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The Tate module of a simple abelian variety of type IV

Grzegorz Banaszak and Aleksandra Kaim-Garnek

Abstract. The aim of this paper is to investigate the Galois module structure of the Tate module of an abelian variety defined over a number field. We focus on simple abelian varieties of type IV in Albert classification. We describe explicitly the decomposition of the $\mathcal{O}_\lambda[G_F]$-module $T_\lambda(A)$ into components that are rationally and residually irreducible. Moreover these components are non-degenerate, hermitian modules that rationally and residually are non-degenerate, hermitian vector spaces.

Contents

1. Introduction

Let A be an abelian variety of dimension g over a number field F. Let $\mathcal{R} := \mathrm{End}_{\overline{F}}(A)$. Put $D := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Q}$ and let E be the center of D. The ring $\mathcal{R}$ is an order in D. Because $\mathcal{R}$ is finitely generated $\mathbb{Z}$-module then $\mathcal{R} \cap E = \mathcal{O}_E^0$ is an order in $\mathcal{O}_E$. Throughout the paper we fix a polarization of A. Let l be a prime number and let $T_l(A)$ be the Tate module of A. Let $G_F := \mathrm{Gal}(\overline{F}/F)$ and let

$$\rho_l : G_F \to GL(T_l(A))$$

be the l-adic representation associated with A.

From now on, we assume that $\mathcal{R}$ is defined over F, i.e. $\mathcal{R} = \mathrm{End}_F(A)$ so $D = \mathrm{End}_F(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

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In this paper we also assume that A is simple, hence D is a division algebra of finite dimension over $\mathbb{Q}$ with a positive involution $'$ [14, p. 193-203]. Let E_0 be the subfield of elements of E fixed by $'$. We put $d^2 := [D : E]$, $e := [E : \mathbb{Q}]$ and $e_0 := [E_0 : \mathbb{Q}]$.

Recall that, due to A. A. Albert, simple abelian varieties can be classified according to the type of their endomorphism algebra (see: [1] and [14, Theorem 2, p. 201-203]):

TYPE I: $D = E = E_0$ is a totally real field.

TYPE II: $E = E_0$ is a totally real field and D is a quaternion division algebra over $\mathbb{Q}$ such that $D \otimes_{E_0} \mathbb{R} \cong M_2(\mathbb{R})$ for any embedding $\sigma : E_0 \to \mathbb{R}$.

TYPE III: $E = E_0$ is a totally real field and D is a quaternion division algebra over $\mathbb{Q}$ such that $D \otimes_{E_0} \mathbb{R} \cong \mathbb{H}$ for any embedding $\sigma : E_0 \to \mathbb{R}$.

TYPE IV: E_0 is a totally real field, E is a totally imaginary quadratic extension of E_0 and D is a division algebra over $\mathbb{Q}$ such that $D \otimes_{E_0} \mathbb{R} \cong M_d(\mathbb{C})$ for any embedding $\sigma : E_0 \to \mathbb{R}$.

If A is of type I then $d = 1$. If A is of type II or III then $d = 2$, and if A is of type IV then $d \geq 1$ can be arbitrary. Moreover, if A is a simple abelian variety of type IV then E is a quadratic imaginary extension of a totally real field E_0 so $e = 2e_0$ cf. [14, Theorem 2, p. 201-203].

Let λ be a prime ideal in $\mathcal{O}_E$ dividing l. Let $\mathcal{O}_\lambda$ be the completion of $\mathcal{O}_E$ at λ, $E_\lambda := \mathrm{Frac}(\mathcal{O}_\lambda)$ and $k_\lambda := \mathcal{O}_\lambda/\lambda$. Observe that for each l :

$$E_l := E \otimes_{\mathbb{Q}} \mathbb{Q}_l = \prod_{\lambda | l} E_\lambda \quad \text{and} \quad \mathcal{O}_{E_l} := \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_l = \prod_{\lambda | l} \mathcal{O}_\lambda. \tag{1.1}$$

For $l \nmid [\mathcal{O}_E : \mathcal{O}_E^0]$, we have $\mathcal{O}_E^0 \otimes_{\mathbb{Z}} \mathbb{Z}_l = \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_l$. Hence, for such an l, the ring $\mathcal{O}_{E_l}$ acts on $T_l(A)$ and we put $T_\lambda(A) := T_l(A) \otimes_{\mathcal{O}_{E_l}} \mathcal{O}_\lambda$. Hence, for each $l \nmid [\mathcal{O}_E : \mathcal{O}_E^0]$:

$$T_l(A) = \bigoplus_{\lambda | l} T_\lambda(A). \tag{1.2}$$

The aim of this paper is to describe explicitly the decomposition of the $\mathcal{O}_\lambda[G_F]$-module $T_\lambda(A)$ for a simple abelian variety A of type IV into components that are rationally and residually irreducible. We also show that these components are compatible with corresponding non-degenerate, hermitian forms. This work is a continuation of the research in [2], [3] and [4] on the Galois l-adic representations for abelian varieties of types I, II and III. In the papers loc. cit., the first author with W. Gajda and P. Krasoń showed that $T_\lambda(A)$ has the following decomposition for $l \gg 0$:

$$T_\lambda(A) \cong \mathcal{W}_\lambda(A)^d,$$

where $\mathcal{W}_\lambda(A)$ is a free $\mathcal{O}_\lambda$-module of rank $\frac{2g}{ed}$, with non-degenerate bilinear, G_F-equivariant form

$$\psi_\lambda : \mathcal{W}_\lambda(A) \times \mathcal{W}_\lambda(A) \to \mathcal{O}_\lambda$$

such that $W_\lambda(A) := \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$ is an absolutely irreducible G_F-module with a non-degenerate, G_F-equivariant bilinear form $\psi_\lambda^0 := \psi_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda$ (resp. $\overline{\mathcal{W}}_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} k_\lambda$ is an absolutely irreducible G_F-module with a non-degenerate, G_F-equivariant bilinear form $\overline{\psi}_\lambda := \psi_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda$). For type I and II, the forms ψ_λ, ψ_λ^0 and $\overline{\psi}_\lambda$ are alternating and for type III the forms ψ_λ, ψ_λ^0 and $\overline{\psi}_\lambda$ are symmetric.

