begin{cases} \mathbf{k} & a \in I \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{I}(a \leq b) = \begin{cases} 1 & a, b \in I \\ 0 & \text{otherwise} \end{cases}$$

A morphism between two persistence modules can be given in the following way.

Definition 2.3 (Morphism between persistence modules)**.** *Let M and N be the two persistence modules. Then a morphism $\varphi : M \to N$ is a set of* $\mathbf{k}$*-linear maps $\{\varphi_a : M(a) \to N(a) \mid a \in \mathbb{R}\}$ such that the following diagram commutes for each pair $a \leq b$:*

$$\begin{array}{ccc} M(a) & \xrightarrow{v_a^b} & M(b) \\ {\scriptstyle\varphi_a}\downarrow & & \downarrow{\scriptstyle\varphi_b} \\ N(a) & \xrightarrow{w_a^b} & N(b) \end{array} \tag{1}$$

Such a morphism is an isomorphism *if and only if each linear map φ_a is an isomorphism.*

Example 2.4. *Consider the interval modules $\mathcal{I} = [3, \infty)$ and $\mathcal{J} = [2, 8)$. Then a nonzero morphism $\varphi : \mathcal{I} \to \mathcal{J}$ is given by*

$$\varphi(a) = \begin{cases} 1 & a \in [3, 8) \\ 0 & \textit{otherwise} \end{cases}.$$

However, any morphism $\psi : \mathcal{J} \to \mathcal{I}$ is zero. To see this, let $a \in [2, 8)$. Then the diagram

$$\begin{array}{ccc} \mathcal{J}(a) & \longrightarrow & \mathcal{J}(8) \\ {\scriptstyle\psi_a}\downarrow & & \downarrow{\scriptstyle\psi_8} \\ \mathcal{I}(a) & \longrightarrow & \mathcal{I}(8) \end{array} \quad \textit{transforms into} \quad \begin{array}{ccc} \mathbf{k} & \xrightarrow{0} & 0 \\ {\scriptstyle\psi_a}\downarrow & & \downarrow{\scriptstyle\psi_8} \\ \mathcal{I}(a) & \xrightarrow[0\ or\ 1]{} & \mathbf{k} \end{array}$$

and is commutative only if $\psi_a = 0$.

Definition 2.5 (Direct sum of persistence modules)**.** *Let M and N be two persistence modules. Then the direct sum of M and N, $M \oplus N$, is a persistence module such that $(M \oplus N)(a) = M(a) \oplus N(a)$ and $(M \oplus N)(a \leq b) = M(a \leq b) \oplus N(a \leq b)$. If $\{M_i \mid i \in A\}$ is a collection of persistence modules indexed by an arbitrary set A, their direct sum $\oplus_{i\in A} M_i$ is defined in the same way that $(\oplus_{i\in A} M_i)(a) = \oplus_{i\in A} M_i(a)$ and $(\oplus_{i\in A} M_i)(a \leq b) = \oplus_{i\in A} M_i(a \leq b)$.*

A persistence module is said to be *indecomposable* if it is not isomorphic to a nontrivial direct sum. Note that interval modules are indecomposable.

Theorem 2.6 (Structure Theorem)**.** [11, 12] *Let M be a persistence module. If $M(a)$ is finite dimensional for each $a \in \mathbb{R}$, then M is isomorphic to a direct sum of interval modules.*

In this study, we will only focus on persistence modules that are isomorphic to the direct sum of interval modules.

2.2. Sets of persistence modules. We have already mentioned that the class of persistence modules is not a set. Since we first need to have a set in order to talk about topology, we want to restrict ourselves with the sets of persistence modules.

In this paper, we work on the subsets of the class of interval decomposable persistence modules given in [5]. These sets and their abbreviations are as follows:

- (rid) is the set of persistence modules isomorphic to $\bigoplus_{\alpha\in A} I_\alpha$ where the cardinality of the index set A is at most the cardinality of $\mathbb{R}$ and each I_α is an interval module.
- (cid), the *countably interval-decomposable* persistence modules, is the subset of (rid) where the index set A is countable.
- (cfid), the *countably finite-interval decomposable* persistence modules, is the subset of (cid) in which each interval I_α is finite.
- (fid), the *finitely interval-decomposable* persistence modules, is the subset of (cid) where the index set A is finite.
- (ffid), the *finitely finite-interval decomposable* persistence modules, is the subset of (fid) in which each I_k is a finite interval.
- Given $c < d$, ($\mathrm{ffid}^{[c,d]}$) is the subset of (ffid) in which each $I_k \subset [c, d]$.
- (pfd), the *pointwise finite dimensional* persistence modules, is the set of all persistence modules M with each $M(a)$ finite dimensional. By Theorem 2.6, any element in (pfd) are interval decomposable so that (pfd) is the subset of (rid).

