

On Y -coordinates of Pell equations which are members of a fixed binary recurrence

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ABSTRACT. In this paper, we show that if \mathbf{u} is a fixed binary recurrent sequence of integers whose characteristic equation has real roots and (X_k, Y_k) is the k th solution of the Pell equation $X^2 - dY^2 = 1$ for some non-square integer $d > 1$, the equation $Y_k \in \mathbf{u}$ has at most two positive integer solutions k provided d exceeds some effectively computable number depending on \mathbf{u} .

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1. Introduction

Let $d > 1$ be an integer which is not a square. The Pell equation

$$X^2 - dY^2 = 1$$

has infinitely many positive integer solutions (X, Y) . Furthermore, putting (X_1, Y_1) for the smallest such, all other solutions are of the form (X_m, Y_m) where

$$X_m + \sqrt{d}Y_m = (X_1 + \sqrt{d}Y_1)^m \quad \text{for } m \geq 1.$$

Let U be some interesting set of positive integers like squares, rep-digits in base 10 or in an arbitrary base $g > 1$, Fibonacci numbers, Tribonacci numbers, factorials, etc. In recent papers, the question of determining all positive integers d such that $X_m \in U$ holds for at least two positive integers m has been investigated. In all cases mentioned above, there are only finitely many such d , meaning that with these finitely many exceptions in d , the equation $X^2 - dY^2 = 1$ has at most one positive integer solution (X, Y)

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with $X \in U$ (see [7], [8], [9], [12], [15] and [16]). That this is best possible follows from the fact that if $u \in U \setminus \{1\}$, then $(X, Y) = (u, 1)$ is a solution to $X^2 - dY^2 = 1$ for $d := u^2 - 1$.

In this paper, we investigate the same question for the coordinate Y . Here, it is easy to construct infinitely many d such that $Y_m \in U$ has two solutions m . Namely, assume that $1 \in U$. Take $d = u^2 - 1$, where u will be determined later. Then $(X_1, Y_1) = (u, 1)$ and $(X_2, Y_2) = (2X_1^2 - 1, 2X_1Y_1) = (2u^2 - 1, 2u)$. Hence, if also $2u \in U$, then for this d , we have $Y_m \in U$ for both $m = 1, 2$. Thus, if U contains 1 and infinitely many even numbers, then there are infinitely many d such that $Y_m \in U$ for both $m = 1, 2$. We ask if this is best possible, meaning whether for particular interesting sets of positive integers U , the containment $Y_m \in U$ holds for three or more values of m only for a finite set of d . We mention that the question of how many solutions m does $Y_m \in U$ has been studied before for a few interesting sets U . For example, if U is the set of squares, then Ljunggren [13] showed that there are at most two such m . Further, if U is the set of Y -coordinates of a Pell equation corresponding to the non-square integer $d_1 > 1$, then for any non-square positive integer $d \neq d_1$, the containment $Y_m \in U$ has at most three solutions m . This is a result of Bennett [3] which improved upon a prior result of Masser and Rickert [17] who had proved an upper bound of at most 16 on the number of such solutions m . Finally, if $U = \{2^n - 1 : n \geq 1\}$, then the equation $Y_m \in U$ has at most two solutions m (see [10]).

In this paper, we let r, s be integers and let $\{u_n\}_{n \geq 1}$ be the binary recurrent sequence of recurrence $u_{n+2} = ru_{n+1} + su_n$ for $n \geq 1$ with $u_1, u_2 \in \mathbb{Z}$. Then

$$u_n = a\alpha^n + b\beta^n \quad \text{for all } n \geq 1, \tag{1}$$

where α, β are the roots of the characteristic equation $x^2 - rx - s = 0$ and $a, b \in \mathbb{K} := \mathbb{Q}(\alpha)$ can be determined in terms of u_1, u_2 . We impose that $r^2 + 4s > 0$. In particular, α, β are real. We put $\mathbf{u} := \{u_n : n \geq 1\}$.

Theorem 1.1. *Let $\mathbf{u} := \{u_n\}_{n \geq 1}$ be a binary recurrent sequence whose characteristic equation has real roots. Let $d > 1$ be an integer which not a square and let (X_m, Y_m) be the sequence of positive integer solutions to $X^2 - dY^2 = 1$. Then the equation $Y_m = u_n$ has at most two positive integer solutions (m, n) provided $d > d_0$, where $d_0 := d_0(\mathbf{u})$ is some effectively computable constant depending on \mathbf{u} .*

Before proceeding to the proofs, let us recall a related result of Bennett and Pintér from [4]. Their result is more general but for our problem it implies there exists a computable positive constant $c := c(\mathbf{u})$ depending on \mathbf{u} such that if $Y_1 > d^{c(\log \log d)^3}$, then the equation $Y_m = u_n$ has at most one positive integer solution (m, n) . It is known that $Y_1 < \exp(3\sqrt{d} \log d)$ and it is believed that up to replacing the number 3 above by some smaller number, say $c_2 > 0$, the inequality $Y_1 > \exp(c_2\sqrt{d} \log d)$ holds for infinitely many d . For such d which are large, the Bennett–Pintér condition is satisfied

so for such d the result of Bennet and Pintér is better than ours. However, there are infinitely many d 's for which the above condition is not satisfied, the easiest parametric family of such being $d = k^2 - 1$ for some positive integer k since for those ones $Y_1 = 1$, and from previous remarks it is these d 's that lead to two solutions to the equation $Y_m = u_n$, when $1 \in \mathbf{u}$ and \mathbf{u} contains infinitely many even numbers. However, the result of Bennett and Pintér applied to X -coordinates of Pell equations gives that if d is sufficiently large with respect to \mathbf{u} , then $X_m = u_n$ has at most one solution. In particular, Corollary 1.4 in [4] shows that if a is a fixed non-square integer, then for all b sufficiently large (with respect to a) which are not squares, the system of equations $x^2 - ay^2 = y^2 - bz^2 = 1$ has at most one positive integer solution (x, y, z) . Under the same hypothesis (that b is not a square and large with respect to a), our result shows that the system of equations $x^2 - ay^2 = z^2 - by^2 = 1$ has at most 2 positive integer solutions (x, y, z) and there are infinitely many a 's for which this system of equations has exactly two solutions for infinitely many b 's, for example the a 's of the form $a = k^2 - 1$ for some positive integer $k \geq 2$. Hence, our results complement the results of Bennett and Pintér in that they give a sharp upper bound for the problem of bounding the cardinality of the intersection of the two sequences \mathbf{u} and \mathbf{Y} in a situation for which the condition (1.4) from the Bennett and Pintér paper does not hold.

2. Preliminaries on Pell equations

In this section, we recall a couple of facts about Pell equations. Let

$$\gamma := X_1 + \sqrt{d}Y_1 \quad \text{and} \quad \delta := X_1 - \sqrt{d}Y_1 = \gamma^{-1}. \quad (2)$$

Then

$$X_k = \frac{\gamma^k + \delta^k}{2} \quad \text{and} \quad Y_k = \frac{\gamma^k - \delta^k}{2\sqrt{d}} \quad \text{hold for all } k \geq 1. \quad (3)$$

In particular,

$$\begin{aligned} Y_k &= \frac{\gamma^k - \delta^k}{2\sqrt{d}} = \left(\frac{\gamma^k - \delta^k}{2\sqrt{d}Y_1} \right) Y_1 = \left(\frac{\gamma^k - \delta^k}{\gamma - \delta} \right) Y_1 \\ &= (\gamma^{k-1} + \gamma^{k-2}\delta + \dots + \delta^{k-1})Y_1 \geq \gamma^{k-1}Y_1. \end{aligned} \quad (4)$$

3. The proof of Theorem 1.1

We assume that $rs \neq 0$ and we will discuss the degenerate cases when $r = 0$ or $s = 0$ at the end.

