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Lehmer's question, graph complexity growth and links

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Abstract. Lehmer's question, an open question about the Mahler measure of monic integral polynomials, is shown to be equivalent to a question about the complexity growth rate of signed 1-periodic graphs. If G is a d-periodic graph (i.e. G has a co-finite free $\mathbb{Z}^d$-action by automorphisms), then a d-variable polynomial Δ_G can be defined with Mahler measure equal to the logarithmic growth rate γ_G of a complexity defined for the finite quotients of G.

A plane 1-periodic graph determines a link via projection and the medial graph construction. The polynomial Δ_G can be determined from the Alexander polynomial of the link. The complexity growth rate γ_G of any d-periodic graph is at least $\log 2$. An investigation of plane 1- and 2-periodic graphs yields more connections with knot theory including work of A. Champanerkar and I. Kofman.

Contents

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1. Introduction

Lehmer's question is about monic integral polynomials that are not products of cyclotomic polynomials. Given such a polynomial, it asks whether the absolute value of the product of its roots outside the unit circle can be arbitrarily close to 1. (The product is today called the "Mahler measure" of the polynomial.) D.H. Lehmer posed the question in [19]. Despite efforts for almost 90 years, Lehmer's question remains unanswered.

Finding questions equivalent to Lehmer's in unexpected places might afford new insights. D. Lind found such an equivalence by observing that a negative answer is equivalent to the existence of an ergodic automorphism of the infinite-dimensional torus $\mathbb{T}^\infty$ with finite entropy [28, p. 161].

In his paper, Lehmer displayed a polynomial of degree 10 with a single root outside the unit circle, a root approximately equal to 1.17628. No polynomial with Mahler measure closer to 1 has since been found. Later it was observed that by replacing the variable of Lehmer's polynomial with its negative, a modification that preserves Mahler measure, one obtains the Alexander polynomial of a well-known knot, the so-called (-2, 3, 7)-pretzel knot, a hyperbolic knot with notable properties [17].

In [35] the authors showed that for an investigation of Lehmer's question one can restrict attention to Alexander polynomials of knots in the 3-sphere and, more generally, lens spaces. Previous work [31] showed that the Mahler measure of an Alexander knot polynomial is the topological entropy of an algebraic dynamical system, a growth rate of the order of torsion in the first homology groups of the the various cyclic branched covers of the knot as the cover index increases. Hence Lehmer's question became a question of knot theory. (See [32, 33] for extensions of this work; see [14, 38] for other connections with Lehmer's question and more background information.)

The central purpose of the present paper is to present an equivalence between Lehmer's question and a question about graphs, specifically graphs in an annulus $\mathbb{S} \times I$ that are signed, that is, with edges weighted by ± 1. Such a graph lifts to an infinite cyclic covering graph in $\mathbb{R} \times I$ for which a polynomial can be defined using a variant of the well-known Laplacian matrix. Lehmer's polynomial arises in a simple fashion. In fact the collection of all polynomials obtained will be shown to suffice for the study of Lehmer's question. As in the case of knots, the Mahler measure of each represents the topological entropy of a dynamical system while also having an interpretation as a growth rate of torsion in abelian groups associated to the intermediate finite cyclic covering graphs.

The graphs are generally not embedded. However, when they are, knots and links again enter our story. The medial graph construction associates a link diagram to an embedded graph, and the torsion groups associated to the graph have a precise topological interpretation.

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2. Outline

We investigate interactions among graphs, knots and dynamical systems. The reader who is interested primarily in Lehmer's question might skip over material about knots and links.

In order to guide the reader, we briefly describe the main contents of each section. Necessary definitions and background material can be found in them.

In Section 3 we introduce the *Laplacian group* of a finite signed graph. It is the finitely generated abelian group presented by the Laplacian matrix. The order of its torsion subgroup is a measure of complexity of the graph, the *torsion complexity*. For graphs that are *unsigned* (i.e., all edges have weight 1), the complexity is the number of spanning trees of the graph, a measure of complexity that is frequently used in graph theory.

The well-known medial graph construction is reviewed in Section 4. Signed plane graphs determine link diagrams by the construction.

Laplacian matrices of plane graphs are closely related to the vector spaces of *Fox p-colorings* (or, equivalently, *Dehn p-colorings*) of the associated link diagrams, for any prime p, as discussed in Section 5. We see that when field $\mathbb{Z}/p$ is replaced by the compact "circle group" $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, a close relationship between the Laplacian groups and the abelian groups of $\mathbb{T}$-colorings of link diagrams results.

In Section 6 we consider graphs in the annulus. Such a graph has a regular covering that is 1-periodic, admitting an action by $\mathbb{Z}$. The covering graph is a special case of a more general d-periodic graph, which admits a $\mathbb{Z}^d$-action. We describe a Laplacian matrix with entries in the polynomial ring $\mathbb{Z}[\mathbb{Z}^d]$, and we use it to define the *Laplacian module*. The determinant of the matrix will be called the *Laplacian (determinant) polynomial* of the graph.

In Section 7 we expand our pallete of colors from $\mathbb{Z}/p$ to the elements of the circle group $\mathbb{T}$. Homomorphisms from the Laplacian module to $\mathbb{T}$ are themselves elements of a compact abelian group. The $\mathbb{Z}^d$ action on the graph induces an action on the group. The homomorphisms correspond to $\mathbb{T}$-colorings of the graph, with the colors of vertices dictated by the Laplacian matrix.

The Laplacian polynomial of a d-periodic graph can be computed directly from its set of *cycle-rooted spanning forests*, a result of R. Forman [11]. We review the idea in Section 8.

In Section 9 we consider the links determined by plane 1-periodic graphs. We show that the Laplacian polynomial of the graph can be obtained from the (multivariable) Alexander polynomial of the link by specializing some of its variables.

We bring algebraic dynamics into our story in Section 10. There we see that for any d-periodic graph, the Mahler measure of the Laplacian polynomial is the logarithmic growth rate of torsion complexity of the graph. (This generalizes a previous result for unsigned graphs.) The quantity will be called the *complexity growth rate of the graph*.

In Section 11 we prove that the Laplacian polynomials of 1-periodic signed graphs suffice for investigating Lehmer's question. Lehmer's polynomial arises from a graph in the annulus with one vertex and four edges.

Section 12 focuses on unsigned graphs. We show that the d-dimensional grid graph has the smallest complexity growth rate of all unsigned d-periodic graphs. Consequently, the complexity growth rate of any d-periodic unsigned graph is at least $\log 2$, and that is the reason that we must place signs our graphs for an investigation of Lehmer's question. Finally, a look at plane 1- and 2-periodic graphs leads us back to knot theory and connections with work of A. Champanerkar, I. Kofman, and J. Purcell.

3. Laplacian groups and complexity of graphs

Let G be a graph, not necessarily planar, with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G) = \{e_1, \dots, e_m\}$. The graph is allowed to have loops and multiple edges, but no isolated vertices. Each edge $e \in E(G)$ is labeled with a sign $\sigma_e = \pm 1$. All graphs that we consider are assumed to have signed edges. The graph is *unsigned* if every $\sigma_e = +1$.

The *signed adjacency matrix* of G is the $n \times n$ matrix $A = (a_{i,j})$ such that $a_{i,j}$ is the sum of the signs of all edges between v_i and v_j, with loops counted twice. Define the *signed degree matrix* $\delta = (\delta_{i,j})$ to be the $n \times n$ diagonal matrix with $\delta_{i,i}$ equal to the sum of signs of edges incident on v_i. Again, loops contribute twice.

An integer matrix presents an abelian group in which columns correspond to the generators, and rows to the relations, of the group. Equivalently, the group is the cokernel of the matrix when regarded as a linear transformation of free abelian groups.

Definition 3.1. The *Laplacian matrix* of a finite graph G is $L_G = \delta - A$, where δ and A are the signed degree and adjacency matrices, respectively, defined above. The abelian group presented by L_G is the *Laplacian group* of G, denoted by $\mathcal{L}_G$. The *(torsion) complexity* κ_G is the order of the torsion subgroup $T\mathcal{L}_G$.

We note that in computing the integer matrix L_G we may ignore loops, since they contribute equally to A and δ.

When we regard the entries of L_G modulo a prime p, the rows represent the relations needed to p-color the graph. A *p-coloring* of G is an assignment of elements ("colors") of $\mathbb{Z}/p$ to the vertices adjacent to v:

$$\delta\alpha = \sum \sigma_{e_i}\alpha_i,$$

where δ, α are the signed degree and color, respectively, of v, and the summation is over all edges e_i incident to v, and α_i is the color of the other vertex of e_i. (A self-loop at v is counted as two edges from v to v.) Here we are extending the notion of p-coloring in [18] to signed graphs.

Clearly the space of p-colorings of G is isomorphic to the null space of L_G. Nontrivial p-colorings exist precisely when its dimension is greater than 1. Note that the abelian group of p-colorings of G is isomorphic to $\mathcal{L}_G \otimes \mathbb{Z}/p$.

Returning to integer coefficients, the nullity of the Laplacian matrix L_G is equal to 1 whenever G is connected and unsigned (see, for example, Lemma 13.1.1 of [12]). In this case the Laplacian group $\mathcal{L}_G$ decomposes as the direct sum of $\mathbb{Z}$ and the torsion subgroup $T\mathcal{L}_G$. The Matrix Tree Theorem (c.f. [39]) implies that $\kappa_G = |T\mathcal{L}_G|$ is equal to the number of spanning trees of G.