For the case of abelian varieties of type II, this result extends integrally and residually the main result of [8, Theorem A] by W. C. Chi.

The Galois module structure of $T_l(A)$ for abelian varieties A of types I, II and III has been widely investigated as well as Galois module structure of $T_l(A)$ for abelian varieties A of type IV with D commutative (in particular see [9] for type IV). Such results are useful for current research. Results in [2], [3] and [4] also found a variety of applications eg. [5], [6], [7], [10], [16], [17], [18] just to mention a few recent papers. Similarly as in [2], [3] and [4], we expect to prove the Mumford-Tate conjecture for some families of abelian varieties of type IV based on results of this paper.

In this paper we address all abelian varieties of type IV, especially those with D noncommutative. In general, endomorphism algebras of abelian varieties of type IV are much more complicated than endomorphism algebras of abelian varieties of types I, II and III. Indeed, the degree of D over E may be arbitrary and the standard involution acts nontrivially on its center which is CM field, c.f. [14, Theorem 2, p. 201-203]. Nevertheless, we obtain new results for abelian varieties of type IV (see Theorem 1.1 below) showing striking similarity with corresponding results for abelian varieties of types I, II and III discussed above.

In Section 2, we describe as explicitly as possible (Lemma 2.1) the endomorphism algebra D of a simple abelian variety of type IV and its splitting field L to obtain an L-algebra isomorphism:

$$s \,:\, D \otimes_{E_0} L^+ = D \otimes_E L \xrightarrow{\sim} M_d(L).$$

Lemma 2.1 is an arithmetic refinement of computations in D. Mumford's book [14, § Application I, Step IV]. It is one of the main technical devices in our paper. Observe how Lemma 2.1 significantly differs from corresponding lemma ([4, Lemma 2.11]) for abelian varieties of type III. Based on this, we also obtain an S-integral splitting of the algebra $\mathcal{R}_S := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_S$ at the end of this section. This result and construction of the finite set S can be found in Corollary 2.3. We also construct a positive involution $x \mapsto x^*$ of D which has the extension $X \mapsto X^* := \overline{X}^{Tr}$ to the ring $M_d(L)$ via splitting s (see Lemma 2.1 for details). In Section 3, we carefully investigate the Tate module of abelian variety of type IV based on results of Section 2. Namely, adding a few assumptions on S and working as far as possible S-integrally, we eventually construct non-degenerate,

G_F-equivariant hermitian forms (see Lemma 3.5) to obtain, in Sections 4 and 5, our main result as follows.

Theorem 1.1 (Theorems 4.3, 5.2)**.** *Let A be an abelian variety of type IV. Let l be a prime outside of a finite set S. Let $\lambda|l$ be a prime of $\mathcal{O}_E$ such that λ is inert over $\lambda_0 := \lambda \cap \mathcal{O}_{E_0}$ and λ splits completely in $\mathcal{O}_L$. The $\mathcal{O}_\lambda[G_F]$-module $T_\lambda(A)$ has the following decomposition:*

$$T_\lambda(A) \cong \mathcal{W}_\lambda(A)^d,$$

where $\mathcal{W}_\lambda(A)$ is a free $\mathcal{O}_\lambda$-module of rank $\frac{2g}{ed}$ with a non-degenerate, hermitian, G_F-equivariant form

$$\psi_\lambda : \mathcal{W}_\lambda(A) \times \mathcal{W}_\lambda(A) \to \mathcal{O}_\lambda$$

such that $W_\lambda(A) := \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$ is an absolutely irreducible G_F-module with a non-degenerate, hermitian, G_F-equivariant form $\psi^0_\lambda := \psi_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda$ (resp. $\overline{\mathcal{W}}_\lambda(A) := \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} k_\lambda$ is an absolutely irreducible G_F-module with a non-degenerate, hermitian, G_F-equivariant form $\overline{\psi}_\lambda := \psi_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda$).

Remark 1.2. The restrictions on prime l in Theorem 1.1 result mainly from complexity of the endomorphism algebra of A.

2. Ring of endomorphisms of an abelian variety of type IV

Let A be a simple abelian variety of type IV satisfying all assumptions stated in Section 1. Let $'$ be the standard Rosati involution on D. This is a positive involution. Any other positive involution $*$ of D is of the form $x^* = \gamma x' \gamma^{-1}$ with $\gamma \in D$ and $\gamma' = \gamma$. There exists a positive involution $x^* = \gamma x' \gamma^{-1}$ of D and an isomorphism [14, Theorem 2, p. 201-203]

$$D \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \underbrace{M_d(\mathbb{C}) \times \cdots \times M_d(\mathbb{C})}_{e_0 \text{ copies}},$$

which carries this involution into $(X_1, \ldots, X_{e_0}) \mapsto (\overline{X_1}^{Tr}, \ldots, \overline{X_{e_0}}^{Tr})$. Under the above isomorphism $\gamma \otimes 1$ maps to $(C_1, \ldots, C_{e_0})$, where each C_i is a hermitian positive definite matrix. It follows that there exists a positive involution $x^* = \gamma x' \gamma^{-1}$ of D and an isomorphism

$$D \otimes_{E_0} \mathbb{R} \xrightarrow{\sim} M_d(\mathbb{C}), \tag{2.1}$$

which carries the involution $*$ of D into the involution $X \mapsto X^* := \overline{X}^{Tr}$ by the isomorphism (2.1) and $\gamma \otimes 1 \mapsto C$, where C is a hermitian positive definite matrix.