We fix some notation. For a non-square integer $d > 1$, we use (X_1, Y_1) for the smallest positive integer solution of the Pell equation $X^2 - dY^2 = 1$. The numbers γ and δ are given by (2) and the general formula of X_k and Y_k is given by (3). We use the Binet formula (1) for u_n . We put $\mathbb{K} := \mathbb{Q}(\alpha)$, $\mathbb{L} := \mathbb{Q}(\gamma)$ and $\mathbb{M} := \mathbb{KL}$. We put c_1, c_2, \dots for computable constants depending in \mathbf{u} . Sometimes we ignore these and write the Landau symbols O

and the Vinogradov symbols \ll and \gg with the convention that the implied constants depend on \mathbf{u} . We also use $A \asymp B$ to express the fact that both $A \ll B$ and $B \ll A$ hold. Now assume that the equation $Y_m = u_n$ has three positive integer solutions (m, n) which are (m_i, n_i) for $i = 1, 2, 3$. We assume $m_1 < m_2 < m_3$. Note that $m_2 \geq 2$, so

$$\sqrt{d} \leq \gamma \leq \gamma^{m_2-1} = Y_{m_2} = u_{n_2} \tag{5}$$

(see (4)). Since d can be made arbitrarily large, we may assume that n_2 is arbitrarily large. Let us discuss the signs of the roots α, β . We label them such that $|\alpha| > |\beta|$. Assume first that $\alpha > 0$. If $a < 0$, it follows that $u_n < 0$ for all n sufficiently large. Since n_2 can be chosen to be arbitrarily large, we get a contradiction. So, if $\alpha > 0$, then we assume that $a > 0$. Suppose next that $\alpha < 0$. Since $u_n = ((-1)^n a)(-\alpha)^n + ((-1)^n b)(-\beta)^n$ for all n and $-\alpha > |\beta| \geq -\beta$, it follows if n_2 is sufficiently large, then the numbers n_2 and n_3 have the same parity. Furthermore, $\text{sign}(a) = (-1)^{n_2} = (-1)^{n_3}$. Thus, we may simultaneously change the signs of both α and β (hence, replace r by $-r$ and keep the same s) and change also the signs of both a and b , therefore assume that $u_n = a\alpha^n + b\beta^n$ holds for all sufficiently large n with both a and α positive. Note that in this last case it is possible that n_1 had a different parity than n_2 and n_3 in which case $u_{n_1} = \varepsilon(a\alpha^{n_1} + b\beta^{n_1})$, where $\varepsilon = -1$, but this is possible only if $n_1 < n_0$, where n_0 is some constant that depends on \mathbf{u} . Thus, we shall assume that a, α are positive, that

$$u_n = a\alpha^n + b\beta^n \quad \text{for } n \in \{n_2, n_3\} \quad \text{and} \quad u_{n_1} = \varepsilon(a\alpha^{n_1} + b\beta^{n_1})$$

with $\varepsilon \in \{\pm 1\}$, but that the possibility $\varepsilon = -1$ occurs only when $n_1 < n_0$. Next, there exists n_0 such that $u_n > \max\{|u_m| : 1 \leq m \leq n-1\}$ holds for all $n > n_0$. We shall assume that $n_2 > n_0$. Hence, $n_1 < n_2 < n_3$ because $m_1 < m_2 < m_3$ therefore

$$Y_{m_1} = |u_{n_1}| < Y_{m_2} = u_{n_2} < Y_{m_3} = u_{n_3}.$$

We proceed in various steps.

3.1. The case when α and γ are multiplicatively dependent. Clearly, $\mathbb{K} = \mathbb{L} = \mathbb{M}$ in this case. Further, $\gamma^k = \alpha^\ell$ holds for some integers k, ℓ not both zero. Since $\min\{\gamma, \alpha\} > 1$, it follows that none of k and ℓ are zero and that they have the same sign. Thus, up to changing the signs of both of them, we may assume that they are positive. We may also assume that they are coprime. Since $\sqrt{d} < X_1 + \sqrt{d}Y_1 = \gamma$, it follows that we may assume that $k < \ell$, otherwise

$$\sqrt{d} < \gamma \leq \alpha$$

so $d < \alpha^2$, and we have bounded d by some number depending on \mathbf{u} . Further, by conjugation in \mathbb{K} , it follows that $\delta^k = \beta^\ell$. Moreover, $d = d_1 v^2$ for some fixed positive square-free integer d_1 depending on \mathbf{u} . Further, there exists a unit $\alpha_1 > 1$ in \mathbb{K} such that $\alpha = \alpha_1^k, \gamma = \alpha_1^\ell$. Let β_1 be the conjugate of α_1 .

Then $\beta_1 = \pm\alpha_1^{-1}$. Note that k is bounded. The only variable is ℓ (or d , or v) and

$$\gamma = \alpha_1^\ell = X_1 + \sqrt{d}Y_1 = X_1 + (\sqrt{d_1v})Y_1.$$

Let us write the equation

$$Y_m = u_n$$

for $(m, n) = (m_i, n_i)$ with $i = 2, 3$ as

$$\frac{\alpha_1^{\ell m} - \beta_1^{\ell m}}{2\sqrt{d_1v}} = a\alpha_1^{kn} + b\beta_1^{kn}. \quad (6)$$

So,

$$\alpha_1^{kn} \left(\frac{\alpha_1^{\ell m - kn}}{2a\sqrt{d_1v}} - 1 \right) = \frac{\beta_1^{\ell m}}{2a\sqrt{d_1v}} + \frac{\varepsilon_1(b/a)}{\alpha_1^{kn}}, \quad \text{where } \varepsilon_1 \in \{\pm 1\}. \quad (7)$$

The case $\varepsilon_1 = -1$ only occurs above if and only if $\beta_1 = -\alpha_1^{-1}$ and kn is odd. Since d (hence, v), can be assumed arbitrarily large, we assume that

$$v > \max \left\{ \frac{2}{a\sqrt{d_1}}, \frac{2}{|b|\sqrt{d_1}} \right\}. \quad (8)$$

Note that b is the conjugate of a in \mathbb{K} . We may also assume that n_2 is such that

$$\alpha_1^{n_2} > \max \left\{ \frac{2|b|}{a}, \left(\frac{|b|}{a} \right)^{10} \right\}.$$

It follows from (6), that

$$\frac{\alpha_1^{\ell m}}{2\sqrt{d_1v}} > \frac{\alpha_1^{\ell m} - \beta_1^{\ell m}}{2\sqrt{d}} > a\alpha_1^{kn} - \frac{a}{2} > \frac{a\alpha_1^{kn}}{2} > \frac{\alpha_1^{kn}}{2\sqrt{d_1v}},$$

so

$$\alpha_1^{\ell m} > \alpha_1^{kn}, \quad \text{therefore } \ell m - kn > 0. \quad (9)$$

In particular, (7) shows that

$$\alpha_1^{kn} \left| \frac{\alpha_1^{\ell m - kn}}{2a\sqrt{d_1v}} - 1 \right| = \left| \frac{\beta_1^{\ell m}}{2a\sqrt{d_1v}} + \frac{\varepsilon_1(b/a)}{\alpha_1^{kn}} \right| < \frac{1}{2a\sqrt{d_1v}\alpha_1^{kn}} + \frac{1}{\alpha_1^{0.9kn}} < \frac{2}{\alpha_1^{0.9kn}} \quad (10)$$

for $(m, n) = (m_i, n_i)$ and $i = 2, 3$. Now let us show that $\ell m - kn < 1.1\ell$ unless d is bounded by a constant depending on \mathbf{u} . Indeed, suppose that $\ell m - kn \geq 1.1\ell$. Then

$$\frac{\alpha_1^{1.1\ell}}{2a\sqrt{d_1v}} - 1 \leq \frac{\alpha_1^{\ell m - kn}}{2a\sqrt{d_1v}} - 1 \leq \left| \frac{\alpha_1^{km - \ell n}}{2a\sqrt{d_1v}} - 1 \right| < \frac{2}{\alpha_1^{1.9kn}} < 1,$$

so

$$\frac{\alpha_1^{1.1\ell}}{4a} < \sqrt{d_1v} < X_1 + (\sqrt{d_1v})Y_1 = \alpha_1^\ell,$$

giving $d < \alpha_1^{2\ell} < (4a)^{20}$, which is a constant depending on \mathbf{u} . We thus have that $0 < \ell m - kn \leq 1.1\ell$. Hence, we get

$$\left| \frac{\alpha_1^{\ell n - km}}{2a\sqrt{d_1}v} - 1 \right| < \frac{2}{\alpha_1^{1.9kn}} \leq \frac{2}{\alpha_1^{1.9(m-1.1)\ell}}.$$

Let $u_3 := \ell m_3 - kn_3$. Since $m_3 \geq 3$, we get

$$\left| \frac{\alpha_1^{u_3}}{2a\sqrt{d_1}v} - 1 \right| < \frac{2}{\alpha_1^{1.9kn_3}} \leq \frac{2}{\alpha_1^{1.9(m_3-1.1)\ell}} < \frac{2}{\alpha_1^{3.6\ell}}.$$

We multiply the above expression with

$$\left| \frac{\beta_1^{u_3}}{2b\sqrt{d_1}v} + 1 \right| < \frac{1}{2|b|\sqrt{d_1}v} + 1 < \frac{5}{4}$$

(since $u_3 > 0$), and we get

$$\left| \left(\frac{\alpha_1^{u_3}}{2a\sqrt{d_1}v} - 1 \right) \left(\frac{\beta_1^{u_3}}{2b\sqrt{d_1}v} + 1 \right) \right| < \frac{5}{2\alpha_1^{3.6\ell}}.$$

We multiply across by $4a|b|d_1v^2$ getting

$$|(\alpha_1^{u_3} - 2a\sqrt{d_1}v)(\beta_1^{u_3} + 2b\sqrt{d_1}v)| < \frac{10a|b|d_1v^2}{\alpha_1^{3.6\ell}}. \quad (11)$$

Let D be the denominator of a . That is, D is the smallest positive integer such that Da is an algebraic integer. Multiplying the above inequality (11) by D^2 , we get

$$|(D\alpha_1^{u_3} - 2(Da)\sqrt{d_1}v)(D\beta_1^{u_3} + 2(Db)\sqrt{d_1}v)| < \frac{10(D^2a|b|)d_1v^2}{\alpha_1^{3.6\ell}}. \quad (12)$$

The expression inside the absolute value on the left-hand side above is an algebraic integer which is invariant under the action of the only non-identical Galois automorphism of \mathbb{K} call it σ , since $\sigma(\alpha_1) = \beta_1$, $\sigma(a) = b$ and $\sigma(\sqrt{d_1}) = -\sqrt{d_1}$. So, the left-hand side is a non-negative integer. If it is not zero, then it is ≥ 1 . If this is the case, we get

$$\alpha_1^{3.6\ell} < 10D^2a|b|d_1v^2 < 10D^2a|b|\alpha_1^{2\ell},$$

giving

$$d = d_1v^2 < \gamma^2 = \alpha_1^{2\ell} < (10D^2a|b|)^{5/4},$$

which bounds d in terms of \mathbf{u} . The other possibility is that the integer in the left-hand side of (11) is zero, in which case we get

$$v = \frac{\alpha_1^{u_3}}{2a\sqrt{d_1}}.$$

Taking norms in \mathbb{K} and absolute values, we get

$$v^2 = \frac{1}{4a|b|d_1},$$

which is false by (8).