More generally, we define *tree complexity* τ_G of a connected graph G by

$$\tau_G = \left|\sum_T \prod_{e\in E(T)} \sigma_e\right|,$$

where the summation is taken over all spanning trees of G. If G is not connected, then we define τ_G to be the product of the tree complexities of its connected components. Again by [39], we have $\tau_G = \kappa_G$ if and only if τ_G is nonzero; for connected G this common value is equal to the absolute value of any $(n-1)\times(n-1)$ principal minor of L_G. However, the following example shows that τ_G can vanish, whereas κ_G is positive by definition.

Example 3.2. Consider the connected graph G in Figure 1. Unlabeled edges here and throughout will be assumed to have sign $+1$. The Laplacian matrix L_G is square of size 8. A routine calculation shows that any principal 7×7 minor of L_G vanishes, and hence $\tau_G = 0$. However, the absolute value of the greatest common divisors of the 6×6 minors of L_G is 9, and so $\kappa_G = 9$.

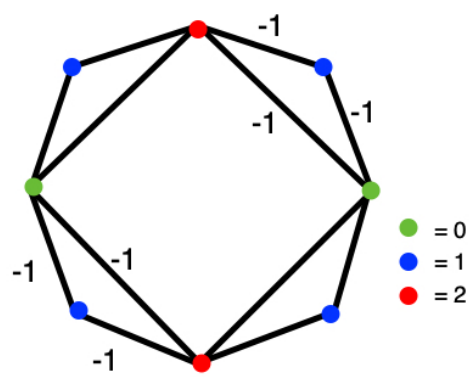

FIGURE 1. Graph G with $\tau_G = 0$ and $\kappa_G = 9$

We can go further by computing the Smith Normal Form of L_G. We then see that $\mathcal{L}_G \cong \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$. The vector space of p-colorings has dimension 2 for all primes $p \neq 3$, while the space of 3-colorings of G has dimension 4. An example of a 3-coloring is shown.

4. Link diagrams and Tait graphs

A *link* ℓ in $\mathbb{R}^3$ (or equivalently $\mathbb{S}^3$) is a set of smoothly embedded, pairwise disjoint simple closed curves. Any link can be described by a *diagram* D, a 4-valent plane graph with a hidden-line device in a neighborhood of each vertex indicating how one strand of the link passes over another.

We will make use of a few other related terms. An *arc* of D is a maximal connected subset. Following [15], we refer to the underlying graph of D as a *universe* of ℓ, denoted here by $|D|$. Finally, a *region* of D is a connected component of $\mathbb{R}^2 \setminus |D|$.

As usual, isotopic links are regarded as the same. It is well known that two links are isotopic if and only if a diagram of one can be transformed into a diagram of the other by a finite sequence of local "Reidemeister moves" as in Figure 2 (see [5] or [20] for details). Consequently, any quantity that is determined by a diagram and is unchanged by Reidemeister moves is a link invariant.

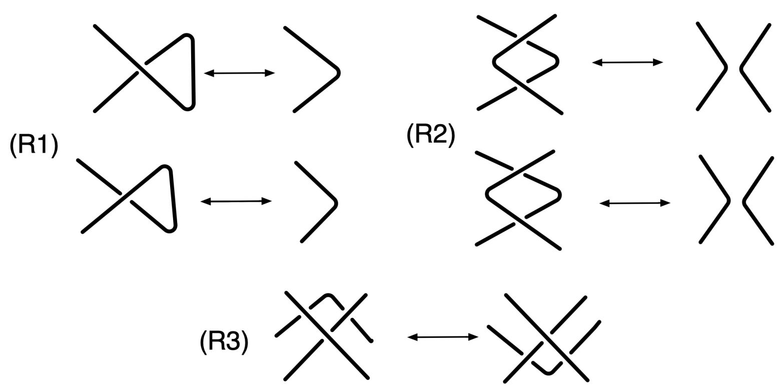

FIGURE 2. Reidemeister moves

We can obtain a plane graph G from a link diagram D by the following familiar procedure. First we *checkerboard shade D*, shading some of the regions so that every edge meets a single shaded region. There are two checkerboard shadings of D, but for the sake of definiteness we will choose the one in which the unbounded region is unshaded. We construct a graph G with a vertex in each shaded region of D and an edge through each crossing, joining the vertices of the regions on both sides. The sign of the edge is determined by the type of crossing, as in Figure 3. Such a graph is often called a "Tait graph," in honor of Peter Guthrie Tait, a 19th century pioneer of knot theory.

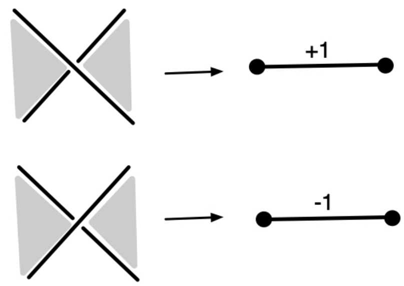

FIGURE 3. Constructing a graph from a link diagram

For some link diagrams it can happen that the Tait graph has isolated vertices. We avoid this by always assuming that $|D|$ is connected, a condition that we can assume without loss of generality by applying Reidemeister moves to D.

For a plane graph G we can reverse the above procedure to obtain a link digram D. For this we use the *medial construction*, replacing each edge of G by a pair of arc segments that run parallel to the edge except at the middle, where they cross, as in Figure 4. We join the segments near vertices in the obvious way without creating any additional crossings.

Figure 7 shows the diagram of a 3-component link ℓ obtained from the graph of Example 3.2. According to [8] the link was introduced by J. Milnor. Original interest in the link came from the fact that it was the first non-alternating boundary link discovered with zero Alexander polynomial, a fact that we will not use here. We will return to the link in the next section.

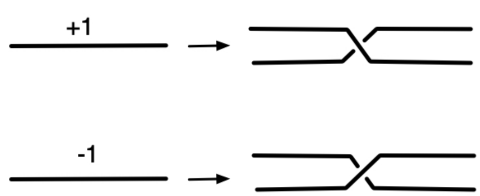

FIGURE 4. Constructing a link diagram from a plane graph

5. Coloring link diagrams

Assume that D is a diagram of a link ℓ. Let p be a prime. A *(Fox) p-coloring* of D is an assignment of elements of $\mathbb{Z}/p$, called *colors*, to the arcs of D such that twice the color of any overcrossing arc is equal to the sum of the colors of its undercrossing arcs, as in Figure 5. A p-coloring is *trivial* if all assigned colors are the same.

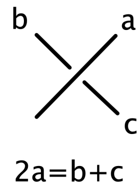

FIGURE 5. Fox coloring relation

We denote the vector space of p-colorings of D by $\mathrm{Col}_p(D)$ and refer to it as the *p-coloring space* of the diagram. It is a vector space over $\mathbb{Z}/p$, with addition and scalar multiplication performed arc-wise. A routine exercise shows that diagrams differing by a Reidemeister move have isomorphic p-coloring spaces. Hence $\mathrm{Col}_p(D)$ is an invariant of the link ℓ, and so we write $\mathrm{Col}_p(\ell)$ and speak of the vector space as the p-coloring space of the link. (For more information about this popular construction of knot theory, there are many excellent expositions such as [22].)

Fox envisioned p-colorings as homomorphisms from the link group $\pi_\ell = \pi_1(\mathbb{R}^3 \setminus \ell)$ onto the dihedral group

$$D_{2p} = \langle \tau, \alpha \mid \tau^2 = 1, \alpha^p = 1, \alpha\tau = \tau\alpha^{-1} \rangle.$$

He described the correspondence using the Wirtinger presentation of π_ℓ, in which generators (resp. relations) are identified with arcs (resp. crossings) of a diagram D. Given a p-colorings of D we obtain a homomorphism from π_ℓ to D_{2p} by sending a Wirtinger generator of an arc colored by $k \in \mathbb{Z}/p$ to the element $\tau\alpha^k$ of D_{2p}.

If instead of the Wirtinger presentation, we use the Dehn presentation of π_ℓ, a presentation in which generators (resp. relations) are identified with bounded regions (resp. crossings), then a p-coloring becomes an assignment of colors to the bounded regions of D. We assign 0 to the unbounded region. The condition at each crossing that corresponds to the Fox p-coloring condition appears in Figure 6. Details can be found in [36]. We will call such an assignment of colors to the regions a *Dehn p-coloring* of the diagram. The collection of all Dehn p-colorings of a diagram is vector space under region-wise addition and scalar multiplication, isomorphic to the space of Fox p-colorings.

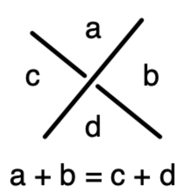

FIGURE 6. Dehn p-coloring condition

Assume that D is a link diagram arising from a plane graph G by the medial construction. Any p-coloring of G determines Dehn and Fox p-colorings of D. To see this, first assign the colors of the vertices of G to the associated shaded regions of the link diagram. Then use the Dehn coloring relations to determine uniquely the colors of the unshaded regions. This is possible since the unbounded region is already labeled (with 0). Uniqueness follows from the observation that if we determine the color of some unshaded region and then follow a simple closed path around a vertex, determining the colors of successive unshaded regions along the way, then when we return to the initial unshaded region, the Laplace relation forces us to arrive at the same color with which we began. (More details can be found in [36].) Finally, assign to each arc of D the sum of the colors of the regions on both sides. It is easy to verify that we obtain in this way is well defined Fox p-coloring of the diagram.