For our applications, we need an arithmetic refinement of the above statements as follows.

Lemma 2.1. *There exist a finite Galois extension L/E_0 containing E, an element $\gamma \in D$ with $\gamma' = \gamma$ and an L-algebra isomorphism*

$$s \,:\, D \otimes_{E_0} L^+ = D \otimes_E L \xrightarrow{\sim} M_d(L) \tag{2.2}$$

such that via this isomorphism the positive involution $x \mapsto x^* := \gamma x' \gamma^{-1}$ of D has the extension $X \mapsto X^* := \overline{X}^{Tr}$ to the ring $M_d(L)$ and $s(\gamma \otimes 1) = C$, where C is a hermitian positive definite matrix. Here, L^+ denotes the maximal real subfield of L. Changing base to $\mathbb{R}$ over L^+ the isomorphism (2.2) naturally extends to an isomorphism of the form (2.1) with the same properties.

Proof. From [13, Theorem 16, Chap. 29], we know that D has maximal subfields which are splitting fields of degree d over E. Hence, let L_0 be a maximal subfield of degree $d = [L_0 : E]$ such that $D \otimes_E L_0 \xrightarrow{\sim} M_d(L_0)$. Let L_1/E_0 be the Galois closure of L_0/E_0. Naturally $D \otimes_E L_1 \xrightarrow{\sim} M_d(L_1)$. Moreover, we obtain:

$$D \otimes_{E_0} L_1^+ \;=\; D \otimes_E E \otimes_{E_0} L_1^+ \;=\; D \otimes_E L_1 \;\xrightarrow{\sim}\; M_d(L_1). \tag{2.3}$$

Now we argue similarly to [14, p. 199-200]. By the Skolem-Noether theorem, the Rosati involution on $D \otimes_{E_0} L_1^+$ (acting trivially on L_1^+) extends to an involution of $M_d(L_1)$ of the following form:

$$X \mapsto A_1 X^* A_1^{-1} \tag{2.4}$$

with $A_1 \in GL_d(L_1)$. Because (2.4) is an involution, we get $A_1^* = \eta A_1$ for an element $\eta \in L_1^\times$ such that $|\eta| = 1$. Let L_2/E_0 be the Galois closure of $L_1(\eta^{\frac{1}{2}})/E_0$. We obtain:

$$D \otimes_{E_0} L_2^+ \;=\; D \otimes_E E \otimes_{E_0} L_2^+ \;=\; D \otimes_E L_2 \;\xrightarrow{\sim}\; M_d(L_2). \tag{2.5}$$

If $\eta \neq 1$ observe that:

$$(\eta^{\frac{1}{2}} A_1)^* = \eta^{-\frac{1}{2}} A_1^* = \eta^{-\frac{1}{2}} \eta A_1 = \eta^{\frac{1}{2}} A_1,$$

$$\eta^{\frac{1}{2}} A_1 X^* (\eta^{\frac{1}{2}} A_1)^{-1} = A_1 X^* A_1^{-1}.$$

Hence, $A_2 := \eta^{\frac{1}{2}} A_1 \in M_d(L_2)$ is a hermitian matrix and the Rosati involution on $D \otimes_{E_0} L_2^+$ (acting trivially on L_2^+) extends to an involution of $M_d(L_2)$ of the following form:

$$X \mapsto A_2 X^* A_2^{-1}. \tag{2.6}$$

Observe that A_2 is a fixed point of the involution (2.6). The set of elements in $D \otimes_{E_0} L_2^+$ fixed by this involution via (2.5) (equivalently fixed by the Rosati involution) is of the form $V \otimes_{E_0} L_2^+$, where V is the E_0-vector space $V = \{\alpha \in D \,:\, \alpha' = \alpha\}$. Indeed, by primitive element theorem there is $\delta \in L_2^+$ such that $L_2^+ = E_0(\delta)$. Let $r := [L_2^+ \,:\, E_0]$. Then every element of $D \otimes_{E_0} L_2^+$ is of the form

$\sum_{i=0}^{r-1} \alpha_i \otimes \delta^i$ for some $\alpha_i \in D$. The element $\sum_{i=0}^{r-1} \alpha_i \otimes \delta^i$ is fixed by the Rosati involution if and only if

$$\sum_{i=0}^{r-1} (\alpha_i' - \alpha_i) \otimes \delta^i = 0$$

and this occurs if and only if $\alpha_i' = \alpha_i$ for each i.

Let $\sum_{i=0}^{r-1} \alpha_i \otimes \delta^i \in V \otimes_{E_0} L_2^+$ be the element sent via (2.5) to A_2. Note that E_0 is dense in $\mathbb{R}$ with respect to the absolute value. Therefore, we can find elements $e_i \in E_0$, close enough to $\delta^i \in L_2^+$, such that the element

$$\alpha \otimes 1 = \sum_{i=0}^{r-1} \alpha_i e_i \otimes 1 = \sum_{i=0}^{r-1} \alpha_i \otimes e_i$$

maps via (2.5) to B_2 such that $A_2 B_2^* A_2^{-1} = B_2$ and $A_3 := B_2^{-1} A_2$ is very close to unit matrix I_d. Observe that A_3 is a hermitian matrix. Indeed, we have $(B_2^{-1})^* = (B_2^*)^{-1} = A_2^{-1} B_2^{-1} A_2$. Hence:

$$A_3^* = (B_2^{-1} A_2)^* = A_2 (B_2^{-1})^* = A_2\, A_2^{-1} B_2^{-1} A_2 = B_2^{-1} A_2 = A_3.$$

The hermitian matrix A_3, being very close to I_d, is positive definite. There exist a finite Galois extension L_3/E_0 with $L_2 \subset L_3$, a unitary matrix $U \in GL_d(L_3)$ and a diagonal matrix $D_3 \in GL_d(L_3^+)$ with positive entries on the diagonal such that $A_3 = U\, D_3^2\, U^*$. Put $B_3 := U\, D_3\, U^* \in GL_d(L_3)$. Observe that $B_3^* = B_3$ and $A_3 = B_3^2$. By (2.5) we obtain:

$$D \otimes_{E_0} L_3^+ \;=\; D \otimes_E E \otimes_{E_0} L_3^+ \;=\; D \otimes_E L_3 \xrightarrow{\sim} M_d(L_3). \tag{2.7}$$