Hence, in all cases, we got a contradiction for $d > d_0(\mathbf{u})$ by assuming that there are at least three positive integer solutions (m, n) to the equation $Y_m = u_n$ in this case.

From now on, we continue under the assumption that α and γ are multiplicatively independent.

3.2. Linear forms in logarithms. We need lower bounds for linear forms in complex logarithms. For an algebraic number α of minimal polynomial

$$f(X) := a_0X^d + a_1X^{d-1} + \cdots + a_d = a_0(X - \alpha^{(1)}) \cdots (X - \alpha^{(d)}) \in \mathbb{Z}[X]$$

($\alpha^{(1)} = \alpha$ and $a_0 > 0$), we put

$$h(\alpha) := \frac{1}{d} \left(\log a_0 + \sum_{\substack{1 \leq i \leq d \\ |\alpha^{(i)}| > 1}} \log |\alpha^{(i)}| \right)$$

for the logarithmic height of α . The following result is referred to in the literature as Baker's lower bound for a non-zero linear form in logarithms.

Theorem 3.1. *Let $\alpha_1, \dots, \alpha_k$ be positive algebraic numbers different from 1 and b_1, \dots, b_k be nonzero integers. Let $B \geq \max\{3, |b_1|, \dots, |b_k|\}$ and let $A_i \geq h(\alpha_i)$ for $i = 1, \dots, k$. Let D be the degree of $\mathbb{Q}(\alpha_1, \dots, \alpha_k)$. There is a computable constant $c_1 := c_1(k, D)$ depending only on k and D such that if we put*

$$\Lambda := \sum_{i=1}^k b_i \log \alpha_i,$$

then $\Lambda \neq 0$ implies

$$|\Lambda| > \exp(-c_1 A_1 \cdots A_k \log B).$$

For an explicit $c_1(k, D)$ one can consult the work of Baker and Wüstholz [2], or Matveev [18].

We now continue with the analysis of the equation $Y_m = u_n$.

We rewrite the equation

$$\frac{\gamma^m - \delta^m}{2\sqrt{d}} = a\alpha^n + b\beta^n \tag{13}$$

for $(m, n) = (m_i, n_i)$ for $i = 2, 3$ (and even for $i = 1$ provided $n_1 > n_0$) as

$$(2a\sqrt{d})^{-1}\gamma^m\alpha^{-n} - 1 = \frac{1}{2a\sqrt{d}\gamma^m\alpha^n} + \frac{(b/a)}{(\alpha/\beta)^n}. \tag{14}$$

We suppose that d and n_2 are large enough so $1/(2a\sqrt{d}\gamma) < 1/4$ and $(|b/a|)/(\alpha/\beta)^{n_2} < 1/4$. If $(m, n) = (m_1, n_1)$, we will assume that the above inequalities hold with n_1 instead of n_2 provided $n_1 > n_0$. Then

$$\left| (2a\sqrt{d})^{-1}\gamma^m\alpha^{-n} - 1 \right| < \frac{1}{2}.$$

The above inequality implies that

$$\frac{\alpha^n}{2\gamma^m} < \frac{1}{2a\sqrt{d}} \quad \text{and} \quad \gamma^m < 3a\sqrt{d}\alpha^n. \tag{15}$$

For large d , we have that $3a\sqrt{d} < \gamma^{1.1}$ so

$$(m - 1.1) \log \gamma < n \log \alpha. \tag{16}$$

Estimate (13) shows that

$$\gamma^m = 2\sqrt{d}a\alpha^n \left(1 + O\left(\frac{1}{(\alpha/|\beta|)^n}\right) \right),$$

so

$$\frac{1}{\gamma^m} = \frac{1}{2\sqrt{d}a\alpha^n} + O\left(\frac{1}{\alpha^n(\alpha/|\beta|)^n}\right). \tag{17}$$

Since also $|\beta| = |s|/\alpha \geq \alpha^{-1}$, it follows that $\gamma^m\alpha^n > \alpha^{2n} \geq (\alpha/|\beta|)^n$. Thus,

$$(2a\sqrt{d})^{-1}\gamma^m\alpha^{-n} - 1 = \frac{(b/a)}{(\alpha/\beta)^n} + \frac{1}{4a^2d\alpha^{2n}} + O\left(\frac{1}{\alpha^{2n}(\alpha/|\beta|)^n}\right). \tag{18}$$

We pass to logarithmic form in (18) to get that

$$\begin{aligned} m \log \gamma - n \log \alpha - \log(2a\sqrt{d}) &= \frac{(b/a)}{(\alpha/\beta)^n} + \frac{1}{4a^2d\alpha^{2n}} \\ &+ O\left(\frac{1}{\alpha^{2n}(\alpha/|\beta|)^n} + \frac{1}{(\alpha/|\beta|)^{2n}}\right). \end{aligned} \tag{19}$$

We shall use the above estimate for $(m, n) = (m_i, n_i)$ with $i = 2, 3$ and also with $i = 1$ assuming that $n_1 > n_0$ is sufficiently large. Sometimes we will use the weaker consequence of (19) that

$$m \log \gamma - n \log \alpha - \log(2a\sqrt{d}) < \frac{c_2}{(\alpha/|\beta|)^n} \tag{20}$$

with $c_2 := 2|b|/a$ for $n > n_0$, but we will have some use for the full-expansion (19) lateron.

3.3. The case when γ , α and $2a\sqrt{d}$ are multiplicatively dependent.

We already know that γ and α are multiplicatively independent. Thus, since there are integers x, y, z not all zero such that

$$\gamma^x \alpha^y (2a\sqrt{d})^z = 1, \tag{21}$$

it follows that $z \neq 0$. Furthermore, assuming that $\gcd(x, y, z) = 1$, and $z > 0$, it follows that the vector $(x, y, z) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ is unique. Computing norms in \mathbb{M} and keeping in mind that γ is a unit, we get

$$(N_{\mathbb{M}/\mathbb{Q}}(\alpha))^y (N_{\mathbb{M}/\mathbb{Q}}(2a))^z (N_{\mathbb{M}/\mathbb{Q}}(\sqrt{d}))^z = 1. \tag{22}$$

Note that $N_{\mathbb{M}/\mathbb{Q}}(\sqrt{d}) = d^{[\mathbb{M}:\mathbb{L}]} \in \{d, d^2\}$. Further, $N_{\mathbb{M}/\mathbb{Q}}(\alpha) \in \{\alpha^2, s, s^2\}$, according to whether $\alpha \in \mathbb{N}$ (so $\mathbb{M} = \mathbb{K}$), or $\mathbb{K} = \mathbb{L}$ (so, again $\mathbb{M} = \mathbb{K}$), or \mathbb{M} has degree 4, respectively. Similarly, $N_{\mathbb{M}/\mathbb{Q}}(a) \in \{a^2, ab, (ab)^2\}$. Let \mathcal{P} be the set of primes dividing s or dividing either the numerator of the

denominator of the number $N_{\mathbb{K}/\mathbb{Q}}(a)$. This is a finite set of primes depending on \mathbf{u} . Formula (22) together with the fact that $z > 0$ implies that all prime factors of d are in \mathcal{P} . Thus, we can write $d = d_1 v$, where d_1 is square-free with prime factors in \mathcal{P} and v is an integer all whose prime factors are also in \mathcal{P} . Clearly, d_1 is bounded so we need to bound v . Fix d_1 . Let (U_1, V_1) be the minimal solution in positive integers of the Pell equation $U^2 - d_1 V^2 = 1$. Put $\gamma_1 := U_1 + \sqrt{d_1} V_1$. Let (U_k, V_k) be the k th solution of the above Pell equation given of course by the formula

$$U_k + \sqrt{d_1} V_k = \gamma_1^k.$$

Now let (X, Y) be a positive integer solution to $X^2 - (d_1 v^2) Y^2 = 1$. Then $X^2 - d_1 (vY)^2 = 1$. Thus, there exists a positive integer k with the property that $(X, vY) = (U_k, V_k)$. Hence,

$$Y = \frac{V_k}{v}.$$

It thus follows that $Y_m = V_{k_m}/v$, where $\{k_m\}_{m \geq 1}$ is the increasing sequence of all positive integers k such that $v \mid Y_k$. But this sequence has been studied. Namely, $k_1 = z(v)$ is called the *index of appearance* of v in $\{Y_k\}_{k \geq 1}$. Furthermore, $v \mid Y_k$ if and only if $z(v) \mid k$. Thus, $k_m = mz(v)$. Additionally,

$$\gamma = \gamma_1^{z(v)}.$$

It remains to recall some of the properties of $z(v)$ which we now do.