Conversely, any Fox p-coloring of a link diagram determines a p-coloring of the associated Tait graph. The Dehn color of any region is the sum of the colors of the arcs that we cross traveling along any path to the unbounded region. The graph coloring is simply a restriction to the shaded regions.

The two processes above are inverses of each other. Hence the vector spaces of p-colorings of G and D are isomorphic. As a result, we have:

Proposition 5.1. *Let G be a plane graph and D an associated link diagram. Then* $\mathcal{L}_G \otimes \mathbb{Z}/p$ *is isomorphic to* $\mathrm{Col}_p(D)$, *for any prime p.*

As an example, consider the diagram of Milnor's boundary link shown in Figure 7. The Fox 3-coloring displayed corresponds to the 3-coloring of the graph G in Figure 1.

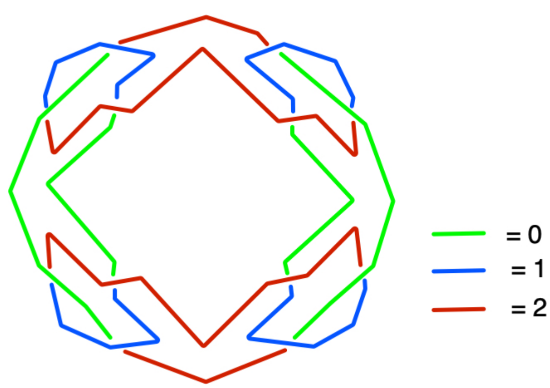

FIGURE 7. 3-colored diagram of Milnor's boundary link

Every finite cyclic group is contained in the compact abelian "circle group" $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as a subgroup. If we replace $\mathbb{Z}/p$ by $\mathbb{T}$, then the coloring vector spaces for graphs and link diagrams become compact abelian groups. (This extension of Fox p-coloring was introduced in [31].) The Laplace group $\mathcal{L}_G$

tensored with $\mathbb{T}$ consists of $|T\mathcal{L}_G|$ tori each of dimension equal to the nullity of L_G. A similar description of the circle-coloring group $\mathrm{Col}_{\mathbb{T}}(\ell)$ applies. In the following sections we will explore this idea for infinite graphs and links with symmetries. The result is a rich structure that brings algebraic dynamics into our story.

6. Periodic graphs and Laplacian modules

A (signed) graph G is *d-periodic* if it admits a cofinite free $\mathbb{Z}^d$-action by automorphisms that preserves the signs of edges. By *cofinite* we mean the quotient graph $\overline{G}$ is finite, while an action is *free* if the stabilizer of any edge or vertex is trivial. Such a graph G is locally finite, and finite if and only if $d = 0$.

We regard $\mathbb{Z}^d, d \geq 1$, as the multiplicative abelian group freely generated by $x_1, \dots, x_d$. We denote the Laurent polynomial ring $\mathbb{Z}[\mathbb{Z}^d] = \mathbb{Z}[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$ by $\mathcal{R}_d$. As an abelian group $\mathcal{R}_d$ is generated freely by monomials $x^{\mathbf{n}} = x_1^{n_1} \dots x_d^{n_d}$, where $\mathbf{n} = (n_1, \cdots, n_d) \in \mathbb{Z}^d$. We represent $(0, \dots, 0)$ by $\mathbf{0}$. For notational convenience, when $d = 1$, we replace x_1 by x.

The vertex set $V(G)$ and the edge set $E(G)$ consist of finitely many vertex orbits $\{v_{1,\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}, \dots, \{v_{n,\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}$ and signed edge orbits $\{e_{1,\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}, \dots, \{e_{m,\mathbf{n}} \mid \mathbf{n} \in \mathbb{Z}^d\}$, respectively. The $\mathbb{Z}^d$-action is determined by

$$x^{\mathbf{n}'} \cdot v_{i,\mathbf{n}} = v_{i,\mathbf{n}+\mathbf{n}'}, \qquad x^{\mathbf{n}'} \cdot e_{j,\mathbf{n}} = e_{j,\mathbf{n}+\mathbf{n}'}, \tag{6.1}$$

where $1 \leq i \leq n$, $1 \leq j \leq m$ and $\mathbf{n}, \mathbf{n}' \in \mathbb{Z}^d$. (When G is embedded in some Euclidean space with $\mathbb{Z}^d$ acting by translation, it is usually called a *lattice graph*. Such graphs arise frequently in physics, for example in studying crystal structures.)

When $d > 1$ we can think of G as covering a finite graph $\overline{G}$ in the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. When $d = 1$, G covers a finite graph $\overline{G}$ in the annulus $\mathbb{A} = I \times \mathbb{S}^1$. In either case the cardinality $|V(\overline{G})|$ is equal to the number n of vertex orbits of G, while $|E(\overline{G})|$ is the number m of edge orbits. The projection map is given by $v_{i,\mathbf{n}} \mapsto v_i$ and $e_{j,\mathbf{n}} \mapsto e_j$.

The *Laplacian matrix* of a d-periodic graph G is defined to be the $n \times n$ matrix $L_G = \delta - A$, where now $A = (a_{i,j})$ is the *adjacency $\mathcal{R}_d$-matrix* with each entry $a_{i,j}$ equal to the sum of monomials $\sigma_e x^{\mathbf{n}}$ for each edge $e \in E(G)$ between $v_{i,\mathbf{0}}$ and $v_{j,\mathbf{n}}$, $\mathbf{n} \in \mathbb{Z}^d$. As in Section 3, the signed degree matrix $\delta = (\delta_{i,j})$ is a diagonal matrix constants, with $\delta_{i,i}$ equal to the sum of signs of edges incident on $v_{i,\mathbf{0}}$. Loops are counted twice in both matrices, so loops in G may be ignored, but note that loops in $\overline{G}$ may lift to non-loop edges in G.

The matrix L_G presents a finitely generated $\mathcal{R}_d$-module, the *Laplacian module* of G, denoted by $\mathcal{L}_G$. The *Laplacian (determinant) polynomial* Δ_G is the determinant of L_G. When $d = 0$, $\mathcal{R}_d = \mathbb{Z}$ and these definitions reduce to the ones in Section 3. Examples appear below; additional examples can be found in [18, 36]. The reader should be aware that in graph theory literature the term

"Laplacian polynomial" is often used for the characteristic polynomial of the integral Laplacian matrix.

7. Coloring periodic graphs

Let G be a d-periodic graph. The collection of all $\mathbb{T}$-colorings of G is the Pontryagin dual group $\widehat{\mathcal{L}_G} = \mathrm{Hom}(\mathcal{L}_G, \mathbb{T})$. Elements are functions $f : V(G) \to \mathbb{T}$ that assign to each vertex $v_{i,\mathbf{n}} \in V(G)$ a color $f(v_{i,\mathbf{n}}) \in \mathbb{T}$ such that the Laplacian condition (corresponding to the ith row of L_G) is satisfied:

$$\delta_{i,i} f(v_{i,\mathbf{n}}) = \sum_e \sigma_e f(v_{j,\mathbf{n}'}), \tag{7.1}$$

where $\delta_{i,i}$ is the signed degree of $v_{i,\mathbf{n}}$, and the summation is taken over all edges e that connect $v_{i,\mathbf{n}}$ with some $v_{j,\mathbf{n}'} \in V(G)$, loops contributing twice.

We regard $\mathcal{L}_G$ with the discrete topology. Endowed with the compact-open topology, $\widehat{\mathcal{L}_G}$ is a compact space (see, for example, Section 2 of [21]). It admits a $\mathbb{Z}^d$-action by automorphisms. Such an action is a homomorphism $\mathbf{s} : \mathbf{n} \mapsto \mathbf{s}_\mathbf{n}$ from $\mathbb{Z}^d$ to the automorphism group of $\widehat{\mathcal{L}_G}$.

We denote $\widehat{\mathcal{L}_G}$ with its $\mathbb{Z}^d$-action by $\mathrm{Col}_{\mathbb{T},\mathbb{Z}^d}(G)$. It is an example of a dynamical system known as a $\mathbb{Z}^d$*-shift*. By the Pontryagin Duality Theorem we recover the Laplace module by taking the dual of $\mathrm{Col}_{\mathbb{T},\mathbb{Z}^d}(G)$. We will say more about the dynamical properties of $\widehat{\mathcal{L}_G}$ in Section 10.

In the case of a finite plane graph and its associated link diagram ($d = 0$), their isomorphic groups of $\mathbb{T}$-colorings are dual respectively to the Laplacian group of the graph and the *abelian core group* of the link. The latter group is generated by the arcs of the diagram with relations given by the Fox coloring condition of Figure 5; it is well known to be isomorphic to the direct sum of the first homology group of the 2-fold branched cover of the link and an infinite cyclic group. (For more about such dynamical systems see [28, 21] or [31]. Information about the core group can be found in [34].)