Observe that the map:

$$x \mapsto x^* := \alpha^{-1} x' \alpha \tag{2.8}$$

is an involution of $D \otimes_{E_0} L_3^+$ and it extends via (2.7) to the following involution of $M_d(L_3)$:

$$X \mapsto A_3 X^* A_3^{-1}. \tag{2.9}$$

Now we put $L := L_3$ and $\gamma := \alpha^{-1}$. Composing the isomorphism (2.7) $x \mapsto X$ with the conjugation by B_3, namely $X \mapsto B_3^{-1} X B_3$, we obtain the isomorphism:

$$s \;:\; D \otimes_{E_0} L^+ \xrightarrow{\sim} M_d(L), \tag{2.10}$$

$$s(x) := B_3^{-1} X B_3. \tag{2.11}$$

Observe that:

$$s(x^*) = s(x)^*. \tag{2.12}$$

Indeed:

$$s(x^*) = s(\gamma x' \gamma^{-1}) = B_3^{-1} A_3 X^* A_3^{-1} B_3 = B_3 X^* B_3^{-1} = (B_3^{-1} X B_3)^* = s(x)^*.$$

Hence the involution $x \mapsto x^* = \gamma x' \gamma^{-1}$ extends via (2.10) to the involution $X \mapsto X^*$ of $M_d(L)$. The last statement of the lemma follows because:

$$D \otimes_{E_0} \mathbb{R} = D \otimes_E E \otimes_{E_0} L^+ \otimes_{L^+} \mathbb{R} = D \otimes_E L \otimes_{L^+} \mathbb{R} \xrightarrow{\sim} M_d(L) \otimes_{L^+} \mathbb{R} = M_d(\mathbb{C}). \tag{2.13}$$

Naturally, the involution $x \mapsto x^* = \gamma x' \gamma^{-1}$ of D extends to the involution $X \mapsto X^*$ of $M_d(\mathbb{C})$.

The following diagram illustrates relations between consecutive extensions of fields E_0 and E used in this proof.

$$\begin{tikzcd}
 & E \arrow[r, dash, "d"] \arrow[d, dash, "2"] & L_0 \arrow[r, dash] \arrow[d, dash, "2"] & L_1 \arrow[r, dash] \arrow[d, dash, "2"] & L_2 \arrow[r, dash] \arrow[d, dash, "2"] & L := L_3 \arrow[r, dash] \arrow[d, dash, "2"] & \mathbb{C} \arrow[d, dash, "2"] \\
\mathbb{Q} \arrow[ur, dash, "e"] \arrow[r, dash, "e_0"'] & E_0 \arrow[r, dash] & L_0^+ \arrow[r, dash] & L_1^+ \arrow[r, dash] & L_2^+ \arrow[r, dash] & L_3^+ \arrow[r, dash] & \mathbb{R}
\end{tikzcd}$$

□

Remark 2.2. Lemma 2.1 is useful for the proof of Proposition 3.7 which is crucial in the proof of Lemma 4.1 ultimately leading to the proof of Theorem 1.1.

Recall that the ring $\mathcal{R}$ is a finitely generated free $\mathbb{Z}$-module. Let $\mathcal{O}_{E_0}^0 := \mathcal{R} \cap \mathcal{O}_{E_0}$ and $\mathcal{O}_E^0 := \mathcal{R} \cap \mathcal{O}_E$. Then $\mathcal{O}_{E_0}^0$ is an order in $\mathcal{O}_{E_0}$ and $\mathcal{O}_E^0$ is an order in $\mathcal{O}_E$.

Let S be a set of primes of $\mathbb{Z}$ containing prime numbers that divide the indexes $[\mathcal{O}_{E_0} : \mathcal{O}_{E_0}^0]$ and $[\mathcal{O}_E : \mathcal{O}_E^0]$.

Corollary 2.3. *One can enlarge S so that the primes not in S are unramified in $\mathcal{O}_L$, all primes dividing the polarization degree of A are in S, the Rosati involution acts on $\mathcal{R}_S := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_S$, $\gamma \in \mathcal{R}_S^{\times}$, and the L-algebra isomorphism (2.2) restricts to an $\mathcal{O}_{L,S}$-algebra isomorphism:*

$$s \, : \, \mathcal{R}_S \otimes_{\mathcal{O}_{E_0,S}} \mathcal{O}_{L^+,S} = \mathcal{R}_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{L,S} \xrightarrow{\sim} M_d(\mathcal{O}_{L,S}). \tag{2.14}$$

Moreover with these assumptions the involution $*$ of $D \otimes_{E_0} L^+$ restricts to the involution $*$ of $\mathcal{R}_S \otimes_{\mathcal{O}_{E_0,S}} \mathcal{O}_{L^+,S}$ which, in turn, extends to the involution $X \mapsto X^* := \overline{X}^{Tr}$ of $M_d(\mathcal{O}_{L,S})$.