Lemma 3.2. *Let $d_1 > 1$ be a positive integer which is not a square. Let (U_k, V_k) be the sequence of positive integer solutions to $U^2 - d_1 V^2 = 1$. For each positive integer k let $z(k)$ be the minimal positive integer ℓ such that $k \mid V_\ell$. The following properties hold:*

- (i) $z(p_1^{t_1} \cdots p_k^{t_k}) = \text{lcm}(z(p_1^{t_1}), z(p_2^{t_2}), \dots, z(p_k^{t_k}))$ for all distinct primes p_1, \dots, p_k and positive integers t_1, \dots, t_k ;
- (ii) If $p \mid d_1$, then $z(p) = p$. Otherwise, $z(p)$ divides one of $p-1$ or $p+1$.
- (iii) Put $e_p = \nu_p(V_{z(p)})$, that is the exponent of p in the factorisation of $V_{z(p)}$. Then $z(p^e) = z(p)p^{\min\{0, e - e_p\}}$.

We now continue with our argument. Since v is formed only of primes from the fixed finite set \mathcal{P} depending on \mathbf{u} , it follows from the above properties that $z(v) \asymp v$. That is, there are constants c_3 and c_4 depending on \mathbf{u} such that $c_3 v < z(v) < c_4 v$. This is for a fixed d_1 but since there are only finitely many choices for d_1 (squarefree integers > 1 formed with primes from \mathcal{P}), it follows that we may assume that c_3 and c_4 are such that the above inequality holds for all possible values of d_1 . We now go to inequality (19) and evaluate it in $(m, n) = (m_i, n_i)$ for $i = 2, 3$ and deduce that

$$|mz(v) \log \gamma_1 - n \log \alpha - \log(2a\sqrt{d_1}v)| < \frac{c_2}{(\alpha/|\beta|)^n}, \quad \text{for } (m, n) = (m_i, n_i), \quad (23)$$

and $i = 2, 3$. The form in the left-hand side might be zero. If it is, then since the vector of integer exponents (x, y, z) realising the equality (21) is unique provided that $z > 0$ and $\gcd(x, y, z) = 1$, it follows that $z = 1$ and $(mz(v), n) = (-x, y)$. Thus, $n = y$ is fixed for the current value of v . It follows that of the two inequalities (23) for $i = 2, 3$, there is at most one of them whose left-hand side is zero. Say it is for $i \in \{2, 3\}$. We then work with the respective inequality for $(m, n) = (m_j, n_j)$ and $j \in \{2, 3\} \setminus \{i\}$ whose left-hand side is non-zero. We apply Theorem 3.1 with

$$k := 3, \alpha_1 := \gamma_1, \alpha_2 := \alpha, \alpha_3 := 2a\sqrt{d_1}v, b_1 := mz(v), b_2 := -n, b_3 := -1.$$

Note that $h(\alpha_1) = O(1)$, $h(\alpha_2) = O(1)$ and $h(\alpha_3) = \log v + O(1)$. Thus, applying Theorem 3.1 and using inequality (23), we get

$$n \log(\alpha/|\beta|) - \log c_2 < c_5(\log v + c_6) \log(\max\{n, mz(v)\}). \tag{24}$$

Assume n realises the maximum in the right-hand side above. Returning to (16), we get

$$v \leq mv \ll (m - 1.1)z(v) \log \gamma_1 = (m - 1.1) \log \gamma \ll n,$$

so $v \leq c_7n$ (here, we used the fact that $m = m_j$ for some $j \in \{2, 3\}$ so $m \geq 2$). Hence, we get that

$$n \log(\alpha/|\beta|) - \log c_2 \leq c_5(\log(c_7n) + c_6) \log n,$$

showing that $n \leq c_8$. Thus, choosing $n_2 > c_8$, we can bypass this situation. Assume now that $mz(v)$ realises the maximum in the right-hand side of (24). Then $mz(v) < c_4mv$, and again by (16), we have

$$mv \ll (m - 1.1)z(v) \ll n.$$

Hence, we get

$$c_9mv < n \log(\alpha/|\beta|) < c_5(\log(mv) + c_6) \log(c_4mv) + \log c_4,$$

which gives $mv \leq c_{10}$, so v , therefore d , is bounded in terms of \mathbf{u} . This completes the analysis of the current situation.

From now on, we assume that γ , α and $2a\sqrt{d}$ are multiplicatively independent. In particular, the left-hand side of (20) does not vanish for any pair of positive integers (m, n) .

3.4. Bounds on n_i and m_i for $i = 1, 2, 3$ in terms of γ . Here, we prove the following lemma.

Lemma 3.3. *We have:*

- (i) $n_i \gg m_i \log \gamma$ for $i = 2, 3$ and even for $i = 1$ if $m_1 > 1$.
- (ii) $n_3 \ll (\log \gamma)^2 \log \log \gamma$ and $m_3 \ll \log \gamma \log \log \gamma$.
- (iii) $n_i - n_j = (m_i - m_j) \log \gamma + O(1)$ holds for indices $i > j$ both in $\{1, 2, 3\}$.

Proof. The first one is immediate from (5) since then

$$(m-1)\log\gamma \leq \log u_n \ll n \quad \text{for} \quad (m, n) = (m_i, n_i) \quad \text{where} \quad i = 1, 2, 3.$$

For the second one, we apply Theorem 3.1 on the left-hand side of (19) for $(m, n) = (m_3, n_3)$ with the obvious choices

$$k := 3, \quad \alpha_1 = \gamma, \quad \alpha_2 := \alpha, \quad \alpha_3 := 2a\sqrt{d}, \quad b_1 = m, \quad b_2 = -n, \quad b_3 = -1.$$

Clearly, $h(\alpha_1) = O(\log\gamma)$, $h(\alpha_2) = O(1)$, $h(\alpha_3) = O(\log d) = O(\log\gamma)$ and $B := n$. Applying Theorem 3.1 and using (19), we get

$$n_3 \log(\alpha/|\beta|) + O(1) \ll (\log\gamma)^2 \log n_3,$$

which gives $n_3 = O((\log\gamma)^2 \log\log\gamma)$. This is the first part of (ii) and the second part of it follows from (i) for $i = 3$. Finally, for (iv), we write

$$Y_{m_i} = u_{n_i} \quad \text{and} \quad Y_{m_j} = u_{n_j}$$

for $i < j$ and divide them side by side. We get

$$\frac{Y_{m_i}}{Y_{m_j}} = \frac{u_{n_i}}{u_{n_j}}.$$

Since $u_n \asymp \alpha^n$, it follows that the right-hands side is $\asymp \alpha^{n_i - n_j}$. Similarly, the left-hand side is $\asymp \gamma^{m_i - m_j}$. Hence, taking logarithms we get

$$(m_i - m_j) \log\gamma = (n_i - n_j) \log\alpha + O(1),$$

which is (iii). □

Remark. One can get slightly better bounds for m_2 and n_2 by applying estimates for linear form in simultaneous logarithms, namely simultaneously for (m_2, n_2) and (m_3, n_3) . See [11] or [14] for the actual statements. These give the slightly better bounds $n_2 \ll (\log\gamma)^{3/2} \log\log\gamma$ and $m_2 \ll (\log\gamma)^{1/2} \log\log\gamma$. However, such better bounds on m_2 and n_2 do not seem to induce any simplifications in the subsequent arguments, which is why we do not formally prove them here.

3.5. The case $m_1 > 1$ or n_1 large. Here, we prove the following lemma.

Lemma 3.4. *The number d is bounded in terms of \mathbf{u} unless the following hold:*

- (i) $m_1 = 1$;
- (ii) $n_1 \ll \log\log\gamma$;

Proof. Assume that n_1 is large. Consider the matrix

$$A = \begin{pmatrix} n_1 & m_1 & 1 \\ n_2 & m_2 & 1 \\ n_3 & m_3 & 1 \end{pmatrix}.$$

Assume first that its rank is 3. Writing (19) for $(m, n) = (m_\ell, n_\ell)$, for $\ell = 1, 2, 3$, subtracting the one for $\ell = 1$ from the ones for $\ell \in \{2, 3\}$ and using the absolute value inequality we get

$$|(m_\ell - m_1) \log \gamma - (n_\ell - n_1) \log \alpha| < \frac{2c_2}{(\alpha/|\beta|)^{n_1}} \quad \text{for } \ell \in \{2, 3\}.$$

Eliminating $\log \gamma$ between the two inequalities above, we get

$$|\Delta| \log \alpha \leq \frac{2(m_2 + m_3)c_2}{(\alpha/|\beta|)^{n_1}},$$

where

$$\Delta := |(m_3 - m_1)(n_2 - n_1) - (m_2 - m_1)(n_3 - n_1)| = |\det A| \geq 1.$$

So, by Lemma 3.3, we get

$$(\alpha/|\beta|)^{n_1} \ll m_3 \ll \log \gamma \log \log \gamma,$$

which in turn shows that $n_1 \ll \log \log \gamma$. If $m_1 > 1$, we then get by Lemma 3.3 that $\log \gamma \ll \log \log \gamma$ which bounds γ . Hence, $\gamma = O(1)$. We now study the case when $\Delta = 0$. Let L_1, L_2, L_3 be the rows of the above matrix. Note that A has rank 2 since otherwise L_2 and L_1 should be multiples of each other, which is not the case since their third component is equal to 1 but their first components are different. Let u, v be rational numbers such that $L_1 = uL_2 + vL_3$. The numbers u, v solve the system

$$\begin{cases} u + v = 1 \\ un_2 + vn_3 = n_1 \end{cases}$$

whose solution is $(u, v) = ((n_3 - n_1)/(n_3 - n_2), (n_1 - n_2)/(n_3 - n_2))$. So, $uv \neq 0$.