8. Computing the Laplacian polynomial

A *cycle-rooted spanning forest* (CRSF) of $\overline{G}$ is a subgraph of $\overline{G}$ containing all of $\overline{V}$ such that each connected component has exactly as many vertices as edges and therefore has a unique cycle. The *connection* ϕ of an oriented cycle is its homology class in $H_1(\mathbb{T}^d; \mathbb{Z}) \cong \mathbb{Z}^d$. See [16] for details.

The following is a consequence of the main theorem of [11]. It is made explicit in Theorem 5.2 of [16].

Theorem 8.1. [16] *Let G be a d-periodic graph. Its Laplacian polynomial has the form*

$$\Delta_G = \sum_{\overline{F}} \prod_{\overline{e} \in E(\overline{F})} \sigma_{\overline{e}} \prod_{\text{Cycles of } \overline{\mathrm{F}}} (2 - \phi - \phi^{-1}), \tag{8.1}$$

where the sum is over all cycle-rooted spanning forests $\overline{F}$ of $\overline{G}$, and ϕ, ϕ^{-1} are the connections of the two orientations of the cycle.

A d-periodic graph need not be connected. In fact, it can have countably many connected components. Nevertheless, the number of $\mathbb{Z}^d$-orbits of components, henceforth called *component orbits*, is necessarily finite.

Proposition 8.2. *If G is a d-periodic graph with component orbits $G_1, \dots, G_t$, then $\Delta_G = \Delta_{G_1} \cdots \Delta_{G_t}$.*

Proof. After suitable relabeling, the Laplacian matrix for G is a block diagonal matrix with diagonal blocks equal to the Laplacian matrices for $G_1, \dots, G_t$. The result follows immediately. Alternatively, it can be deduced from Theorem 8.1. □

Proposition 8.3. *Let G be a d-periodic graph. If G contains a finite component, then its Laplacian polynomial Δ_G is identically zero. The converse statement is true if G is unsigned.*

Proof. If G contains a finite component, then some component orbit G_i consists of finite components. We have $\Delta_{G_i} = 0$ by Theorem 8.1, since all cycles of $\overline{G_i}$ represent trivial homology classes and hence have vanishing connection. By Proposition 8.2, Δ_G is identically zero.

Conversely, assume G is unsigned and every component is infinite. Each component of $\overline{G}$ must contain a nontrivial cycle. We can extend this collection of cycles to a cycle rooted spanning forest F with no additional cycles. The corresponding summand in Theorem 8.1 has positive constant coefficient. Since every summand has nonnegative constant coefficient, Δ_G is not identically zero. □

9. Plane 1-periodic graphs and links in solid tori

When a plane graph G is 1- or 2-periodic, the medial construction in Section 5 produces a diagram D of an infinite link ℓ. It has a finite quotient diagram $\overline{D}$ modulo the $\mathbb{Z}$- or $\mathbb{Z}^2$-action induced by the action on G.

For $d = 1$ we regard $\overline{D}$ in an annulus $\mathbb{A}$. It describes a link $\overline{\ell} = \overline{\ell}_1 \cup \dots \cup \overline{\ell}_\mu$ in a solid unknotted torus V. The complement $(\operatorname{int} V) \setminus \overline{\ell}$ is homeomorphic to $\mathbb{S}^3 \setminus \hat{\ell}$, where $\hat{\ell} = \overline{\ell} \cup C$ is the link formed by the union of $\overline{\ell}$ with a meridian C of V. The meridian acquires an orientation induced by the infinite cyclic action on D. It is easy to see that every link with an unknotted component that has even linking number with the rest of the link arises in this way.

The following result relates the Laplacian polynomial Δ_G to the Alexander polynomial $\Delta_{\hat{\ell}}$.

Theorem 9.1. *Let G be a plane 1-periodic graph and $\hat{\ell}$ the encircled link $\overline{\ell} \cup C$. Then*

$$\Delta_G(x) \doteq (x-1)\, \Delta_{\hat{\ell}}(-1, \dots, -1, x),$$

where $\doteq$ indicates equality up to multiplication by units in $\mathbb{Z}[x^{\pm 1}]$.

Proof. The argument at the end of section 7 shows that the Laplacian group of a finite plane graph is isomorphic to the abelian core group of the associated link. The same argument can be applied to any 1-periodic graph G and its associated link ℓ. (Either of the two unbounded regions of the link diagram can be used as the base region as we pass from from vertex colorings to Fox colorings via Dehn colorings.)

Let D be the diagram of ℓ obtained from G by the medial construction. We regard D as lying in a strip $I \times \mathbb{R} \subset \mathbb{R}^2$, with $R = I \times [0, 1]$ a fundamental domain for the $\mathbb{Z}$-action. Then D meets R in a tangle diagram D_0. We label the arcs of D_0 meeting the "top," $R \times \{1\}$, by $a_1, \dots, a_n$ and those meeting the "bottom," $R \times \{0\}$, by $a'_1, \dots, a'_n$, with the $\mathbb{Z}$-action taking a'_i to a_i. (It can happen that some a_i and a'_j are identical.) Let $b, c, \dots$ be labels for the remaining arcs of D_0.

Define B to be the quotient of the free abelian group on $a_1, \dots, a_n, a'_1, \dots, a'_n$, $b, c \dots$ by the Fox relations (Figure 5) of the crossings in D_0. Let U be the free abelian on $u_1, \dots, u_n$, and $f : U \to B$ (resp. $g : U \to B$) the homomorphisms mapping each u_i to a_i (resp. u_i to a'_i). The Laplacian module has the form

$$\cdots \oplus_U B \oplus_U B \oplus_U \cdots \tag{9.1}$$

with identical amalgamations $B \xleftarrow{g} U \xrightarrow{f} B$. The module action of x merely shifts each summand B one place to the right. Thus the Laplacian module $\mathcal{L}_G$ is the cokernel of the square matrix A with columns corresponding to the arcs of D_0 and rows recording the Fox relations as well as the relations $xa'_1 = a_1, \dots, xa'_n = a_n$.

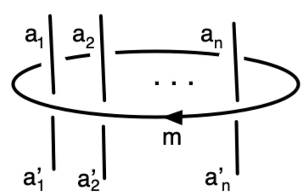

FIGURE 8. Detail of $\bar{\ell} \cup C$

We claim that the matrix A is an Alexander matrix of the link $\bar{\ell} \cup C$ with x corresponding to a meridian m of C while the meridianal variables of ℓ are set equal to -1 (cf. [5]). To see this, consider Figure 8. The Alexander matrix has columns corresponding to $a_1, \dots, a_n, a'_1, \dots, a'_n, b, c \dots$ and the meridian m. There are n rows corresponding to the crossing relations in D_0, and additional rows for relations $m + xa'_i = a_i - m$, where $i = 1, \dots, n$. (The relations, which can be determined by Fox calculus or by considering the appropriate infinite cyclic cover, are unaffected by the directions of the arcs of D_0. The $n - 1$ arcs in the back of C correspond to generators defined by the relations that arise from $n - 1$ of the n crossings in the back; the last relation is redundant and can be ignored. Hence the extra generators and relations can be disregarded.) We delete the column corresponding to m in order to obtain an Alexander matrix. The rows that we have added now correspond to the relations $xa_i' = a_i$. The result is the matrix A. The Alexander polynomial of $\bar{\ell} \cup C$ is the determinant of A divided by $x - 1$. □

Proposition 9.2. *Let G be a plane 1-periodic graph, D its associated link diagram, and ℓ the associated link. The Laplacian polynomial Δ_G is nonzero if and only if the Laplacian module $\mathcal{L}_G$ is a torsion module.*

Proof. The proposition follows from basic facts of commutative algebra. Since $\mathcal{L}_G$ has a square matrix presentation, Δ_G generates annihilator ideal of $\mathcal{L}_G$. □

Proposition 9.3. *Let G be a plane 1-periodic graph, D its associated link diagram, and ℓ the associated link. The following are equivalent.*

(1) *The link ℓ has closed components.*
(2) *The group of 2-colorings of D is infinite.*
(3) *The Laplacian polynomial Δ_G reduced modulo 2 is identically zero.*

Proof. Any 2-coloring of D assigns a single color to every arc corresponding to a component of ℓ. If the link has no closed components, then it has only finitely many components, and conversely. The equivalence of the first two statements follows.

Regard $\mathcal{L}_G$ as an abelian group. The vector space of 2-colorings of D is isomorphic to $\mathcal{L}_G \otimes \mathbb{Z}/2$. Its dimension is the degree of the mod 2 reduction of Δ_G, provided the reduced polynomial is nonzero; otherwise the dimension is infinite. However, the dimension is also equal to the number of components of ℓ. Hence the first and third statements are equivalent. □

The next proposition characterizes the leading coefficient of the Laplacian polynomial of a plane 1-periodic graph in terms of $\mathbb{T}$-colorings of its associated link. We will use the notation in the proof of Theorem 9.1. Note that $\Delta_G(x) = \Delta_G(x^{-1})$, by Theorem 8.1.