Proof. Follows immediately from Lemma 2.1 and its proof. □

3. Weil pairing of an abelian variety of type IV

Let $T(A) = H_1(A(\mathbb{C}), \mathbb{Z})$ and $V(A) = T(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The polarization on A induces a non-degenerate alternating $\mathbb{Z}$-bilinear form, the Riemann form of A:

$$\kappa \, : \, T(A) \times T(A) \to \mathbb{Z}. \tag{3.1}$$

Let $\kappa^0 := \kappa \otimes_{\mathbb{Z}} \mathbb{Q} \, : \, V(A) \times V(A) \to \mathbb{Q}$. Then for all $v_1, v_2 \in V(A)$ and $x \in D$ we have:

$$\kappa^0(x v_1, v_2) = \kappa^0(v_1, x' v_2). \tag{3.2}$$

There exists a unique E-bilinear form (E acts on factor $V(A)$ on the right by complex conjugation):

$$\phi^0 \,:\, V(A) \times V(A) \to E \tag{3.3}$$

with $\kappa^0(v_1, v_2) = Tr_{E/\mathbb{Q}}(f\phi^0(v_1, v_2))$ where $f \in E$ and $\overline{f} = -f$ [11, Lemma 4.6]. In addition, it is also proven loc. cit. that ϕ^0 is E-hermitian. Now let $T_S := T(A) \otimes_{\mathcal{O}_E^0} \mathcal{O}_{E,S}$ and $V_S := T_S \otimes_{\mathcal{O}_{E,S}} E$. Observe that:

$$\begin{aligned} V(A) &= T(A) \otimes_{\mathbb{Z}} \mathbb{Q} = T(A) \otimes_{\mathcal{O}_E^0} \mathcal{O}_E^0 \otimes_{\mathbb{Z}} \mathbb{Q} = T(A) \otimes_{\mathcal{O}_E^0} E \\ &= T(A) \otimes_{\mathcal{O}_E^0} \mathcal{O}_{E,S} \otimes_{\mathcal{O}_{E,S}} E = T_S \otimes_{\mathcal{O}_{E,S}} E = V_S. \end{aligned}$$

We can enlarge the set S from previous section, if necessary, so that $f \in \mathcal{O}_{E,S}^{\times}$ and the E-hermitian form (3.3) restricts to the following $\mathcal{O}_{E,S}$-hermitian form

$$\phi_S \,:\, T_S \times T_S \to \mathcal{O}_{E,S} \tag{3.4}$$

such that $\kappa_S(v_1, v_2) = Tr_{E/\mathbb{Q}}(f\phi_S(v_1, v_2))$ where $\kappa_S := \kappa \otimes_{\mathbb{Z}} \mathbb{Z}_S$. Observe that $\phi^0 = \phi_S \otimes_{\mathcal{O}_{E,S}} E$.

Recall that we put $E_l := E \otimes_{\mathbb{Q}} \mathbb{Q}_l$ and $\mathcal{O}_{E_l} := \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_l$. Note that $E_l = \mathcal{O}_{E_l} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$.

From now on till the end of this paper we assume that $l \notin S$.

Then $\mathcal{O}_{E_l} = \mathcal{O}_{E,S} \otimes_{\mathbb{Z}_S} \mathbb{Z}_l$. We can naturally extend the action of complex conjugation on E to the action on rings E_l and $\mathcal{O}_{E_l}$ imposing trivial action on $\mathbb{Q}_l$. The action will be denoted in the same way as complex conjugation i.e. the action on $x \in E_l$ will be denoted $\overline{x}$. Observe that $f \in \mathcal{O}_{E_l}^{\times}$. By [3, Lemma 3.1] and the idea of the proof of [11, Lemma 4.6] there is a unique $\mathcal{O}_{E_l}$-bilinear form ($\mathcal{O}_{E_l}$ acts on factor $T_l(A)$ on the right by complex conjugation)

$$\phi_l \,:\, T_l(A) \times T_l(A) \to \mathcal{O}_{E_l} \tag{3.5}$$

such that the $\mathbb{Z}_l$-bilinear form $\kappa_l := \kappa \otimes_{\mathbb{Z}} \mathbb{Z}_l$:

$$\kappa_l \,:\, T_l(A) \times T_l(A) \to \mathbb{Z}_l \tag{3.6}$$

has the following property:

$$\kappa_l(v_1, v_2) = Tr_{E_l/\mathbb{Q}_l}(f\phi_l(v_1, v_2)). \tag{3.7}$$

We can also prove as in loc.cit. that ϕ_l is $\mathcal{O}_{E_l}$-hermitian.

Lemma 3.1. *There is the following isomorphism of $\mathcal{O}_{E_l}$-hermitian forms:*

$$\phi_l = \phi_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E_l}. \tag{3.8}$$

Proof. There is the following equality in $\mathbb{Z}_l$:

$$\begin{aligned} &\kappa_S(u_1, u_2) \otimes_{\mathbb{Z}_S} 1 = Tr_{E/\mathbb{Q}}(f\phi_S(u_1, u_2)) \otimes_{\mathbb{Z}_S} 1 \\ &= Tr_{E_l/\mathbb{Q}_l}(f\, \phi_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E_l}(u_1 \otimes_{\mathcal{O}_{E,S}} 1,\, u_2 \otimes_{\mathcal{O}_{E,S}} 1)). \end{aligned} \tag{3.9}$$

Since $\kappa_S \otimes_{\mathbb{Z}_S} \mathbb{Z}_l = \kappa_l$ and $\mathcal{O}_{E_l} = \mathcal{O}_{E,S} \otimes_{\mathbb{Z}_S} \mathbb{Z}_l$, we obtain by (3.9) the following equality in $\mathbb{Z}_l$ for all $u_1, u_2 \in T_S$ and $\alpha_1, \alpha_2 \in \mathcal{O}_{E_l}$:

$$\kappa_l(u_1 \otimes \alpha_1, u_2 \otimes \alpha_2) = Tr_{E_l/\mathbb{Q}_l}(f\,\phi_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E_l}\,(u_1 \otimes_{\mathcal{O}_{E,S}} \alpha_1,\ u_2 \otimes_{\mathcal{O}_{E,S}} \alpha_2)). \quad (3.10)$$