Assume first that $|s| > 1$.

In this case, $|\beta| = |s|/\alpha > \alpha^{-1}$. We put $\kappa := \log(\alpha/|\beta|)/\log \alpha$. Then $\kappa \in (0, 2)$. We return to estimates (19) which we write in the much simpler form

$$m \log \gamma - n \log \alpha - \log(2a\sqrt{d}) = \frac{b/a}{(\alpha/\beta)^n} + O\left(\frac{1}{\alpha^{2n_1}}\right) \quad \text{for } (m, n) = (m_i, n_i) \tag{25}$$

and $i = 1, 2, 3$. Multiplying estimates (25), the one corresponding to $i = 2$ with u , the one corresponding to $i = 3$ with v , adding them and subtracting the one corresponding to $i = 1$, we get

$$0 = b/a \left(\frac{u}{(\alpha/\beta)^{n_2}} + \frac{v}{(\alpha/\beta)^{n_3}} - \frac{1}{(\alpha/\beta)^{n_1}} \right) + O\left(\frac{n_3}{\alpha^{2n_1}}\right).$$

Simplifying a factor of $(\alpha/\beta)^{n_1}$ and using the fact that for integer ℓ we have $(\alpha/\beta)^\ell = \pm \alpha^{\kappa \ell}$ (here, the negative sign occurs only when ℓ is odd and β is

negative), we get

$$1 - \frac{\pm u}{\alpha^{\kappa(n_2-n_1)}} - \frac{\pm v}{\alpha^{\kappa(n_3-n_1)}} = O\left(\frac{n_3}{\alpha^{(2-\kappa)n_1}}\right). \quad (26)$$

Assume first that either $|u/\alpha^{\kappa(n_2-n_1)}| > 1/3$ or $|v/\alpha^{\kappa(n_3-n_1)}| > 1/3$. In this case, we have

$$\alpha^{\kappa(n_2-n_1)} \ll \max\{|u|, |v|\} \ll n_3,$$

and taking logarithms we get

$$\log \gamma \ll n_3 - n_2 \ll \log n_3 + O(1) \ll \log \log \gamma, \quad (27)$$

where the left and the right estimates above follow from Lemma 3.3 (ii) and (iii). But this gives $\gamma = O(1)$. So, let us assume that

$$|u/\alpha^{\kappa(n_2-n_1)}| < 1/3 \quad \text{and} \quad |v/\alpha^{\kappa(n_3-n_1)}| < 1/3.$$

In this case, we get that the left-hand side in (26) is $\geq 1/3$ in absolute value, so (26) leads to

$$\alpha^{(2-\kappa)n_1} \ll n_3,$$

therefore $n_1 \ll \log n_3 + O(1) \ll \log \log \gamma$, which together with Lemma 3.3 (i) implies now that $(m_1 - 1) \log \gamma \ll \log \log \gamma$, so $\gamma = O(1)$, unless $m_1 = 1$. This gives (i) and (ii) under the current assumption on s .

Assume next that $|s| = 1$. In this case, $\beta = \pm\alpha^{-1}$, and estimates (19) take the shape

$$m \log \gamma - n \log \alpha - \log(2a\sqrt{d}) = \left(\frac{b\varepsilon_n}{a} - \frac{1}{4a^2d}\right) \frac{1}{\alpha^{2n}} + O\left(\frac{1}{\alpha^{4n}}\right). \quad (28)$$

Here, $\varepsilon_n \in \{\pm 1\}$. We assume that d is sufficiently large so that the coefficient of $1/\alpha^{2n}$ above is in absolute value $\geq 1/(2|a|)$. We multiply again the estimate (28) for $i = 2$ with u , for $i = 3$ with v , and subtract the one for $i = 1$, getting

$$\begin{aligned} 0 &= u \left(\frac{b\varepsilon_{n_2}}{a} - \frac{1}{4a^2d}\right) \frac{1}{\alpha^{2n_2}} + v \left(\frac{b\varepsilon_{n_3}}{a} - \frac{1}{4a^2d}\right) \frac{1}{\alpha^{2n_3}} \\ &\quad - \left(\frac{b\varepsilon_{n_1}}{a} - \frac{1}{4a^2d}\right) \frac{1}{\alpha^{2n_1}} + O\left(\frac{1}{\alpha^{4n_1}}\right). \end{aligned}$$

We thus get that

$$\begin{aligned} \left(\frac{b\varepsilon_{n_1}}{a} - \frac{1}{4a^2d}\right) &= u \left(\frac{b\varepsilon_{n_2}}{a} - \frac{1}{4a^2d}\right) \frac{1}{\alpha^{2(n_2-n_1)}} \\ &\quad + v \left(\frac{b\varepsilon_{n_3}}{a} - \frac{1}{4a^2d}\right) \frac{1}{\alpha^{2(n_3-n_1)}} + O\left(\frac{n_3}{\alpha^{2n_1}}\right). \end{aligned}$$

We use the same argument as before. Namely, if the first term in the right-hand side above is in absolute value $> 1/(6|a|)$, we then get that $\alpha^{2(n_2-n_3)} \ll n_3$, so $n_2 - n_3 \ll \log n_3 \ll \log \log \gamma$, therefore $\log \gamma \ll \log \log \gamma$ by Lemma 3.3, so $\gamma = O(1)$. A similar conclusion holds if the second term on the right-hand side above is $> 1/(6|a|)$. In case both terms first and second terms

on the right are smaller than $1/(6|a|)$ in absolute value, then the left-hand side of the expression

$$\begin{aligned} & \left(\frac{b\varepsilon_{n_1}}{a} - \frac{1}{4a^2d} \right) - u \left(\frac{b\varepsilon_{n_2}}{a} - \frac{1}{4a^2d} \right) \frac{1}{\alpha^{2(n_2-n_1)}} \\ & - v \left(\frac{b\varepsilon_{n_3}}{a} - \frac{1}{4a^2d} \right) \frac{1}{\alpha^{2(n_3-n_1)}} = O \left(\frac{n_3}{\alpha^{2n_1}} \right) \end{aligned}$$

is $> 1/(6|a|)$ in absolute value. This gives $\alpha^{2n_1} \ll n_3$, so $n_1 \ll \log \log \gamma$, which implies that $\gamma = O(1)$ unless $m_1 = 1$, and the conclusions (i) and (ii) again follow under the current assumption on s . The lemma is therefore proved. \square

3.6. The case when $\gcd(r, s) > 1$. Let $\ell := \gcd(r, s)$. Put $\alpha_1 := \alpha^2/\ell$ and $\beta_1 := \beta^2/\ell$. Then α_1, β_1 are integers and $\alpha_1 + \beta_1 = (r^2 + 2s)/\ell$ and $\alpha_1\beta_1 = s^2/\ell^2$ are coprime integers. Further

$$u_n = \ell^{\lfloor n/2 \rfloor} (a_1\alpha_1^n + b_1\beta_1^n), \quad \text{where } (a_1, b_1) \in \{(a, b), (\alpha\alpha, b\beta)\}$$

according to whether n is even or odd (see Lemma A10 in [19]). Let p be a prime factor of ℓ and let $u := \nu_p(\ell)$. Then

$$\nu_p(u_n) = \nu_p(\ell^{\lfloor n/2 \rfloor}) + \nu_p(a_1\alpha_1^n + b_1\beta_1^n) = nu/2 + O(\log n),$$

where the error term above appears as a result of applying a linear form in p -adic logarithms to $a_1\alpha_1^n + b_1\beta_1^n$. Now let us return to our equations and look at $Y_m = u_n$ for $(m, n) = (m_i, n_i)$ for $i = 2, 3$. We have

$$Y_m = \ell^{\lfloor n/2 \rfloor} (a_1\alpha_1^n + b_1\beta_1^n) \quad \text{for } (m, n) = (m_i, n_i) \quad \text{and } i = 2, 3.$$