The Figure 1 is the Hasse diagram for the sets of persistence modules under consideration in this study [5]. Note that the arrows in this diagram represent the inclusions.

2.3. Interleaving Distance. We now define the interleaving distance between two persistence modules [9]. Note that the interleaving distance is also given from a categorical point of view in [1].

Definition 2.7. *Let M and N be two persistence modules and $\varepsilon \geq 0$. An ε-interleaving between M and N is a pair of morphisms $\varphi_a : M(a) \to N(a+\varepsilon)$ and $\psi_a : N(a) \to M(a+\varepsilon)$ for all a such that the following four diagrams commute for all $a \leq b$, where the horizontal maps are given by the respective persistence modules.*

$$\begin{tikzcd} M(a) \arrow[r] \arrow[dr, "\varphi_a"'] & M(b) \arrow[dr, "\varphi_b"] & \\ & N(a+\varepsilon) \arrow[r] & N(b+\varepsilon) \end{tikzcd} \qquad \begin{tikzcd} & M(a+\varepsilon) \arrow[r] & M(b+\varepsilon) \\ N(a) \arrow[ur, "\psi_a"] \arrow[r] & N(b) \arrow[ur, "\psi_b"'] & \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} M(a) \arrow[rr] \arrow[dr, "\varphi_a"'] & & M(a+2\varepsilon) \\ & N(a+\varepsilon) \arrow[ur, "\psi_{a+\varepsilon}"'] & \end{tikzcd} \qquad \begin{tikzcd} & M(a+\varepsilon) \arrow[dr, "\varphi_{a+\varepsilon}"] & \\ N(a) \arrow[ur, "\psi_a"] \arrow[rr] & & N(a+2\varepsilon) \end{tikzcd} \tag{3}$$

$$\begin{tikzcd}
 & & \text{(rid)} & \\
 & \text{(cid)} \arrow[ur] & & \text{(pfd)} \arrow[ul] \\
\text{(cfid)} \arrow[ur] & & \text{(fid)} \arrow[ul] \arrow[ur] & \\
 & \text{(ffid)} \arrow[ul] \arrow[ur] & & \\
 & \text{(ffid}^{[\text{c,d}]}\text{)} \arrow[u] & &
\end{tikzcd}$$

FIGURE 1. Sets of persistence modules

Note that M and N are isomorphic persistence modules if and only if M and N are 0*-interleaved. Then the* interleaving distance $d_I(M,N)$ *between M and N is defined as*

$$d_I(M,N) := \inf\big(\varepsilon \in [0,\infty) \mid M \textit{ and } N \textit{ are } \ \varepsilon\textit{-interleaved}\ \big)$$

If no such ε exists, then $d_I(M,N)=\infty$.

Example 2.8. *Consider the interval modules $\mathcal{I}=[1,6)$ and $\mathcal{J}=[2,\infty)$. Let ϵ be a nonnegative real number. We'll show that $\mathcal{I}$ and $\mathcal{J}$ can not be ε-interleaved hence $d_I(\mathcal{I},\mathcal{J})=\infty$. If $\varepsilon \leq 4$, then the diagram*

$$\begin{tikzcd}
 & \mathcal{I}(6)=0 \arrow[dr, "0"] & \\
\mathcal{J}(6-\varepsilon)=\mathbf{k} \arrow[ur, "0"] \arrow[rr, "1"] & & \mathcal{J}(6+\varepsilon)=\mathbf{k}
\end{tikzcd}$$

does not commute. If $\varepsilon > 4$, then the diagram

$$\begin{tikzcd}
 & \mathcal{I}(10+\varepsilon)=0 \arrow[dr, "0"] & \\
\mathcal{J}(10)=\mathbf{k} \arrow[ur, "0"] \arrow[rr, "1"] & & \mathcal{J}(10+2\varepsilon)=\mathbf{k}
\end{tikzcd}$$

does not commute.