Observe that

$$T_l(A) = T(A) \otimes_{\mathbb{Z}} \mathbb{Z}_l = T(A) \otimes_{\mathcal{O}_E^0} \mathcal{O}_E^0 \otimes_{\mathbb{Z}} \mathbb{Z}_l = T(A) \otimes_{\mathcal{O}_E^0} \mathcal{O}_{E,S} \otimes_{\mathbb{Z}_S} \mathbb{Z}_l =$$
$$= T_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E,S} \otimes_{\mathbb{Z}_S} \mathbb{Z}_l = T_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E_l}. \quad (3.11)$$

Hence by (3.10), for all $v_1, v_2 \in T_l(A)$ we obtain:

$$\kappa_l(v_1, v_2) = Tr_{E_l/\mathbb{Q}_l}(f\,\phi_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E_l}\,(v_1,\ v_2)). \quad (3.12)$$

By uniqueness of the form ϕ_l (3.5) and by equality (3.12), we obtain the equality (3.8). □

Lemmas 3.2 and 3.3 below extend [8, Lemma (2.3)] to abelian varieties of type IV.

Lemma 3.2. *For all $v_1, v_2 \in V(A)$, $u_1, u_2 \in T_S$, $x \in D$, $y \in \mathcal{R}_S$ there are the following equalities:*

$$\phi^0(xv_1, v_2) = \phi^0(v_1, x'v_2),$$
$$\phi_S(yu_1, u_2) = \phi_S(u_1, y'u_2).$$

Proof. Fix $x \in D$. Consider the $\mathbb{Q}$-bilinear form $\kappa_x(v_1, v_2) : V(A) \times V(A) \to \mathbb{Q}$, defined as follows:

$$\kappa_x(v_1, v_2) := \kappa^0(xv_1, v_2) = \kappa^0(v_1, x'v_2).$$

Consider two E-bilinear forms $\phi_1^0, \phi_2^0 : V(A) \times V(A) \to E$:

$$\phi_1^0(v_1, v_2) := \phi^0(xv_1, v_2) \quad \text{and} \quad \phi_2^0(v_1, v_2) := \phi^0(v_1, x'v_2).$$

Recall that

$$\kappa^0(xv_1, v_2) = Tr_{E/\mathbb{Q}}(f\,\phi^0(xv_1, v_2))$$

and

$$\kappa^0(v_1, x'v_2) = Tr_{E/\mathbb{Q}}(f\,\phi^0(v_1, x'v_2)),$$

where $f \in E$. Hence

$$\kappa_x(v_1, v_2) = Tr_{E/\mathbb{Q}}(f\,\phi_1^0(v_1, v_2)) = Tr_{E/\mathbb{Q}}(f\,\phi_2^0(v_1, v_2)).$$

We have $\phi_1^0 = \phi_2^0$ by [11, Lemma 4.6]. So the first equality follows. Fix $y \in \mathcal{R}_S$. Consider two $\mathcal{O}_{E,S}$-bilinear forms $\phi_1, \phi_2 : T_S \times T_S \to \mathcal{O}_{E,S}$ defined as follows:

$$\phi_1(u_1, u_2) := \phi_S(yu_1, u_2) \quad \text{and} \quad \phi_2(u_1, u_2) := \phi_S(u_1, y'u_2).$$

Observe that bilinear forms ϕ_i are restrictions to $T_S \times T_S$ of E-bilinear forms ϕ_i^0 with y in place of x. Hence, the second equality of the lemma follows from the first. □

Lemma 3.3. *For all $v_1, v_2 \in T_l(A)$ and $g \in G_F$ we have the following equality:*

$$\phi_l(gv_1, gv_2) = \chi_c(g)\phi_l(v_1, v_2).$$

Here χ_c is the cyclotomic character $\chi_c : G_F \to \mathbb{Z}_l$.

Proof. By Galois equivariance of the Weil pairing for all $v_1, v_2 \in T_l(A)$ and all $g \in G_F$ we have:

$$\kappa_l(gv_1, gv_2) = \chi_c(g)\,\kappa_l(v_1, v_2). \tag{3.13}$$

Fix $g \in G_F$ and consider $\mathbb{Z}_l$-bilinear form: $\kappa_g(v_1, v_2) : T_l(A) \times T_l(A) \to \mathbb{Z}_l$ defined as follows:

$$\kappa_g(v_1, v_2) := \kappa_l(gv_1, gv_2) = \chi_c(g)\,\kappa_l(v_1, v_2).$$

Consider two $\mathcal{O}_{E_l}$-bilinear forms: $\phi_l^1, \phi_l^2 : T_l(A) \times T_l(A) \to \mathcal{O}_{E_l}$:

$$\phi_l^1(v_1, v_2) := \phi_l(gv_1, gv_2) \quad \text{and} \quad \phi_l^2(v_1, v_2) := \chi_c(g)\phi_l(v_1, v_2).$$

By (3.7) we obtain

$$\kappa_g(v_1, v_2) = Tr_{E_l/\mathbb{Q}_l}(f\phi_l^1(v_1, v_2)) = Tr_{E_l/\mathbb{Q}_l}(f\phi_l^2(v_1, v_2))$$

for $f \in \mathcal{O}_{E_l}^{\times}$. Hence, we obtain $\phi_l^1 = \phi_l^2$ by [11, Lemma 4.6]. □

Now define the following $\mathcal{O}_{E,S}$-hermitian form

$$\psi_S : T_S \times T_S \to \mathcal{O}_{E,S}, \tag{3.14}$$

$$\psi_S(v_1, v_2) = \phi_S(\gamma^{-1}v_1, v_2).$$

Let

$$\psi^0 := \psi_S \otimes_{\mathcal{O}_{E,S}} E : V \times V \to E. \tag{3.15}$$

Because the form (3.1) is non-degenerate, the forms ϕ, ϕ^0, ψ and ψ^0 are also non-degenerate.