Clearly, $\nu_p(Y_m) = \nu_p(u_n) = un/2 + O(\log n)$ for $(m, n) = (m_i, n_i)$ and $i = 2, 3$. By Lemma 3.2, we have that $z(p) \mid p(p^2 - 1)$. Since $p \mid \ell$ depends only on \mathbf{u} , it follows that $z(p) = O(1)$. Let $e_p = \nu_p(Y_{z(p)})$. Write $m_2 = z(p)p^{\ell_2}m'_2$ and $m_3 = z(p)p^{\ell_3}m'_3$, where m'_2, m'_3 are coprime to p . Now

$$\begin{aligned} \nu_p(Y_{m_2}) &= un_2/2 + O(\log n_2) = e_p + \max\{\ell_2 - e_p, 0\}, \\ \nu_p(Y_{m_3}) &= un_3/2 + O(\log n_3) = e_p + \max\{\ell_3 - e_p, 0\}. \end{aligned}$$

Thus,

$$u(n_3 - n_2)/2 + O(\log n_3) = \max\{\ell_3 - e_p, 0\} - \max\{\ell_2 - e_p, 0\}. \quad (29)$$

Assume first that the right-hand side above is 0. Then

$$n_3 - n_2 \ll \log n_3 \ll \log \log \gamma,$$

which is (27) and implies that $\log \gamma \ll \log \log \gamma$ by Lemma 3.3 (iii). Assume next that the maximum in the right-hand side of (29) is positive. Then $\ell_3 > e_p$. If $\ell_2 \leq e_p$, then the right-hand side in (29) is $\ell_3 - e_p$. Further,

$$e_p = \nu_p(Y_{m_2}) = un_2/2 + O(\log n_2).$$

Hence, we get that

$$u(n_3 - n_2)/2 + O(\log n_3) = \ell_3 - e_p = \ell_3 - un_2/2 + O(\log n_2),$$

obtaining that

$$un_3/2 = \ell_3 + O(\log n_3).$$

Hence, $n_3 \ll \ell_3 \ll \log m_3 \ll \log \log \gamma$ by Lemma 3.3 (ii), so inequality (27) holds in this case as well. Finally, assume that $\ell_3 > e_p$ and $\ell_2 > e_p$. We then get

$$u(n_3 - n_2)/2 + O(\log n_3) = \ell_3 - \ell_2 = O(\log m_3),$$

so

$$n_3 - n_2 = O(\log n_3 + \log m_3) = O(\log \log \gamma),$$

which is again inequality (27) and implies $\gamma = O(1)$. This finishes the analysis in the current case. From now on, we assume that $\gcd(r, s) = 1$.

3.7. Expressing X_1 in terms of u_{n_i} and u_{n_1} for any $i \in \{2, 3\}$. We use the fact that

$$Y_k = \frac{\gamma^k - \delta^k}{2\sqrt{d}} = Y_1 \left(\frac{(X_1 + \sqrt{X_1^2 - 1})^k - (X_1 - \sqrt{X_1^2 - 1})^k}{2\sqrt{X_1^2 - 1}} \right) := Y_1 P_k(X_1),$$

where

$$\begin{aligned} P_k(X_1) &= \frac{(X_1 + \sqrt{X_1^2 - 1})^k - (X_1 - \sqrt{X_1^2 - 1})^k}{2\sqrt{X_1^2 - 1}} \\ &= \sum_{\substack{0 \leq i \leq k \\ i \equiv k-1 \pmod{2}}} \binom{k}{i} X_1^i (X_1^2 - 1)^{(k-1-i)/2} \end{aligned}$$

is in $\mathbb{Z}[X_1]$. So, we take $(m, n) = (n_i, m_i)$ for $i = 2, 3$ and write

$$P_{m_i}(X_1) = \frac{Y_{m_i}}{Y_1} = \frac{u_{n_i}}{u_{n_1}} \quad \text{for } i = 2, 3. \quad (30)$$

The following lemma gives the exact value of X_1 for large d .

Lemma 3.5. *For $d > d_0(\mathbf{u})$, in (30) we have*

$$X_1 = 0.5 \left(\frac{u_{n_i}}{u_{n_1}} \right)^{1/(m_i-1)} + \frac{(m_i - 2)}{4(m_i - 1)X_1} + O\left(\frac{1}{X_1^3} \right), \quad i = 2, 3. \quad (31)$$

Proof. We have that

$$\begin{aligned}
\frac{u_{n_i}}{u_{n_1}} &= P_{m_i}(X_1) = \frac{(X_1 + \sqrt{X_1^2 - 1})^{m_i} - (X_1 - \sqrt{X_1^2 - 1})^{m_i}}{2\sqrt{X_1^2 - 1}} \\
&= \frac{1}{2\sqrt{X_1^2 - 1}} \left((X_1 + \sqrt{X_1^2 - 1})^{m_i} \right) + O\left(\frac{1}{X_1^{m_i+1}}\right) \\
&= \frac{1}{2\sqrt{X_1^2 - 1}} \left(2\sqrt{X_1^2 - 1} + (X_1 - \sqrt{X_1^2 - 1}) \right)^{m_i} + O\left(\frac{1}{X_1^3}\right) \\
&= \frac{1}{2\sqrt{X_1^2 - 1}} \left(2\sqrt{X_1^2 - 1} + \frac{1}{2X_1} + O\left(\frac{1}{X_1^3}\right) \right)^{m_i} + O\left(\frac{1}{X_1^3}\right) \\
&= (2\sqrt{X_1^2 - 1})^{m_i-1} \left(1 + \frac{1}{4X_1^2} + O\left(\frac{1}{X_1^4}\right) \right)^{m_i} + O\left(\frac{1}{X_1^3}\right).
\end{aligned}$$

Extracting $m_i - 1$ roots, we get

$$\begin{aligned}
\left(\frac{u_{n_i}}{u_{n_1}}\right)^{1/(m_i-1)} &= 2\sqrt{X_1^2 - 1} \left(1 + \frac{1}{4X_1^2} + O\left(\frac{1}{X_1^4}\right) \right)^{m_i/(m_i-1)} \\
&\quad \times \left(1 + O\left(\frac{1}{X_1^4}\right) \right)^{1/(m_i-1)} \\
&= 2X_1 \left(1 - \frac{1}{2X_1^2} + O\left(\frac{1}{X_1^4}\right) \right) \\
&\quad \times \left(1 + \frac{m_i}{4(m_i-1)X_1^2} + O\left(\frac{1}{X_1^4}\right) \right) \left(1 + O\left(\frac{1}{X_1^4}\right) \right) \\
&= 2X_1 + \left(\frac{m_i - 2}{2(m_i - 1)} \right) \frac{1}{X_1} + O\left(\frac{1}{X_1^3}\right),
\end{aligned}$$

which is the desired estimate. \square

3.8. A reduction to two special cases. We use Lemma 3.5 for $i = 2, 3$, expressing X_1 both in terms of u_{n_2}/u_{n_1} and in terms of u_{n_3}/u_{n_1} and get that

$$\left| \left(\frac{u_{n_2}}{u_{n_1}}\right)^{1/(m_2-1)} - \left(\frac{u_{n_3}}{u_{n_1}}\right)^{1/(m_3-1)} \right| = \frac{m_3 - m_2}{2(m_3 - 1)(m_2 - 1)X_1} + O\left(\frac{1}{X_1^3}\right) \tag{32}$$

for $d > d_0(\mathbf{u})$. Let $M_i := (u_{n_i}/u_{n_1})^{1/(m_i-1)}$ for $i = 2, 3$. We have

$$\begin{aligned}
M_i &= \left(\frac{\alpha^{n_i}}{(u_{n_1}/a)} \right) \left(1 + \frac{(b/a)}{(\alpha/\beta)^{n_i}} \right)^{1/(m_i-1)} \\
&= \left(\frac{\alpha^{n_i/(m_i-1)}}{(u_{n_1}/a)^{1/(m_i-1)}} \right) \left(1 + \frac{(b/a)}{(m_i - 1)(\alpha/\beta)^{n_i}} + O\left(\frac{1}{(\alpha/|\beta|)^{2n_i}}\right) \right)
\end{aligned}$$

for $i = 2, 3$. Hence, putting $N_i := \alpha^{n_i/(m_i-1)}/(u_{n_1}/a)^{1/(m_i-1)}$ for $i = 2, 3$, we get that

$$M_i = N_i \left(1 + \frac{(b/a)}{(m_i-1)(\alpha/\beta)^{n_i}} + O\left(\frac{1}{(\alpha/|\beta|)^{2n_i}}\right) \right) \quad (33)$$

for $i = 2, 3$. Note also that

$$\log N_i = \left(\frac{n_i}{m_i-1} \right) \log \alpha + O(n_1) = \log \gamma + O(n_1) = \log \gamma + O(\log \log \gamma),$$

so N_1 and N_2 tend to infinity as γ becomes large. Since $M_i = N_i(1 + o(1))$ for $i = 1, 2$ and also $M_2 = M_3(1 + o(1))$ as γ becomes large, it follows that $N_3/N_2 \in [1/2, 2]$ for $d > d_0(\mathbf{u})$. We thus get that

$$|N_2 N_3^{-1} - 1| = \frac{m_3 - m_2}{4(m_3 - 1)(m_2 - 1)N_3 X_1} + O\left(\frac{1}{N_3 X_1^3} + \frac{1}{(\alpha/|\beta|)^{n_2}}\right). \quad (34)$$

