New York Journal of Mathematics

New York J. Math. **27** (2021) 918–922.

$\mathcal{C}u$-nuclearity implies LLP and exactness

Cristian Ivanescu and **Dan Kučerovský**

ABSTRACT. The $\mathcal{C}u$-nuclearity property is an analogue of Skandalis's notion of K-nuclearity, adapted to the case of Cuntz semigroups of C^*-algebras. We prove that this implies nuclearity, and we introduce a weaker form of the condition. We prove that the new condition weak $\mathcal{C}u$-nuclearity, for simple separable C^*-algebras, implies exactness and the local lifting property (LLP). We also prove that if A is a simple C^*-algebra with the weak $\mathcal{C}u$-nuclearity property, and B is any simple C^*-algebra, then $A \otimes_{\min} B = A \otimes_{\max} B$. We prove that $\mathcal{C}u$-nuclearity does imply nuclearity, and that in some cases this is also true for weak $\mathcal{C}u$-nuclearity.

A unital C^*-algebra is a noncommutative generalization of the algebra $C(X)$ of continuous functions on a compact topological space. In this commutative case there is a straightforward and essentially unique way to define a tensor product of these objects. In the noncommutative case, the situation is complicated by the fact that there is, in general, more than one C^*-norm that one can put on the tensor product of C^*-algebras.

The Cuntz semigroup has become a standard technical tool for the study of C^*-algebras. The Cuntz semigroup $\mathcal{C}u(A)$ can be defined to be an ordered semigroup with elements given by the positive elements of $A \otimes \mathcal{K}$, modulo an equivalence relation [10]. The projection-class elements are those elements that are equivalent to projections of $A \otimes \mathcal{K}$.

The following definition for the Cuntz semigroups was suggested in [7]:

Definition 1. A C^*-algebra A is $\mathcal{C}u$-nuclear if the canonical quotient map from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ induces isomorphisms at the level of Cuntz semigroups for all C^*-algebras B. We further say that a C^*-algebra A is weakly $\mathcal{C}u$-nuclear if $\mathcal{C}u(A \otimes_{\max} B)$ is isomorphic as a Cuntz semigroup to $\mathcal{C}u(A \otimes_{\min} B)$ for all C^*-algebras B.

At first glance, these conditions might seem very similar, but the difference is that in the first definition, we are adding conditions to a specific C^*-algebraic map, whereas in the second definition, the given data consists of a collection of Cuntz semigroup isomorphisms. It is generally quite hard to deduce C^*-algebraic statements from being given a $\mathcal{C}u$-isomorphism.

The question was posed there of whether either or both of these properties might imply nuclearity. Due to recent work by Pisier [11], this now seems less

Received January 3, 2020.

2010 *Mathematics Subject Classification.* Primary 46A32, 46L06, Secondary 47L07, 47L50.

Key words and phrases. C*-algebra, nuclearity.

The second-named authour thanks NSERC for financial support.

ISSN 1076-9803/2021

likely to be true in full generality, however, we obtained several positive results in this direction. We present the short proofs in this note. These results give ways in which C*-algebra isomorphisms (of full and reduced tensor products) can be deduced from Cuntz semigroup isomorphisms, and this is of course very much in the spirit of the classification program for C*-algebras.

Theorem 2. *Let A and B be C*-algebras with finitely many ideals. Let the C*-algebra A have the weak $\mathcal{C}u$-nuclear property. Then*

$$A \otimes_{min} B = A \otimes_{max} B.$$

Proof. Suppose that it is not true that $A\otimes_{\min}B = A\otimes_{\max}B$. Then the canonical quotient map

$$h: \ A\otimes_{\max}B \to A\otimes_{\min}B$$

has a nontrivial kernel, denoted, say, $I \subset A\otimes_{\max}B$. Combes [3] has shown that ideals of a C*-algebra correspond to order ideals of the positive cone. We now observe that this mapping is well-defined at the level of Cuntz semigroups, *i.e.,* we claim that if I is an ideal in a C*-algebra C then

$$\left\{[a] \in \mathcal{C}u(C) \mid a \in (I \otimes \mathcal{K})\right\}$$

is an order ideal in $\mathcal{C}u(C)$, and conversely that if J is an order ideal in $\mathcal{C}u(C)$ then the set $\{c \in C \mid [cc^*] \in J\}$ is an ideal in C. On the other hand, [1, II.9.6.7], if A and B are simple, or more generally, have finitely many ideals, then a finite set is dense in the primitive ideal space of the minimal tensor product $A\otimes_{\min}B$, implying that the minimal tensor product itself has finitely many order ideals. From the hypothesis that A has the weak $\mathcal{C}u$-nuclearity property, it follows that $\mathcal{C}u(A\otimes_{\min}B)$ is isomorphic as an ordered semigroup to $\mathcal{C}u(A\otimes_{\max}B)$. This isomorphism will preserve order ideals, but by the remarks we have made, one of these semigroups has a nontrivial order ideal that the other does not. This is a contradiction. □