Lemma 3.4. *For every $x \in \mathcal{R}_S$ and all $v_1, v_2 \in T_S$ we have:*

$$\psi_S(xv_1, v_2) = \psi_S(v_1, x^*v_2),$$

where, as defined in previous section, $x^* = \gamma x' \gamma^{-1}$.

Proof. Recall that $\gamma' = \gamma$ and let $x \in \mathcal{R}_S$. We obtain the following equality for all $v_1, v_2 \in T_S$ from the property of Rosati involution, Lemma 3.2, the definition of S and the fact that ϕ_S and ψ_S are $\mathcal{O}_{E,S}$-hermitian forms.

$$\psi_S(xv_1, v_2) = \phi_S(\gamma^{-1}xv_1, v_2) = \phi_S(v_1, x'\gamma^{-1}v_2) = \phi_S(v_1, \gamma^{-1}\gamma x'\gamma^{-1}v_2)$$

$$= \phi_S(v_1, \gamma^{-1}x^*v_2) = \psi_S(v_1, x^*v_2).$$

□

It follows from Lemma 2.1 that the involution $*$ induced on $D_L := D \otimes_E L \cong M_d(L)$ from D is of the form $B^* = \overline{B}^{Tr}$ for each $B \in M_d(L)$. Consider the complex conjugation $\tau \in \mathrm{Gal}(L/E_0)$. Take a prime number l and $\lambda_0 | l$ in $\mathcal{O}_{E_0}$ such that $\mathrm{Frob}_{\omega/\lambda_0} = \tau$ for a prime ideal $\omega \in \mathrm{Spec}(\mathcal{O}_L)$. Let $\lambda = \mathcal{O}_E \cap \omega$ be the prime ideal in $\mathcal{O}_E$ below ω and over λ_0. Because the order of τ is 2 in $\mathrm{Gal}(L/E_0)$ and $\tau \in \mathrm{Gal}(E/E_0)$ is also of order 2, hence λ is inert over λ_0 and λ splits completely in $\mathcal{O}_L$. There are infinitely many such primes λ by Chebotarev's theorem. Then we have

$$[L_\omega : E_\lambda] = 1 \quad \text{and} \quad \mathcal{O}_\lambda = \mathcal{O}_\omega. \tag{3.16}$$

Put

$$T_\lambda(A) := T_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_\lambda, \quad V_\lambda(A) := T_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda, \quad A[\lambda] := T_\lambda(A)/\lambda T_\lambda(A). \tag{3.17}$$

Note that $A[\lambda]$ is a $k_\lambda[G_F]$-module. Define a λ-adic hermitian form as follows:

$$\phi_\lambda := \phi_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_\lambda : T_\lambda(A) \times T_\lambda(A) \to \mathcal{O}_\lambda. \tag{3.18}$$

Observe that $\phi_\lambda = \phi_l \otimes_{\mathcal{O}_{E_l}} \mathcal{O}_\lambda$ by Lemma 3.8. We also obtain the following hermitian forms:

$$\phi_\lambda^0 := \phi_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda : V_\lambda(A) \times V_\lambda(A) \to E_\lambda,$$
$$\overline{\phi}_\lambda := \phi_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda : A[\lambda] \times A[\lambda] \to k_\lambda.$$

Since $\gamma' = \gamma$, the following forms are also hermitian:

$$\begin{aligned} \psi_\lambda &: T_\lambda(A) \times T_\lambda(A) \to \mathcal{O}_\lambda, \\ \psi_\lambda(v_1, v_2) &= \phi_\lambda(\gamma^{-1} v_1, v_2), \end{aligned} \tag{3.19}$$

$$\psi_\lambda^0 := \psi_\lambda \otimes_{\mathcal{O}_\lambda} E_\lambda : V_\lambda(A) \times V_\lambda(A) \to E_\lambda, \tag{3.20}$$

$$\overline{\psi}_\lambda := \psi_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda : A[\lambda] \times A[\lambda] \to k_\lambda. \tag{3.21}$$

Lemma 3.5. *Hermitian forms ϕ_λ, ϕ_λ^0, $\overline{\phi}_\lambda$, ψ_λ, ψ_λ^0, $\overline{\psi}_\lambda$ are non-degenerate and G_F-equivariant.*

Proof. Since the form κ_l (3.6) is non-degenerate, the form ϕ_l (3.5) is also non-degenerate by property (3.7). Consider the bilinear forms:

$$\overline{\kappa}_l := \kappa_l \otimes_{\mathbb{Z}_l} \mathbb{Z}/l : A[l] \times A[l] \to \mathbb{Z}/l,$$
$$\overline{\phi}_l := \phi_l \otimes_{\mathcal{O}_{E_l}} \mathcal{O}_{E_l}/l : A[l] \times A[l] \to \mathcal{O}_{E_l}/l$$

related by the following equality

$$\overline{\kappa}_l(v_1, v_2) = Tr_{E_l/\mathbb{Q}_l}(f \overline{\phi}_l(v_1, v_2)). \tag{3.22}$$

where $f \in \mathcal{O}_{E_l}^\times$. Because l does not divide the polarisation of A, then $\overline{\kappa}_l(v_1, v_2)$ is non-degenerate. Hence, $\overline{\phi}_l(v_1, v_2)$ is non-degenerate by (3.22). By [3, Lemma 3.2], forms ϕ_λ, ϕ_λ^0, $\overline{\phi}_\lambda$ are non-degenerate. Hence, it is obvious that the forms