Note that the left-hand side is above is

$$|\alpha^{n_2/(m_2-1)-n_3/(m_3-1)}(a/u_{n_1})^{1/(m_2-1)-1/(m_3-1)} - 1|.$$

Passing to logarithmic form for $d > d_0(\mathbf{u})$ in the left-hand side of (34), we get that

$$\left| \left(\frac{n_2}{m_2-1} - \frac{n_3}{m_3-1} \right) \log \alpha + \left(\frac{1}{m_2-1} - \frac{1}{m_3-1} \right) \log(a/u_{n_1}) \right| \ll \frac{1}{\gamma}.$$

We clear denominators in the left getting

$$|n_2(m_3-1)-n_3(m_2-1) \log \alpha + (m_3-m_2) \log(a/u_{n_1})| \ll \exp(-\log \gamma + \log n_3). \quad (35)$$

In the left-hand side above we have a linear form in two logarithms. We first treat the case when it is not zero. We apply Theorem 3.1 to it for $k := 2$, $\alpha_1 := \alpha$, $\alpha_2 := a/u_{n_1}$, $b_1 := n_2(m_3-1) - n_3(m_2-1)$, $b_2 := m_3 - m_2$, and we can take $B := 2n_3m_3 \ll (\log \gamma)^3(\log \log \gamma)^2$. Thus, we get that the left-hand side in (35) above is bounded from below as

$$> \exp(-c_{13}h(a/u_{n_1}) \log \log \gamma).$$

Comparing it with (35), we get

$$\log \gamma \ll h(a/u_{n_1}) \log \log \gamma.$$

Clearly, $h(a/u_{n_1}) \leq h(a) + h(u_{n_1}) = n_1 \log \alpha + O(1) \ll n_1$. We thus get that

$$\log \gamma \ll h(a/u_{n_1}) \ll n_1 \log \log \gamma, \quad \text{so} \quad n_1 \gg \frac{\log \gamma}{\log \log \gamma}.$$

Together with Lemma 3.4 (ii), this gives

$$\log \log \gamma \gg n_1 \gg \frac{\log \gamma}{\log \log \gamma}, \quad \text{so} \quad \log \gamma \ll (\log \log \gamma)^2,$$

therefore $\gamma = O(1)$, which takes care of the current case and completes the proof of the theorem.

So, it remains to consider the case when the left-hand side in (35) is zero. We record this as follows.

Lemma 3.6. *We have that $d < d_0(\mathbf{u})$ unless*

$$\left(\frac{\alpha^{n_2}}{(u_{n_1}/a)}\right)^{1/(m_2-1)} = \left(\frac{\alpha^{n_3}}{(u_{n_1}/a)}\right)^{1/(m_3-1)}.$$

We work in the remaining case. We have $N_2 = N_3$. Further, a/u_{n_1} and α are multiplicatively dependent. We insert estimates (33) into (32) and get that

$$\left| \frac{1}{(m_2-1)(\alpha/\beta)^{n_2}} - \frac{1}{(m_3-1)(\alpha/\beta)^{n_3}} \right| = \frac{(m_3-m_2)}{2(m_2-1)(m_3-1)N_2\gamma} + O\left(\frac{1}{X_1^3} + \frac{1}{(\alpha/|\beta|)^{2n_2}}\right).$$

Multiplying across by $(m_2-1)(\alpha/\beta)^{n_2}$, we get

$$\left| 1 - \left(\frac{m_2-1}{m_3-1}\right) \frac{1}{(\alpha/\beta)^{n_3-n_2}} \right| = \frac{(m_3-m_2)(\alpha/\beta)^{n_2}}{(m_3-1)N_2X_1} + O\left(\frac{(m_2-1)(\alpha/\beta)^{n_2}}{X_1^3} + \frac{m_2}{(\alpha/|\beta|)^{n_2}}\right).$$

The left-hand side is in $[1/2, 2]$ for $d > d_0(\mathbf{u})$. Indeed, otherwise we get that

$$(\alpha/|\beta|)^{n_3-n_2} \ll (m_3-1)/(m_2-1) \ll \log \gamma,$$

which gives $n_3-n_2 \ll \log \log \gamma$, which together with Lemma 3.3 (iii) shows that $\log \gamma \ll \log \log \gamma$, so $\gamma = O(1)$. This shows that

$$(m_3-m_2)(\alpha/|\beta|)^{n_2} \asymp (m_3-1)N_2X_1.$$

Taking logarithms we get

$$n_2 \log(\alpha/|\beta|) = \left(\frac{n_2-n_1}{m_2-1}\right) \log \alpha + \log \gamma + O(1).$$

Since we are in the case when a/u_{n_1} and α are multiplicatively dependent, it follows that $P(u_{n_1})$ is bounded, where $P(m)$ is the largest prime factor of m . Hence, $n_1 = O(1)$ (see Theorem 3.6 in [19]). We get

$$n_2 \log(\alpha/|\beta|) = \left(\frac{n_2}{m_2-1}\right) \log \alpha + \log \gamma + O(1).$$

But

$$\log \gamma = \left(\frac{n_2}{m_2-1}\right) \log \alpha + O(1),$$

by Lemma 3.3 (iii). We thus get that

$$n_2 \log(\alpha/|\beta|) = \left(\frac{2n_2}{m_2-1}\right) \log \alpha + O(1).$$

First, this shows that $m_2 = O(1)$. Secondly, for large n_2 it shows that $\log(\alpha/\beta)/\log \alpha$ is rational. Hence, α and $|\beta|$ are multiplicatively dependent. If $\mathbb{K} = \mathbb{Q}$, then $\alpha \in \mathbb{N}$. So $\beta \in \mathbb{Z}$. Since r and s are coprime and β and α are multiplicatively dependent, it follows that $\beta = \pm 1$. It thus follows that $\log(\alpha/|\beta|) = \log \alpha$, and we get $m_2 = 3$. Otherwise, if \mathbb{K} is quadratic, then since α and β are multiplicatively dependent, we get that $s = \pm 1$ and $\beta = \pm \alpha^{-1}$. Hence, $\log(\alpha/|\beta|) = 2 \log \alpha$ in this case and we get $m_2 = 2$.

To summarise, we have arrived at the following scenario.

Lemma 3.7. *The following holds:*

- (i) a/u_{n_1} and α are multiplicatively dependent.
- (ii) $(\beta, m_2) = (\pm 1, 3), (\pm \alpha^{-1}, 2)$.

3.9. The special cases. The case $(\beta, m_2) = (\pm 1, 3)$.

In this case,

$$(2X_1)^2 - 1 = \frac{Y_3}{Y_1} = \frac{u_{n_2}}{u_{n_1}} = a_1 \alpha^{n_2} + b_1 \beta^{n_2}, \quad \text{where } (a_1, b_1) := (a/u_{n_1}, b/u_{n_1})$$

and $\beta \in \{\pm 1\}$. Since this has solutions with arbitrarily large values of the even integer $2X_1$, it follows that $b_1 = \pm 1$ and

$$2X_1 = (a_1 \alpha)^{n_2/2} = (\alpha/(u_{n_1}/a))^{n_2/(m_2-1)}$$

for large n_2 . Inserting this into the asymptotic of Lemma 3.5, we get

$$\begin{aligned} 2X_1 &= \left(\frac{u_{n_3}}{(u_{n_1}/a)} \right)^{1/(m_3-1)} + \frac{(m_3-2)}{4(m_3-1)X_1} + O\left(\frac{1}{X_1^3} \right) \\ &= \left(\frac{\alpha^{n_3}}{(u_{n_1}/a)} \right)^{1/(m_3-1)} \left(1 \pm \frac{1}{a_1 \alpha^{n_3}} \right)^{1/(m_3-1)} + \frac{(m_3-2)}{4(m_3-1)X_1} \\ &\quad + O\left(\frac{1}{X_1^3} \right). \end{aligned}$$

The first term in the right-hand side above is $2X_1$. We thus get,

$$2X_1 = 2X_1 \pm \frac{2X_1}{(m_3-1)a_1 \alpha^{n_3}} + \frac{(m_3-2)}{4(m_3-1)X_1} + O\left(\frac{X_1}{\alpha^{2n_3}} + \frac{1}{X_1^3} \right).$$

The above gives us

$$0 = \frac{2}{(m_3-1)a_1 \alpha^{n_3}} \pm \frac{(m_3-2)}{4(m_3-1)X_1^2} + O\left(\frac{1}{\alpha^{2n_3}} + \frac{1}{X_1^4} \right).$$

Since $m_3 \geq 4$ and $\alpha^{n_3} \gg Y_{m_3}/Y_1 \gg \gamma^{m_3-1} \gg \gamma^3 \gg X_1^3$, we get

$$\frac{m_3-2}{4(m_3-1)X_1^2} = \pm \frac{2}{(m_3-1)a_1 \alpha^{2n_3}} + O\left(\frac{1}{\alpha^{2n_3}} + \frac{1}{X_1^4} \right) = O\left(\frac{1}{X_1^3} \right),$$

which gives $X_1 = O(1)$.