Following Pisier [11, pg. 517, bottom], a concrete C*-algebra $A \subset B(H)$ has the WEP if and only if there is a contractive projection $P: \ B(H)^{**} \to A^{**}$. Kirchberg has shown [8, pg. 451] that this is equivalent to

$$A\otimes_{\min}F(G) = A\otimes_{\max}F(G),$$

where $F(G)$ is the full C*-algebra associated with a countably generated free group. The LLP has a complex definition, however, we may as well define it as the property

$$A\otimes_{\min}B(H) = A\otimes_{\max}B(H),$$

where $B(H)$ denotes the classic algebra of operators on an infinite-dimensional separable Hilbert space. This is justified by a theorem of Kirchberg's [8, Prop. 1.1].

Corollary 3. *If a C*-algebra with finitely many ideals is weakly $\mathcal{C}u$-nuclear then it is exact and has the LLP.*

Proof. By Theorem 2, choosing $B(H)$ as a test algebra, we immediately have the LLP. To show exactness requires a little more. By Theorem 2, this time choosing the classic Calkin algebra as a test algebra, we deduce that

$$A \otimes_{\min} (B(H)/K(H)) = A \otimes_{\max} (B(H)/K(H)).$$

But because the maximal tensor product is always exact, it follows that

$$A \otimes_{\max} (B(H)/K(H)) = (A \otimes_{\max} B(H))/(A \otimes_{\max} K(H)).$$

From the first sentence of the proof, it follows that on the right hand side, $A \otimes_{\max} B(H)$ equals $A \otimes_{\min} B(H)$, and of course $K(H)$ is nuclear, so we can conclude that

$$A \otimes_{\min} (B(H)/K(H)) = (A \otimes_{\min} B(H))/(A \otimes_{\min} K(H)).$$

But this is one of the definitions of exactness, and thus we conclude that A is exact, as claimed. $\square$

Kirchberg conjectured that all C^*-algebras have the QWEP, and if this did hold it would be very useful here. However, there are now two counterexamples to Kirchberg's conjectures, one of which shows that it is not even true that the WEP together with the LLP imply nuclearity [11]. Thus, at the moment we do not know if weak $\mathcal{C}u$-nuclearity does imply nuclearity. A relevant C^*-algebraic question is whether the test algebras for the nuclearity property can be taken to be simple: in other words:

Question 4. If, for some simple C^*-algebra A, it is true that for every simple C^*-algebra B, we have

$$A \otimes_{\min} B = A \otimes_{\max} B,$$

then does it follow that the algebra A is nuclear?

Even more optimistically, one could ask if the reduced free group is sufficient as a test object:

Question 5. If, for some simple exact C^*-algebra A, it is true that

$$A \otimes_{\min} B = A \otimes_{\max} B,$$

where B is the reduced free group $C^*_r(F_\infty)$, does it follow that the algebra A is nuclear?

Motivated by Question 4, we prove the following key corollary:

Corollary 6. *For C^*-algebras with finitely many ideals, weak $\mathcal{C}u$-nuclearity implies nuclearity.*

Proof. Suppose we are given a weakly $\mathcal{C}u$-nuclear C^*-algebra A with finitely many ideals. Corollary 3 implies that A is exact; and it is known [8, pg. 453, middle] that an exact C^*-algebra A will be nuclear if it satisfies

$$A \otimes_{\min} A^{\mathrm{op}} = A \otimes_{\max} A^{\mathrm{op}}, \tag{1}$$

where A^{op} denotes the C^*-algebra A with reversed multiplication. Since A^{op} also has finitely many ideals, Theorem 2 can be applied with $B = A^{\mathrm{op}}$ to show that equation (1) holds. Thus the C^*-algebra A is nuclear. □

If we drop the adjective weak, then $\mathcal{C}u$-nuclearity does imply nuclearity, without assumptions on the number of ideals:

Corollary 7. *If a C^*-algebra is $\mathcal{C}u$-nuclear, then it is nuclear.*

Proof. We are given that the canonical quotient map from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ induces isomorphisms at the level of Cuntz semigroups. But then, the Cuntz subsemigroup of the kernel of this map in $A \otimes_{\max} B$ is zero, for example because the Cuntz semigroup functor is exact on short exact sequences (see [2]), implying that positive elements of the kernel are zero. But then the canonical quotient map from $A \otimes_{\max} B$ to $A \otimes_{\min} B$ is an isomorphism. □

References

[1] Blackadar, Bruce. Operator algebras. Theory of C^*-algebras and von Neumann algebras. Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. *Springer-Verlag, Berlin*, 2006. xx+517 pp. ISBN: 978-3-540-28486-4; 3-540-28486-9. MR2188261, Zbl 1092.46003, doi: 10.1007/3-540-28517-2. 919

[2] Ciuperca, Alin; Robert, Leonel; Santiago, Luis. The Cuntz semigroup of ideals and quotients and a generalized Kasparov stabilization theorem. *J. Operator Theory* **64** (2010), no. 1, 155–169. MR2669433, Zbl 1212.46084, arXiv:0710.5800. 921

[3] Combes, François. Sur les faces d'une C*-algèbre. *Bull. Sci. Math.* (2) **93** (1969), 37–62. MR0265947, Zbl 0177.17801. 919

[4] Coward, Kristofer T.; Elliott, George A.; Ivanescu, Cristian. The Cuntz semigroup as an invariant for C*-algebras. *J. Reine Angew. Math.* **623** (2008), 61–193. MR2458043, Zbl 1161.46029, arXiv:0705.0341, doi: 10.1515/CRELLE.2008.075.

[5] Grothendieck, Alexandre. Produits tensoriels topologiques et espaces nucléaires. *Mem. Amer. Math. Soc.* **16** (1955), 1–191. MR0075539, Zbl 0123.30301, doi: 10.1090/memo/0016.

[6] Guichardet, Alain. Tensor products of C*-algebras. *Dokl. Akad. Nauk SSSR* **160**, (1965), 986–989; translation in *Sov. Math., Dokl.* **6** (1965), 210–213. MR0176351, Zbl 0127.07303

[7] Ivanescu, Cristian; Kučerovský, Dan. Traces and Pedersen ideals of tensor products of nonunital C*-algebras. *New York J. Math.* **25** (2019), 423–450. MR3982248, Zbl 1439.46042. 918

[8] Kirchberg, Eberhard. On nonsemisplit extensions, tensor products and exactness of group C*-algebras. *Invent. Math* **112** (1993), no. 3, 449–489. MR1218321, Zbl 0803.46071, doi: 10.1007/BF01232444. 919, 920

[9] Lance, Christopher. On nuclear C*-algebras. *J. Funct. Anal.* **12** (1973), 157–176. MR0344901, Zbl 0252.46065, doi: 10.1016/0022-1236(73)90021-9.

[10] Ortega, Eduard; Rørdam, Mikael; Thiel, Hannes. The Cuntz semigroup and comparison of open projections. *J. Funct. Anal.* **260** (2011), no. 12, 3474–3493. MR2781968, Zbl 1222.46043, doi: 10.1016/j.jfa.2011.02.017. 918

[11] Pisier, Gilles. A non-nuclear C*-algebra with the weak expectation property and the local lifting property. *Invent. Math* **222** (2020), no. 2, 513–544. MR4160874, Zbl 07269002, doi: 10.1007/s00222-020-00977-4. 918, 919, 920

[12] Schaefer, Helmut H. Topological vector spaces. *The Macmillan Co., New York; Collier-Macmillan Ltd., London*, 1966. ix+294 pp. MR0193469, Zbl 0141.30503.

[13] SKANDALIS, GEORGES. Une notion de nucléarité en K-théorie (d'après J. Cuntz). *K-Theory* **1** (1988), no. 6, 549–573. MR0953916, Zbl 0653.46065, doi: 10.1007/BF00533786.

(Cristian Ivanescu) DEPARTMENT OF MATHEMATICS AND STATISTICS, GRANT MACEWAN UNIVERSITY, EDMONTON, ALBERTA, T5J 4S2, CANADA
ivanescuc@macewan.ca

(Dan Kučerovský) DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW BRUNSWICK, FREDERICTON, NEW BRUNSWICK, E3B 5A3, CANADA
dkucerov@unb.ca

This paper is available via http://nyjm.albany.edu/j/2021/27-36.html.