Proposition 9.4. *Let G be a plane 1-periodic graph, D its associated link diagram, and $D_0 \subset R$ a tangle diagram representing a fundamental region of D. Suppose Δ_G is nonzero. If we assign $0 \in \mathbb{T}$ to the arcs at the top of D_0, then the number of extensions to $\mathbb{T}$-colorings of D_0 is equal to the absolute value of the leading coefficient of Δ_G.*

Proof. The $\mathbb{T}$-colorings of D that assign 0 to arcs labeled $a_1, \dots, a_n$ are the elements of the dual group of the quotient $B/f(U)$. The group $B/f(U)$ is the cokernel of the specialized Alexander matrix A constructed in the proof of Theorem 9.1 with variable x set equal to 0. The determinant of A is the order of $B/f(U)$ as well as the absolute values of both the trailing and leading coefficients of Δ_G. □

Proposition 9.5. *Let G be a plane 1-periodic graph, D its associated link diagram, and $D_0 \subset R$ a tangle diagram representing a fundamental region of D.*

Suppose that the coefficients of Δ_G are coprime. Then any two distinct $\mathbb{T}$-colorings of D_0 with the same color assignments of the top arcs have different color assignments of the bottom arcs.

Proof. We prove the proposition by contradiction. Assume that there exist two $\mathbb{T}$-colorings of D_0 with the same color assignments to the top arcs and also the same assignments to the bottom. We subtract to get a nontrivial $\mathbb{T}$-coloring with all arcs on both top and bottom colored trivially. By duality, the group quotient $\overline{B} = B/(f(U) + g(U))$ must be nonzero. Consider the quotient module $\overline{\mathcal{L}_G}$ of $\mathcal{L}_G$ described by (9.1) with all elements of $x^i f(U)$ and $x^i g(U)$ set equal to 0; it is a direct sum of countably many copies of $\overline{B}$ with the module action of x given by translation. An integral matrix presenting $\overline{B}$ as an abelian group also presents $\overline{\mathcal{L}_G}$ as module over $\mathcal{R}_1 = \mathbb{Z}[x, x^{-1}]$. The group must be torsion since $\mathcal{L}_G$ is, and its 0th-characteristic polynomial $\overline{\Delta_G}$ must divide Δ_G. Since $\overline{\Delta_G}$ is a nonzero constant and the coefficients of Δ_G are coprime, $\overline{\Delta_G} = \pm 1$. Hence $\overline{B}$ is a trivial group, a contradiction. □

The following corollary follows form Propositions 9.4 and 9.5.

Corollary 9.6. *Assume that G is a plane 1-periodic graph, D its associated link diagram, and $D_0 \subset R$ a tangle diagram representing a fundamental region of D. Assume that Δ_G is monic. Given any color assignment to the top arcs of D_0 that extends to a $\mathbb{T}$-coloring of D_0, the colors of the bottom arcs are uniquely determined.*

Example 9.7. Consider the 1-periodic graph G and its associated tangle diagram D_0 in Figure 9. It is easy to see that $\Delta_G(x) = -3x + 6 - 3x^{-1}$, which has leading coefficient -3. The group B of the associated tangle has generators a_0, a_1, b_0, b_1, c and relations $2b_0 = a_0 + c, 2c = b_0 + a_1, 2a_1 = c + b_1$.

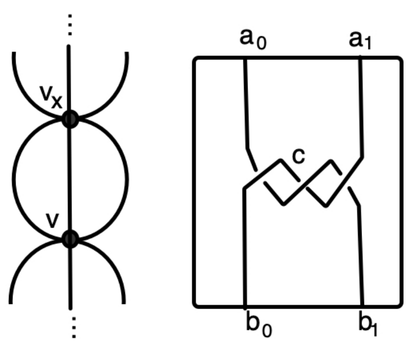

FIGURE 9. 1-peridodic graph G and associated tangle diagram D_0

Since the subgroup $f(U)$ is generated by a_0, a_1, the quotient $B/f(U)$ is generated by b_0, b_1, c, with relations $2b_0 = c, 2c = b_0, 0 = c + b_1$. If we choose a color $\gamma \in \mathbb{T}$ for the arc of D_0 labeled c, then the bottom arcs, labeled b_0, b_1, must receive colors $2\gamma, -\gamma$, respectively. Moreover, the assignment is a $\mathbb{T}$-coloring of D_0 provided that $3\gamma = 0$. Hence $\gamma = 0, 1/3, 2/3 \pmod 1$, and there are exactly three $\mathbb{T}$-colorings of D_0 with the top arcs colored 0, as expected from Proposition 9.4. The three $\mathbb{T}$-colorings appear in Figure 10.

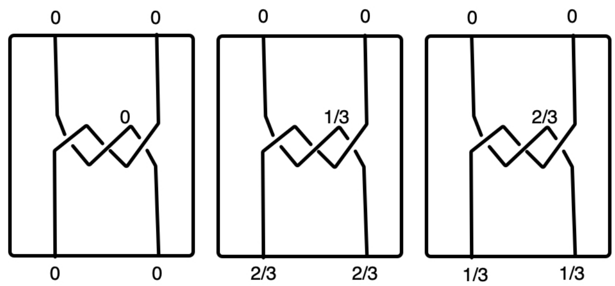

FIGURE 10. $\mathbb{T}$-colorings with top arcs colored trivially

Example 9.8. The closure $\overline{\ell}$ of any $2n$-braid arises from graph $\overline{G}$ embedded in the annulus $\mathbb{A}$, since we can checkerboard shade a diagram of $\overline{\ell}$ in $\mathbb{A}$ so that the border regions are unshaded.

The graph $\overline{G}$ lifts to a 1-periodic graph G in the plane. Consider its Laplacian polynomial Δ_G. We recall that the Burau representation associates to each generator σ_i of the $2n$-braid group a block diagonal matrix

$$I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{2n-i-1},$$

where I_k denotes the $k \times k$ identity matrix. Setting $t = -1$ produces presentation matrix for the group B in the proof of Theorem 9.1. The Laplacian polynomial of G is the characteristic polynomial of the Burau matrix of the braid with $t = -1$.

10. Complexity growth of periodic graphs

When G is a d-periodic graph with quotient $\overline{G}$, we can consider the intermediate covering graphs G_Λ in $\mathbb{R}^d/\Lambda$, where $\Lambda \subset \mathbb{Z}^d$ is a subgroup of Λ having finite index. In this section we see that the growth of the torsion complexity κ_{G_Λ} as the index of Λ goes to infinity is determined by the Mahler measure of the Laplacian polynomial Δ_G.

We begin by reviewing the Mahler measure of polynomials.

Definition 10.1. The *Mahler measure* of a nonzero polynomial $f(x_1, \dots, x_d)$ in $\mathcal{R}_d$ is

$$M(f) = \exp \int_0^1 \dots \int_0^1 \log |f(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_d})| d\theta_1 \cdots d\theta_d.$$

Remark 10.2. (1) The integral in Definition 10.1 can be singular, but nevertheless it converges. (See [10] for two different proofs.) If $u_1, \dots, u_d$ is another basis for $\mathbb{Z}^d$, then $f(u_1, \dots, u_d)$ has the same logarithmic Mahler measure as $f(x_1, \dots, x_d)$.

(2)If $f, g \in \mathcal{R}_d$, then $M(fg) = M(f)M(g)$. Moreover, $M(f) = 1$ if and only if f is a unit or a unit times a product of 1-variable cyclotomic polynomials, each evaluated at a monomial of $\mathcal{R}_d$ (see [28]).

(3)When $d = 1$, Jensen's formula shows that $M(f)$ can be described in a simple way. If $f(x) = c_s x^s + \cdots c_1 x + c_0$, $c_0 c_s \neq 0$, c is the leading coefficient of f, then

$$M(f) = |c_s| \prod_{i=1}^{s} \max\{|\lambda_i|, 1\},$$

where $\lambda_1, \dots, \lambda_s$ are the roots of f.

Theorem 10.3. *If G is a signed d-periodic graph with nonzero Laplacian polynomial Δ_G, then*

$$\limsup_{\langle\Lambda\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log \kappa_{G_\Lambda} = \log M(\Delta_G), \tag{10.1}$$

where Λ ranges over all finite-index subgroups of $\mathbb{Z}^d$, and $\langle\Lambda\rangle$ denotes the minimum length of a nonzero vector in Λ. When $d = 1$, the limit superior can be replaced by an ordinary limit.

We call this limit the *complexity growth rate* of G, and denote it by γ_G. Its relationship to the *thermodynamic limit* or *bulk limit* defined for a wide class of unsigned lattice graphs is discussed in [18], and also below in Section 12.

Remark 10.4. (1) The condition $\langle\Lambda\rangle \to \infty$ ensures that fundamental region of Λ grows in all directions.

(2) If G is unsigned, $\kappa_{G_\Lambda} = \tau_{G_\Lambda}$ for every Λ. In this case, Theorem 10.3 is proven in [23] with the limit superior replaced by ordinary limit.

(3) When $d = 1$, the finite-index subgroups Λ are simply $\mathbb{Z}/r\mathbb{Z}$, for $r > 0$. In this case, we write G_r instead of G_Λ.

(4) When $d > 1$, a recent result of V. Dimitrov [9] asserts that the limit superior in Theorem 10.3 is equal to the ordinary limit along sequences of sublattices Λ of the form $N \cdot \mathbb{Z}^d$, where N is a positive integer.

Before proving Theorem 10.3 we give an example that demonstrates the need for defining graph complexity as we do.