The case $(\beta, m_2) = (\pm \alpha^{-1}, 2)$. We start by finding a structure result for u_n/u_{n_1} for $n \geq n_1$.

Lemma 3.8. *Assume $s = -1$ and a/u_{n_1} and α and multiplicatively dependent. Then there exist an integer C such that $\alpha = \alpha_1^C$, $a/u_{n_1} = \alpha_1^{-n_1 C + \delta}$, where $\alpha_1 = (1 + \sqrt{5})/2$ and $\delta \in \{\pm 1\}$. Furthermore,*

$$u_n = u_{n_1}(\alpha_1^{(n-n_1)C+\delta} + \beta_1^{(n-n_1)C+\delta})$$

holds for all $n \geq n_1$, where $\beta_1 = (1 - \sqrt{5})/2$ is the conjugate of α_1 .

Proof. Let A and B be integers not both zero such that $(a/u_{n_1})^A = \alpha^B$. If $A = 0$ then also $B = 0$, a contradiction. If $B = 0$, it follows that $A \neq 0$ and since $a/u_{n_1} > 0$, it follows that $a/u_{n_1} = 1$, By conjugation in \mathbb{K} , we have that $b/u_{n_1} = 1$. Thus, the equation

$$u_{n_1} = a\alpha^{n_1} + b\beta^{n_1} \quad \text{becomes} \quad \alpha^{n_1} + \beta^{n_1} = 1. \quad (36)$$

Let us show that a similar equation as (36) is obtained when $AB \neq 0$. Assume that $AB \neq 0$. Hence, $a/u_{n_1} = \alpha^{A/B}$. Let α_1 be the fundamental unit in $\mathcal{O}_{\mathbb{K}}$ and write $\alpha = \alpha_1^C$ for some positive integer C . We then get that $a/u_{n_1} = \alpha_1^{AC/B}$. Since α_1 is fundamental, we have that $B \mid AC$, and since A and B are coprime, it follows that $B \mid C$. Thus, $a/u_{n_1} = \alpha_1^D$, where D is an integer. Conjugating in \mathbb{K} we get $b/u_{n_1} = \beta_1^D$, where β_1 is the conjugate of α_1 in \mathbb{K} . Now let us write

$$a\alpha^{n_1} + b\beta^{n_1} = u_{n_1}$$

as

$$\left(\frac{a}{u_{n_1}}\right)\alpha^{n_1} + \left(\frac{b}{u_{n_1}}\right)\beta^{n_1} = 1 \quad \text{or} \quad \alpha_1^{n_1 C + D} + \beta_1^{n_1 C + D} = 1. \quad (37)$$

Note that $n_1 C + D \neq 0$. Assume first that $\beta_1 > 0$. Then the above relation is impossible since one of $\alpha_1^{n_1 C + D}$ and $\beta_1^{n_1 C + D}$ is larger than 1 and the other is positive. Thus, $\beta_1 = -\alpha_1^{-1}$. Furthermore, the exponent $n_1 C + D$ is odd, since otherwise we may work with the relation

$$(\alpha_1^2)^{(n_1 C + D)/2} + (\beta_1^2)^{(n_1 C + D)/2} = 1$$

and get a similar contradiction as before. If $\alpha_1 \neq (1 + \sqrt{5})/2$, then $\alpha_1 > 2$ and $|\beta_1| < 1$, so the right-most relation in (37) is impossible since among $\alpha_1^{n_1 C + D}$, $\beta_1^{n_1 C + D}$ one of them will be larger than 2 in absolute value and the other smaller than 1 in absolute value so their sum cannot be 1. Thus, $(\alpha_1, \beta_1) = ((1 + \sqrt{5})/2, (1 - \sqrt{5})/2)$. In particular, the right-most relation (37) shows that $L_k = 1$ for $k := |C_1 n + D|$ where $\{L_k\}_{k \geq 0}$ is the sequence of Lucas numbers given by $L_0 = 2$, $L_1 = 1$ and $L_{k+2} = L_{k+1} + L_k$ for $k \geq 0$. The only solution is $n_1 C + D = \delta \in \{\pm 1\}$, so $D = -n_1 C + \delta$. Thus, for $n \geq n_1$, we have

$$u_n = \alpha^n + b\beta^n = u_{n_1}(\alpha_1^D(\alpha_1)^{Cn} + \beta_1^D\beta_1^{Cn}) = u_{n_1}(\alpha_1^{C(n-n_1)+\delta} + \beta_1^{C(n-n_1)+\delta}),$$

which is what we wanted. A similar conclusion is obtained in the case $B = 0$, since in that case, the above arguments show that $n_1 = 1$ and so $u_n = u_{n_1}(\alpha^n + \beta^n)$. Hence, in the case we can take $C = 1$ and $\delta = 1$. \square

Reindexing our sequence $\{u_n\}_{n \geq 1}$, replacing d by dY_1 (note that we have $Y_1 = u_{n_1} = O(1)$), we may assume that $\{u_n\}_{n \geq 1}$ is the sequence of Lucas numbers, that $Y_1 = 1 = L_1$, $2X_1 = Y_2 = L_{n_2}$, and $Y_{m_3} = P_{m_3-1}(X_1) = L_{n_3}$. We go back to Lemma 3.5, and write

$$\begin{aligned} \alpha^{n_2} \pm \frac{1}{\alpha^{n_2}} &= L_{n_2} = 2X_1 = L_{n_3}^{1/(n_3-1)} + \frac{(m_3-2)}{4(m_3-1)X_1} + O\left(\frac{1}{X_1^3}\right) \\ &= \alpha^{n_3/(m_3-1)} \left(1 \pm \frac{1}{\alpha^{2n_3}}\right)^{1/(m_3-1)} + \frac{(m_3-2)}{4(m_3-1)X_1} + O\left(\frac{1}{X_1^3}\right). \end{aligned}$$

The first term in the right-hand side is $\alpha^{n_2} = 2X_1 + O(1/X_1)$. Thus, $n_3 = n_2(m_3 - 1)$. Hence, we get

$$\pm \frac{1}{\alpha^{n_2}} = \pm \frac{2X_1}{(m_3-1)\alpha^{2n_3}} + \frac{(m_3-2)}{4(m_3-1)X_1} + O\left(\frac{1}{X_1^3}\right).$$

The left-hand side is $1/(2X_1) + O(1/X_1^3)$. Since $n_3 = n_2(m_3 - 1)$ and $m_3 \geq 3$, it follows that $\alpha^{2n_3} \gg \alpha^{4n_2} \gg X_1^4$. Thus, we get

$$\pm \frac{1}{2X_1} = \frac{(m_3-2)}{4(m_3-1)X_1} + O\left(\frac{1}{X_1^3}\right),$$

so

$$\left(\pm \frac{1}{2} \pm \frac{(m_3-2)}{4(m_3-1)}\right) \frac{1}{X_1} = O\left(\frac{1}{X_1^3}\right).$$

The coefficient of $1/X_1$ in the left-hand side is non-zero. Thus, we get $X_1 = O(1)$, which finishes the analysis for this case and the proof of the theorem.

3.10. The degenerate cases. Consider now the degenerate cases $r = 0$ or $s = 0$. In this case, $\beta = \pm\alpha$ or $\beta = 0$, so $u_n = a_1\alpha^n$ where $a_1 = a$ if $\beta = 0$ and $a_1 \in \{a - b, a + b\}$. Indeed, assume that $Y_m = u_n$ has three solutions $(m, n) = (m_i, n_i)$ for $i = 1, 2, 3$. As before, we may assume that n_2 is large. Then $Y_{m_2} = a_1\alpha^{n_2}$ and $Y_{m_3} = a_1\alpha^{m_3}$ have the same prime factors. Hence, Y_{m_3} has no primitive divisors in the sense that every prime factor of Y_{m_3} divides Y_m for some $m < m_3$. More than 100 years ago, Carmichael [6] showed that $m_3 \leq 12$ (see [5] for the state of the art in this problem). In fact, the classification from [5] shows that apart from a few specific sequences $\{Y_k\}_{k \geq 1}$, which can be avoided by making $d > d_0(\mathbf{u})$, we have $m_3 \in \{2, 3, 4, 6\}$. Since in fact $m_3 \geq 3$, it follows that $m_3 \in \{3, 4, 6\}$. If $m_3 = 4$, then $2X_1^2 - 1 = X_2$ divides Y_4 and so its largest prime factor is bounded. This shows that $X_1 = O(1)$, so $d = O(1)$. If $m_3 \in \{3, 6\}$, then $Y_3/Y_1 = 4X_1^2 - 1$ is a divisor of Y_{m_3} so its largest prime factor is bounded, showing again that $X_1 = O(1)$.

4. Comments

Slight modifications of the above arguments will show that the theorem remains valid if instead of working with the Y -coordinates of the Pell equation $X^2 - dY^2 = 1$, we work with the Y -coordinates of the Pell equation $X^2 - dY^2 = \pm 1$ or $X^2 - dY^2 = \pm 4$. We preferred to give the proof for the case when the right-hand side is 1 since this case is typographically easier but the other cases can be treated in the same way without any new ideas.

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