Example 2.9. *Consider the interval modules $\mathcal{I}_1=[a_1,\infty)$ and $\mathcal{I}_2=[a_2,\infty)$ by assuming $a_1<a_2$. Then for an $\varepsilon \geq |a_1-a_2|$ the diagrams in (2) and (3) commute hence $\mathcal{I}_1$ and $\mathcal{I}_2$ are ε-interleaved. Assume that $\varepsilon < |a_1-a_2|$.*

Then the diagram

$$\begin{tikzcd}
\mathcal{I}_1(a_1) = \mathbf{k} \arrow[rr, "1"] \arrow[dr, "0"'] & & \mathcal{I}_1(a_1 + 2\varepsilon) = \mathbf{k} \\
& \mathcal{I}_2(a_1 + \varepsilon) = 0 \arrow[ur, "0"'] &
\end{tikzcd}$$

does not commute. Hence $d_I(\mathcal{I}_1, \mathcal{I}_2) = |a_1 - a_2|$.

Note that for two families of persistence modules $\{M_i \mid i \in A\}$ and $\{N_i \mid i \in A\}$ indexed by the same set A, we have [8, Proposition 5.5],

$$d_I(\oplus_{i\in A} M_i, \oplus_{i\in A} N_i) \leq \sup_{i\in A} d_I(M_i, N_i).$$

2.4. Pseudometric spaces. The interleaving metric d_I on a set of persistence modules X satisfies the (extended) pseudometric conditions: for persistence modules M, N, K in X,

M1): $d_I(M, M) = 0$
M2): $d_I(M, N) = 0$ implies M and N being isomorphic
M3): $d_I(M, N) = d_I(N, M)$
M4): $d_I(M, N) \leq d_I(M, K) + d_I(K, N)$.

Here, the word "extended" means that d_I may take infinite value.

For any set X of persistence modules, the interleaving distance induces a topology generated by the open balls

$$B_r(x) = \{y \in X \mid d_I(x, y) < r\}$$

where $x \in X$ and $r > 0$ [13].

Thus, each set of persistence modules in Figure 1 is a topological space together with the interleaving metric d_I. Various topological characteristics of these spaces were studied in [5]. In this study, connectedness of these spaces and their LS-categories will be discussed.

Example 2.10. *Consider the subset* $S = \{[a, \infty) \mid a \in \mathbb{R}\}$ *of* (pfd). *For* $\mathcal{I}_1 = [a_1, \infty)$ *and* $\mathcal{I}_1 = [a_2, \infty)$ *in* S, *we have* $d_I(\mathcal{I}_1, \mathcal{I}_2) = |a_1 - a_2|$ *by Example 2.9. The continuous map* $f : S \to \mathbb{R}$ *defined by* $f([a, \infty)) = a$ *has a continuous inverse so that* S *has all topological properties which* $\mathbb{R}$ *with the Euclidean metric has.*

3. Connectedness of the spaces of persistence modules

Note that all but (ffid) and (ffid$^{[c,d]}$) in Figure 1 are not path connected [5, Corollary 8]. In this section we determine whether the spaces are connected. A topological space is said to be *connected* if it is not the union of a pair of disjoint non-empty open sets. Note that the only clopen (both open and closed) subsets in a connected space are the empty set and the space itself [13].

The corollary below follows from the fact the spaces (ffid) and (ffid$^{[c,d]}$) are path connected since they are contractible to the zero module 0 [5, Proposition 15].

Corollary 3.1. (ffid) *and* (ffid$^{[c,d]}$) *are connected.*

We'll show that the rest of the spaces in the Figure 1 are not connected.

Theorem 3.2. (fid) *is not connected.*

Proof. Let M be a persistence module in (fid) and consider $N = M \oplus [a, \infty)$ so that N is also an element of (fid) and $d_I(0, N) = \infty$. Also for any persistence module K in (ffid), $d_I(0, K)$ is finite. This is true since in an extended pseudometric space having an infinite distance between two elements in that space implies that no path can exist between them [5, Lemma 13] while (ffid) is path connected. Thus the triangle inequality $d_I(0, N) \leq d_I(0, K) + d_I(K, N)$ implies $d_I(K, N) = \infty$. This shows that N cannot be in the closure of (ffid): $N \notin \overline{\text{(ffid)}}$. This concludes that any persistence module N in (fid) containing an infinite interval module can not be in the closure of (ffid). Hence $\overline{\text{(ffid)}} = \text{(ffid)}$, so that (ffid) is closed in (fid). By [5, Proposition 9], we know that (ffid) is also an open subset of (fid), hence (ffid) is both open and closed subset of (fid). $\square$

Theorem 3.3. (cid) *is not connected.*

Proof. The proof is the same as the proof of Theorem 3.2 replacing (fid) with (cid) and (ffid) with (cfid). $\square$