Example 10.5. Consider the 1-periodic graph G in Figure 11. Generators for the Laplacian module are indicated. The Laplacian matrix is

$$L_G = \begin{pmatrix} 0 & 1-x^{-1} & 1 & -x^{-1} \\ 1-x & 0 & 1 & -1 \\ 1 & 1 & -2 & 0 \\ -x & -1 & 0 & 2 \end{pmatrix},$$

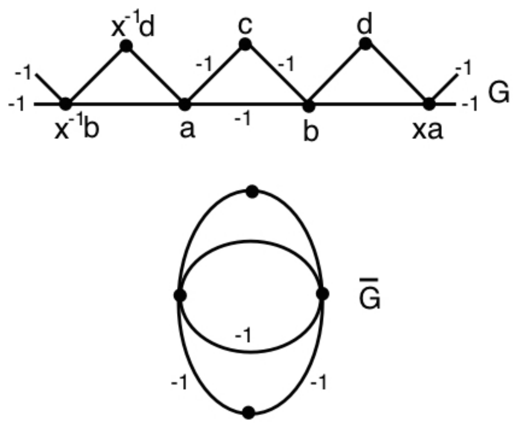

FIGURE 11. 1-Periodic graph G with $\tau_{G_r} = 0$ for all $r \geq 1$

and $\Delta_G(x) = 9(x - 2 + x^{-1})$.

The quotient G_2 is the finite graph in Example 3.2. The Laplacian matrix of any G_r can be described as a block matrix obtained from L_G by replacing x by the companion (permutation) matrix for $x^r - 1$, and any scalar c by cI_r (see [31]). It is conjugate to the diagonal block matrix $\mathrm{Diag}[L_G|_{x=1}, \dots, L_G|_{x=\zeta^{r-1}}]$, where ζ is a primitive rth root of unity. The matrix $L_G|_{x=1}$ is the 4×4 Laplacian matrix of $\overline{G}$,

$$L_{\overline{G}} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix},$$

which has nullity 2. Hence the tree complexity τ_{G_r} vanishes for every r. Nevertheless, by Theorem 10.3 the (torsion) complexity $\varkappa_{G_r}$ is nontrivial and has exponential growth rate equal to 9. One can verify directly that the Laplacian subgroup $\mathcal{L}_{G_r}$ is isomorphic to $\mathbb{Z}^2 \times (\mathbb{Z}/3^{r-1}\mathbb{Z})^2$.

We proceed with the proof of Theorem 10.3.

Proof. The proof that we present is a direct application of a theorem of D. Lind, K. Schmidt and T. Ward (see [21] or Theorem 21.1 of [28]). We review the ideas for the reader's convenience.

Recall that the Laplacian module $\mathcal{L}_G$ is the finitely generated module over the ring $\mathcal{R}_d$ with presentation matrix equal to the $n \times n$ Laplacian matrix L_G, and its Pontryagin dual group $\widehat{\mathcal{L}_G}$ is $\mathrm{Hom}(\mathcal{L}_G, \mathbb{T})$. The module actions of $x_1, \dots, x_d$ determine commuting homeomorphisms $\mathbf{s}_1, \dots, \mathbf{s}_d$ of $\widehat{\mathcal{L}_G}$. Explicitly, $(\mathbf{s}_j\rho)(a) = \rho(x_j a)$ for every $a \in \mathcal{L}_G$. Consequently, $\widehat{\Gamma}_G$ has a $\mathbb{Z}^d$-action $\mathbf{s} : \mathbb{Z}^d \to \mathrm{Aut}(\widehat{\mathcal{L}_G})$.

The pair $(\widehat{\mathcal{L}_G}, \sigma)$ is an algebraic dynamical system, well defined up to topological conjugacy (that is, up to a homeomorphism of $\widehat{\mathcal{L}_G}$ respecting the $\mathbb{Z}^d$ action). In particular its periodic point structure is well defined.

Topological entropy $h(\mathbf{s})$ is a well-defined quantity associated to $(\widehat{\mathcal{L}_G}, \mathbf{s})$, a measure of complexity of the $\mathbb{Z}^d$-action $\mathbf{s}$. We refer the reader to [21] or [28] for the definition.

For any subgroup Λ of $\mathbb{Z}^d$, a Λ-periodic point is a member of $\widehat{\mathcal{L}_G}$ that is fixed by every element of Λ. The set of Λ-periodic points is a finitely generated abelian group that is isomorphic to the Pontryagin dual group $\mathrm{Hom}(\mathcal{L}_G/\Lambda\mathcal{L}_G), \mathbb{T})$.

The group $\mathcal{L}_G/\Lambda\mathcal{L}_G$ is the Laplacian module of the quotient graph G_Λ. As a finitely generated abelian group, it decomposes as $\mathbb{Z}^{\beta_\Lambda} \oplus T(\mathcal{L}_G/\Lambda\mathcal{L}_G)$, where β_Λ is the rank of $\mathcal{L}_G/\Lambda\mathcal{L}_G$ and $T(\cdots)$ denotes the (finite) torsion subgroup. The Pontryagin dual group consists of $P_\Lambda = |T(\mathcal{L}_G/\Lambda\mathcal{L}_G)|$ tori each of dimension β_Λ. By Theorem 21.1 of [28], the topological entropy $h(\mathbf{s})$ is:

$$h(\mathbf{s}) = \limsup_{\langle\Lambda\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log P_\Lambda = \limsup_{\langle\Lambda\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log \kappa_\Lambda.$$

Since the matrix L_G that presents $\mathcal{L}_G$ is square, $h(\mathbf{s})$ can be computed also as the logarithm of the Mahler measure $M(\det L_G)$ (see Example 18.7(1) of [28]). The determinant of L_G is, by definition, the Laplacian polynomial Δ_G. Hence the proof is complete. □

11. Lehmer's question

In [19] D.H. Lehmer asked the following question.

Question 11.1. Do there exist integral polynomials with Mahler measures arbitrarily close but not equal to 1?

Lehmer discovered the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, which has Mahler measure equal to 1.17628.... Despite great effort including extensive computer-aided searches [3, 4, 24, 25, 27], no smaller value greater than 1 has been found, and Lehmer's question remains unanswered.

Topological and geometric perspectives of Lehmer's question have been found [13]. In [35] we showed that Lehmer's question is equivalent to a question about Alexander polynomials of fibered hyperbolic knots in the lens spaces $L(n, 1), n > 0$. (Lens spaces arose from the need to consider polynomials $f(x)$ with $f(1) = n \neq 1$.) Here we present another, more elementary equivalence, in terms of graph complexity.

An integer polynomial $f(x)$ is *reciprocal* if $x^{\deg f} f(x^{-1}) = f(x)$. We will say that a Laurent polynomial $f(x) \in \mathcal{R}_1$ is *palindromic* if $f(x^{-1}) = f(x)$. Any reciprocal polynomial becomes palindromic after it is multiplied by x^j or $x^j(x+1)$, for suitable j. In [37] C. Smyth proved that any irreducible integral non-reciprocal polynomial other than x or $x - 1$ has Mahler measure at least as large as the real root of $x^3 - x - 1$ (approximately 1.324). Since Mahler measure is multiplicative, it suffices to restrict our attention to palindromic Laurent polynomials when investigating Lehmer's question.

Proposition 11.2. *A polynomial $\Delta(x)$ is the Laplacian polynomial of a 1-periodic graph if and only if it has the form $(x-2+x^{-1})f(x)$, where $f(x)$ is a palindromic polynomial.*

Proof. The Laplacian polynomial $\Delta(x)$ of any 1-periodic graph is palindromic. This follows from the fact that the transpose of L_G is L_G with x replaced by x^{-1}. Since the row-sums of L_G become zero when we set $x = 1$, $x - 1$ divides $\Delta(x)$. (Both observations follow also from Theorem 8.1.) Palindromicity requires that the multiplicity of $x - 1$ be even. Hence $\Delta(x)$ has the form $(x - 2 + x^{-1})f(x)$, where $f(x)$ is palindromic.

In order to see the converse assertion, consider any polynomial of the form $p(x) = (x - 2 + x^{-1})f(x)$, where $f(x)$ is palindromic. Then $p(x)$ is also palindromic. Clearly, we can write $p(x)$ as a constant plus a sum of terms $\pm(x^s - 2 + x^{-s})$; but the constant must be 0 since $p(1) = 0$. Then $p(x)$ is the Laplacian polynomial of a 1-periodic graph, constructed as in the following example. □

Example 11.3. Multiplying Lehmer's polynomial

$$f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

by the unit x^{-5} and then by $x - 2 + x^{-1}$ yields

$$x^6 - x^5 - x^4 + x^2 + x^{-2} - x^{-4} - x^{-5} + x^{-6},$$

which in turn can be written as

$$(x^2 - 2 + x^{-2}) - (x^4 - 2 + x^{-4}) - (x^5 - 2 + x^{-5}) + (x^6 - 2 + x^{-6}).$$

This is the Laplacian polynomial of a 1-periodic graph G. The quotient graph $\overline{G}$ is easily described. It has a single vertex, two edges with sign $+1$ and two with -1. The $(+1)$-signed edges wind twice and six times, respectively, around the annulus in the direction corresponding to x. The (-1)-signed edges wind four and five times, respectively, in the opposite direction.

Theorem 11.4. *Lehmer's question is equivalent to the following. Given $\epsilon > 0$, does there exist a 1-periodic graph G such that*

$$1 < \lim_{r\to\infty}(\tau_{G_r})^{1/r} < 1 + \epsilon?$$

Proof. When investigating Lehmer's question it suffices to consider polynomials of the form $(x-2+x^{-1})f(x)$, where $f(x)$ is palindromic and irreducible. By Proposition 11.2 any such polynomial is realized as the Laplacian polynomial of a 1-periodic graph G with a single vertex orbit. As in Example 10.5 the Laplacian matrix L_{G_r} of any finite quotient G_r can be obtained from $(x-2+x^{-1})f(x)$ by substituting for x the companion matrix for $x^r - 1$. Hence the nullity of L_{G_r} is 1 provided that $f(x)$ is not a cyclotomic polynomial (multiplied by a unit), a condition that we can assume without loss of generality. Hence $\kappa_{G_r} = \tau_{G_r}$ for each r (see discussion following Definition 3.) Theorem 10.3 completes the proof.

□

Remark 11.5. (1) The closure of the 16-braid

$$(\sigma_1\sigma_2)^2(\sigma_1\sigma_2\sigma_3\sigma_4)^4(\sigma_1\sigma_2\cdots\sigma_{15})^5$$

is a link $\overline{\ell}$ associated with a plane graph $\overline{G}$ in the annulus (see Example 9.8). The Laplacian polynomial Δ_G is

$$x^{-7}(x-1)(x^2+1)(x^{10}-x^9-x^6+x^5-x^4-x+1).$$

Its Mahler measure is 1.35098.... This is the smallest Mahler measure greater than 1 that we have yet found for any plane graph.

(2) The conclusion of Theorem 11.4 does not hold if we restrict ourselves to unsigned graphs. By Theorem 12.7 below, the Mahler measure of the Laplacian polynomial of any 1-periodic graph with all edge signs equal to 1 is at least 2.

(3) If a 1-periodic graph G as in Example 11.3 can be found with $M(D_G)$ less than Lehmer's value 1.17628..., then by results of [26] some edge of $\overline{G}$ must wind around the annulus at least 29 times.

The cyclic 5-fold cover of the graph in Example 11.3 contains the complete graph on 5 vertices, and hence it is nonplanar. If the answer to the following question is yes, then Lehmer's question is equivalent to a question about determinant density of links (see Remark 12.3(3)).

Question 11.6. Is Theorem 11.4 still true if we require that the graphs G be planar?

We conclude this section with a result that will be used in the next section, but holds for signed as well as unsigned graphs. It concerns complexity growth of a d-periodic graph that is a union of disjoint d'-periodic graphs for some $d' < d$.

Suppose H is a subgraph of a d-periodic graph G consisting of one or more connected components of G, such that the orbit of H under $\mathbb{Z}^d$ is all of G. Let $\Gamma < \mathbb{Z}^d$ be the stabilizer of H. Then $\Gamma \cong \mathbb{Z}^{d'}$ for some $d' < d$, and its action on H can be regarded as a cofinite free action of $\mathbb{Z}^{d'}$. Consider the limit

$$\gamma_H = \lim_{\langle\Lambda\rangle\to\infty} \frac{1}{|\Gamma/\Lambda|} \log \kappa_{H_\Lambda}$$

where Λ ranges over finite-index subgroups of Γ.

Lemma 11.7. *Under the above conditions we have $\gamma_G = \gamma_H$.*

Proof. Let Λ be any finite-index subgroup of $\mathbb{Z}^d$. Then H is invariant under $\Lambda \cap \Gamma$. The image of H in the quotient graph G_Λ is isomorphic to $H_{\Lambda\cap\Gamma}$.

Note that the quotient $\overline{H}$ of H by the action of Γ is isomorphic to $\overline{G}$, since the $\mathbb{Z}^d$ orbit of H is all of G. Since G_Λ is a $|\mathbb{Z}^d/\Lambda|$-fold cover of $\overline{G}$ and $H_{\Lambda\cap\Gamma}$ is

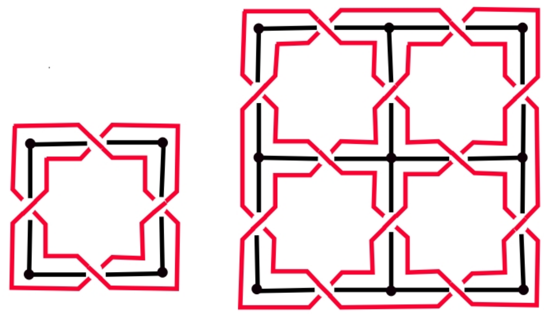

FIGURE 12. Graphs $(\mathbb{G}_2)_R$ and associated links, $\Lambda = \langle x_1^2, x_2^2 \rangle$ and $\langle x_1^3, x_2^3 \rangle$

a $|\Gamma/(\Lambda \cap \Gamma)|$-fold cover of $\overline{H}$, G_Λ comprises $k = |\mathbb{Z}^d/\Lambda|/|\Gamma/(\Lambda \cap \Gamma)|$ mutually disjoint translates of a graph that is isomorphic to $H_{\Lambda\cap\Gamma}$. Hence $\kappa_{G_\Lambda} = \kappa^k_{H_{\Lambda\cap\Gamma}}$ and

$$\frac{1}{|\mathbb{Z}^d/\Lambda|} \log \kappa_{G_\Lambda} = \frac{1}{|\Gamma/(\Lambda\cap\Gamma)|} \log \kappa_{H_{\Lambda\cap\Gamma}}.$$

Since $\langle \Lambda \cap \Gamma \rangle \to \infty$ as $\langle \Lambda \rangle \to \infty$, we have $\gamma_G = \gamma_H$. □

12. Complexity growth of unsigned periodic graphs

It is natural to ask whether the Mahler measure of Laplacian polynomials of signed graphs differs in appreciable ways from unsigned graphs. Proposition 12.7 answers emphatically yes.

Throughout the section G denotes an unsigned d-periodic graph. For this case, the complexity growth rate γ_G is also the growth rate of the number of spanning trees of finite quotients G_Λ. Thus contracting or deleting an edge orbit of G will not increase γ_G.

Denote by $R = R(\Lambda)$ a fundamental domain of Λ. Let $G|_R$ be the full unsigned subgraph of G on vertices $v_{i,\mathbf{n}}$, $\mathbf{n} \in R$. We denote by ℓ_R the corresponding medial link.

If $G|_R$ is connected for each R, then $\{\tau_{G_\Lambda}\}$ and $\{\tau_{G|_R}\}$ have the same exponential growth rates. (See Theorem 7.10 of [18] for a short, elementary proof. A more general result is Corollary 3.8 of [23].) The *bulk limit* is defined by $\gamma_G/|V(\overline{G})|$.

Example 12.1. The *d-dimensional grid graph* $\mathbb{G}_d$ is the unsigned graph with vertex set $\mathbb{Z}^d$ and single edges connecting each pair of vertices of distance 1. Its Laplacian polynomial is

$$\Delta(\mathbb{G}_d) = 2d - x_1 - x_1^{-1} - \cdots - x_d - x_d^{-1}.$$

When $d = 2$, it is a plane graph. The graphs links ℓ_R are indicated in Figure 12 for $\Lambda = \langle x_1^2, x_2^2 \rangle$ on left and $\Lambda = \langle x_1^3, x_2^3 \rangle$ on right.

The *determinant* of a link ℓ, denoted here by $\det(\ell)$, is the absolute value of its 1-variable Alexander polynomial evaluated at -1. It follows from the Mayberry-Mott theorem [2] that if ℓ is an alternating link that arises by the medial construction from a finite plane graph, edge signs ± 1 allowed, then $\det(\ell)$ is equal to the tree complexity of the graph (see Appendix A.4 in [5]). The following corollary is an immediate consequence of Theorem 10.3. It has been proven independently by Champanerkar and Kofman [6].

Corollary 12.2. *Let G be a connected d-periodic unsigned plane graph, $d = 1$ or 2. Then*

$$\lim_{\langle\Lambda\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log\det(\ell_R) = \gamma_{\Delta_G}.$$

Remark 12.3. (1) We regard the limit in that statement of Corollary 12.2 as a *determinant density* of the collection of links $\{\ell_R\}$. There are other ways to define it (e.g., dividing by the number of crossings of the diagram for ℓ_R).

(2) In [7] the authors consider as well more general sequences of links. When $G = \mathbb{G}_2$, their results imply that:

$$\lim_{\langle\Lambda\rangle\to\infty} \frac{2\pi}{c(\ell_R)} \log\det(\ell_R) = v_{oct},$$

where $c(\ell_R)$ is the number of crossings of ℓ_R and $v_{oct} \approx 3.66386$ is the volume of the regular ideal octohedron.

(3) If Question 11.6 has an affirmative answer then Lehmer's question becomes a question about link determinants.

Grid graphs are the simplest unsigned d-periodic graphs, as the following theorem shows.

Theorem 12.4. *If G is an unsigned connected d-periodic graph, then $\gamma_G \geq \gamma_{\mathbb{G}_d}$.*

Asymptotic results about the Mahler measure of certain families of polynomials have been obtained elsewhere. However, the graph theoretic methods that we employ to prove Theorem 12.4 are different from techniques used previously.