Theorem 3.4. (pfd) *is not connected.*

Proof. Consider the path component (hence the connected component) of the zero persistence module 0 in (pfd) and denote this by $\mathcal{O}$. Then any element N in $\mathcal{O}$ is of the form $\oplus_{\alpha \in A} I_\alpha \in$ (pfd) such that $\sup_{\alpha \in A} \text{length}(I_\alpha) < \infty$ [5, Proposition 14]. Note that $d_I(N, 0)$ is finite by [5, Lemma 13] since $\mathcal{O}$ is path connected. We claim that $\mathcal{O}$ is both open and closed in (pfd). Let M be a persistence module in (pfd) which contains an interval module with an infinite length so that $d_I(0, M) = \infty$. Then the triangle inequality $d_I(0, M) \leq d_I(0, N) + d_I(N, M)$ implies $d_I(M, N) = \infty$. Therefore for a persistence module N in $\mathcal{O}$, $B_1(N) \subset \mathcal{O}$. This implies that $\mathcal{O}$ is an open subset in (pfd). Next we show that $\mathcal{O}$ is closed in (pfd). Let $N \in \overline{\mathcal{O}}$. Then $B_\varepsilon(N) \cap \mathcal{O} \neq \emptyset$ for any $\varepsilon > 0$. If $M \in B_\varepsilon(N) \cap \mathcal{O}$, then $d_I(M, N) < \varepsilon$ so that N must be in $\mathcal{O}$. This implies that $\mathcal{O}$ is closed. $\square$

Theorem 3.5. (cfid) *and* (rid) *are not connected.*

Proof. The proof is the same as the proof of Theorem 3.4 replacing (pfd) with (cfid) and (pfd) with (rid). $\square$

4. LS-Categories of the spaces of persistence modules

The *Lusternik-Schnirelmann (or LS-) category* of a topological space X is the least integer n such that there exists an open covering $U_1, U_2, \ldots, U_{n+1}$ of X with each inclusion map $\iota_i : U_i \hookrightarrow X$ is nullhomotopic. We denote the LS-category of X by $\mathrm{cat}(X) = n$ and call such a covering $\{U_i\}$ *categorical*. Write $\mathrm{cat}(X) = \infty$ if no such integer exists [10].

Note that if a topological space X is contractible, then $\mathrm{cat}(X) = 0$. Hence we have the following corollary.

Corollary 4.1. $\mathrm{cat}((\mathrm{ffid})) = 0 = \mathrm{cat}((\mathrm{ffid}^{[c,d]}))$.

Let X be a topological space with $\mathrm{cat}(X) = n$ and U be subset of X in the categorical covering. Then the inclusion map $\iota : U \to X$ induces a path $\alpha : [0,1] \to X$ from any element in U to a fixed element in X defined by $\alpha(t) = H(u,t)$ where H is the homotopy between ι and the constant map at the fixed point. This concludes that there exists a path in X for any pair of elements in U.

Theorem 4.2. *The LS-category of the space* (fid) *is infinite.*

Proof. Consider the persistence modules $N^n = [0,\infty)^n$ for $n \in \mathbb{N}$ in (fid). Since $d_I(N^n, N^k) = \infty$ for $n \neq k$, there exists no path between N^n and N^k by [5, Lemma 13]. Let (fid) have a categorical cover $\{U_i\}$. By the observation given above, N^n and N^k must be contained in a distinct pair of U's for $n \neq k$. This concludes that no such a finite open cover exists for (fid). □

Corollary 4.3. *The LS-categories of the spaces* (pfd)*,* (cid)*, and* (rid) *are infinite.*

Proof. Let X be one of the spaces. Since (fid) is a subset of X, we can choose the interval modules $N^n = [0,\infty)^n$ for $n \in \mathbb{N}$ in the proof of Theorem 4.2 for X and observe that N^n and N^k must be containd in a distinct pair of U's for $n \neq k$. □

Theorem 4.4. *The LS-category of the space* (cfid) *is infinite.*

Proof. Consider the persistence modules $N^i = \bigoplus_{m\in\mathbb{N}}[m, 2^i m)$ for $i \in \mathbb{N}$ in (cfid). Then again, there exists no path between N^i and N^j for $i \neq j$ since $d_I(N^i, N^j) = \infty$. If $\{U_n\}$ is a categorical covering of (cfid), N^i and N^j must be contained in a distinct pair of U's for $i \neq j$. This concludes that no such a finite open cover exists for (cfid). □

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