Proof. Consider the case in which G has a single vertex orbit. Then for some $u_1, \ldots, u_m \in \mathbb{Z}^d$, with $m \geq d$, the edge set $E(G)$ consists of edges from v to $u_i \cdot v$ for each $v \in V$ and $i = 1, \ldots, m$. Since G is connected, we can assume after relabeling that $u_1, \ldots, u_d$ generate a finite-index subgroup of $\mathbb{Z}^d$. Let G' be the $\mathbb{Z}^d$-invariant subgraph of G with edges from v to $u_i \cdot v$ for each $v \in V$ and $i = 1, \ldots, d$. Then G' is the orbit of a subgraph of G that is isomorphic to $\mathbb{G}_d$, and so by Lemma 11.7, $\gamma(\mathbb{G}_d) = \gamma(G') \leq \gamma(G)$.

We now consider a connected graph G having vertex families $v_{1,\mathbf{n}}, \ldots, v_{n,\mathbf{n}}$, where $n > 1$. Since G is connected, there exists an edge e joining $v_{1,\mathbf{0}}$ to some $v_{i,\mathbf{n}}$. Contract the edge orbit $\mathbb{Z}^d \cdot e$ to obtain a new graph G' having cofinite free $\mathbb{Z}^d$-symmetry and complexity growth rate no greater than that of G. Repeat the procedure with the remaining vertex families so that only $v_{1,\mathbf{n}}$ remains. The proof in the previous case of a graph with a single vertex orbit now applies. □

Remark 12.5. The conclusion of Theorem 12.4 does not hold without the hypothesis that G is connected. Consider the 2-periodic graph G obtained from $\mathbb{G}_2$ by removing all vertical edges, so that G consists of countably many copies of $\mathbb{G}_1$. Then $\gamma_G = \gamma_{\mathbb{G}_1} = 0$ while $\gamma_{\mathbb{G}_2} > 0$.

The following lemma, needed for the proof Proposition 12.7, is of independent interest.

Lemma 12.6. *The sequence of complexity growth rates $\gamma_{\Delta_{\mathbb{G}_d}}$ is nondecreasing.*

Proof. Consider the grid graph $\mathbb{G}_d$. Deleting all edges in parallel to the dth coordinate axis yields a subgraph G consisting of countably many mutually disjoint translates of $\mathbb{G}_{d-1}$. By Lemma 11.7, $\gamma_{\mathbb{G}_{d-1}} = \gamma_G \leq \gamma_{\mathbb{G}_d}$. □

Doubling each edge of $\mathbb{G}_1$ results in a graph with Laplacian polynomial $2(x-2+x^{-1})$, which has Mahler measure $2M(x-2+x^{-1}) = 2$. We show that this graph realizes the minimum nonzero complexity growth rate.

Proposition 12.7. *(Complexity Growth Rate Gap) Let G be any unsigned d-periodic graph. If $\gamma_G \neq 0$, then*

$$\gamma_G \geq \log 2.$$

Although $\Delta_{\mathbb{G}_d}$ is relatively simple, the task of computing its Mahler measure is not. It is well known and not difficult to see that $\gamma_{\mathbb{G}_d} \leq \log 2d$. We will use a theorem of N. Alon [1] to show that $\gamma_{\mathbb{G}_d}$ approaches $\log 2d$ asymptotically.

Theorem 12.8. *We have*

(1) $\gamma_{\mathbb{G}_d} \leq \log 2d$, *for all $d \geq 1$.*
(2) $\lim_{d\to\infty} \gamma_{\mathbb{G}_d} - \log 2d = 0$.

Proof of Proposition 12.7. By Lemma 11.7 it suffices to consider a connected d-periodic graph G with γ_G nonzero. Note that $\gamma_{\mathbb{G}_1} = 0$ while $\gamma_{\mathbb{G}_2} \approx 1.165$ is greater than $\log 2$. By Theorem 12.4 and Lemma 12.6 we can assume that $d = 1$.

If G has an orbit of parallel edges, we see easily that $\gamma_G \geq \log 2$. Otherwise, we proceed as in the proof of Theorem 12.4, contracting edge orbits to reduce the number of vertex orbits without increasing the complexity growth rate. If at any step we obtain an orbit of parallel edges, we are done; otherwise we will obtain a graph G' with a single vertex orbit and no loops. If G' is isomorphic to $\mathbb{G}_1$ then G must be a tree; but then $\gamma_G = 0$, contrary to our hypothesis. So G' must have at least two edge orbits. Deleting excess edges, we may suppose G' has exactly two edge orbits.

The Laplacian polynomial $\Delta_{G'}$ has the form $4 - x^r - x^{-r} - x^s - x^{-s}$, for some positive integers r, s. Reordering the vertex set of G', we can assume without loss of generality that $r = 1$. The following calculation is based on an idea suggested to us by Matilde Lalin.

$$\log M(\Delta_{G'}) = \int_0^1 \log|4 - 2\cos(2\pi\theta) - 2\cos(2\pi s\theta)|\, d\theta$$

$$= \int_0^1 \log|2(1-\cos(2\pi\theta)) + 2(1-\cos(2\pi s\theta))|\, d\theta$$

$$= \int_0^1 \log\left(4\sin^2(\pi\theta) + 4\sin^2(\pi s\theta)\right) d\theta.$$

Using the inequality $(u^2 + v^2) \geq 2uv$, for any nonnegative u, v, we have:

$$\log M(\Delta_{G'}) \geq \int_0^1 \log\left(8|\sin(\pi\theta)|\,|\sin(\pi s\theta)|\right) d\theta$$

$$= \log 8 + \int_0^1 \log|\sin(\pi\theta)|\, d\theta + \int_0^1 \log|\sin(\pi s\theta)|\, d\theta$$

$$= \log 8 + \int_0^1 \log\sqrt{\frac{1-\cos(2\pi\theta)}{2}}\, d\theta + \int_0^1 \log\sqrt{\frac{1-\cos(2\pi s\theta)}{2}}\, d\theta$$

$$= \log 8 + \int_0^1 \tfrac{1}{2}\log\left(\frac{2-2\cos(2\pi\theta)}{4}\right) d\theta + \int_0^1 \tfrac{1}{2}\log\left(\frac{2-2\cos(2\pi s\theta)}{4}\right) d\theta$$

$$= \log 8 + \tfrac{1}{2}m(2 - x - x^{-1}) - \tfrac{1}{2}\log 4 + \tfrac{1}{2}m(2 - x^s - x^{-s}) - \tfrac{1}{2}\log 4$$

$$= 3\log 2 + 0 - \log 2 + 0 - \log 2 = \log 2.$$

□

Our proof of Theorem 12.8 depends on the following result of Alon.

Theorem 12.9. [1] *If G is a finite connected ρ-regular unsigned graph, then*

$$\tau_G \geq [\rho(1 - \epsilon(\rho))]^{|V(G)|},$$

where $\epsilon(\rho)$ is a nonnegative function with $\epsilon(\rho) \to 0$ as $\rho \to \infty$.

Proof of Theorem 12.8. (1) The integral representing the logarithm of the Mahler measure of $\Delta_{\mathbb{G}_d}$ can be written

$$\int_0^1 \cdots \int_0^1 \log\left|2d - \sum_{i=1}^{d} 2\cos(2\pi\theta_i)\right| d\theta_1 \cdots d\theta_d$$

$$= \log 2d + \int_0^1 \cdots \int_0^1 \log\left|1 + \sum_{i=1}^{d} \frac{\cos(2\pi\theta_i)}{d}\right| d\theta_1 \cdots d\theta_d$$

$$= \log 2d + \int_0^1 \cdots \int_0^1 -\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\left(\frac{\sum_{i=1}^d \cos(2\pi\theta_i)}{d}\right)^k d\theta_1 \cdots d\theta_d.$$

By symmetry, odd powers of k in the summation contribute zero to the integration. Hence

$$\log(\Delta_{\mathbb{G}_d}) = \log 2d - \int_0^1 \cdots \int_0^1 \sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{\sum_{i=1}^d \cos(2\pi\theta_i)}{d}\right)^{2k} d\theta_1 \cdots d\theta_d,$$

which cannot exceed $\log 2d$.

(2) Let Λ be a finite-index subgroup of $\mathbb{Z}^d$. Consider the quotient graph $(\mathbb{G}_d)_\Lambda$. The cardinality of its vertex set is $|\mathbb{Z}^d/\Lambda|$. The main result of [1], cited above as Theorem 12.9, implies that

$$\tau_{(\mathbb{G}_d)_\Lambda} = \Big((2d)(1-\mu(d))\Big)^{|\mathbb{Z}^d/\Lambda|},$$

where μ is a nonnegative function such that $\lim_{d\to\infty}\mu(d) = 0$. Hence

$$\lim_{d\to\infty}\left(\frac{1}{|\mathbb{Z}^d/\Lambda|}\log\tau_{(\mathbb{G}_d)_\Lambda} - \log 2d\right) = \lim_{d\to\infty}\log(1-\mu(d)) = 0.$$

Theorem 10.3 completes the proof. □

Remark 12.10. One can evaluate $\log M(\Delta(\mathbb{G}_d))$ numerically and obtain an infinite series representing $\gamma_{\mathbb{G}_d} - \log 2d$. However, showing rigorously that the sum of the series approaches zero as d goes to infinity appears to be difficult. (See [29], p. 3893 for a heuristic argument.)

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