ψ_λ, ψ_λ^0, $\overline{\psi}_\lambda$ are non-degenerate. It follows immediately from Lemma 3.3 that ϕ_λ, ϕ_λ^0, $\overline{\phi}_\lambda$ are G_F-equivariant. By definition of ψ_λ, it follows that the forms ψ_λ, ψ_λ^0, $\overline{\psi}_\lambda$ are G_F-equivariant, because G_F commutes with $\mathrm{End}_{\overline{F}}(A)$. □

Observe that we have the following isomorphism

$$D_\lambda := D \otimes_E E_\lambda \cong M_d(E_\lambda). \tag{3.23}$$

Indeed, by (3.16) we have $D_\lambda = D \otimes_E E_\lambda = D \otimes_E L_\omega = D \otimes_E L \otimes_L L_\omega = M_d(L_\omega) = M_d(E_\lambda)$. Then (3.23) induces the following isomorphism of $\mathcal{O}_\lambda$-modules

$$\mathcal{R}_\lambda := \mathcal{R}_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_\lambda \cong M_d(\mathcal{O}_\lambda). \tag{3.24}$$

By (1.1), we have

$$\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_l = \mathcal{R} \otimes_{\mathcal{O}_E^0} \mathcal{O}_{E_l} = \mathcal{R}_S \otimes_{\mathcal{O}_{E,S}} \mathcal{O}_{E_l} = \prod_{\lambda | l} \mathcal{R}_\lambda. \tag{3.25}$$

On the other hand, by [12, Satz 4]:

$$\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Z}_l \xrightarrow{\sim} \mathrm{End}_{\mathbb{Z}_l[G_F]}(T_l(A)). \tag{3.26}$$

By (1.2), (3.25) and (3.26), we obtain the following isomorphism of $\mathcal{O}_\lambda$-algebras.

$$\mathcal{R}_\lambda \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_\lambda[G_F]}(T_\lambda(A)). \tag{3.27}$$

Finally, (3.24) and (3.27) give the following isomorphism of $\mathcal{O}_\lambda$-algebras:

$$\mathrm{End}_{\mathcal{O}_\lambda[G_F]}(T_\lambda(A)) \xrightarrow{\sim} M_d(\mathcal{O}_\lambda). \tag{3.28}$$

Remark 3.6. Since $\lambda | \lambda_0$ is unramified and inert, we have

$$\mathrm{Gal}(E/E_0) \cong \mathrm{Gal}(E_\lambda/E_{0,\lambda_0}) \cong \mathrm{Gal}(k_\lambda/k_{\lambda_0}).$$

Hence, the element $\mathrm{Frob}_{\lambda/\lambda_0} = \tau \in \mathrm{Gal}(E/E_0)$ can be considered as an element in $\mathrm{Gal}(E_\lambda/E_{0,\lambda_0})$. Thus if a matrix $B \in M_d(E)$ is considered as an element of $M_d(E_\lambda)$, τ acts on B via $\mathrm{Frob}_{\lambda/\lambda_0}$ and we will denote $\overline{B} := \mathrm{Frob}_{\lambda/\lambda_0}(B)$.

Proposition 3.7. *(i) For every $v_1, v_2 \in T_\lambda(A)$ and $B \in \mathcal{R}_\lambda$, we have*

$$\psi_\lambda(Bv_1, v_2) = \psi_\lambda(v_1, \overline{B}^{Tr} v_2).$$

(ii) For every $v_1, v_2 \in V_\lambda(A)$ and $B \in D_\lambda$, we have

$$\psi_\lambda^0(Bv_1, v_2) = \psi_\lambda^0(v_1, \overline{B}^{Tr} v_2).$$

(iii) For every $v_1, v_2 \in A[\lambda]$ and $B \in \mathcal{R}_\lambda \otimes_{\mathcal{O}_\lambda} k_\lambda \cong M_d(k_\lambda)$, we have

$$\overline{\psi}_\lambda(Bv_1, v_2) = \overline{\psi}_\lambda(v_1, \overline{B}^{Tr} v_2).$$

Proof. It follows from Lemmas 2.1, 3.4 and the isomorphism (3.24). □

Definition 3.8. Let $\mathcal{P}$ be the set of prime numbers $l \notin S$ such that there is $\lambda_0 | l$ in $\mathcal{O}_{E_0}$ and λ inert over λ_0 in $\mathcal{O}_E$ and λ splits completely in $\mathcal{O}_L$ (see the discussion below Lemma 3.4).

Remark 3.9. Observe that the set $\mathcal{P}$ has a positive Dirichlet's density because of Chebotarev's theorem. Our main results Theorems 4.3 and 5.2 will be formulated for primes $l \in \mathcal{P}$.

4. Main theorem for $d \leq 2$

Based on results of previous sections, we construct the Tate module decomposition for an abelian variety of type IV when $d = [D : E]^{\frac{1}{2}} = 2$. We observe that we can prove Theorem 4.3 applying the same idempotents as for types II and III as well as the standard idempotents. This observation is the key for proving our main result for arbitrary degree $d = [D : E]^{\frac{1}{2}}$ which will be shortly described in the next section. We also briefly explain, at the beginning of the proof of Theorem 4.3, how the construction works for the case $d = 1$.

Following [8, p. 91-92], consider the following matrices:

$$t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the idempotent $e = \frac{1}{2}(1+t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Define:

$$\mathcal{X} := e\, T_\lambda(A) \quad \text{and} \quad \mathcal{Y} = (1-e)\, T_\lambda(A),$$

$$X := \mathcal{X} \otimes_{\mathcal{O}_\lambda} E_\lambda, \quad Y := \mathcal{Y} \otimes_{\mathcal{O}_\lambda} E_\lambda, \quad \overline{\mathcal{X}} := \mathcal{X} \otimes_{\mathcal{O}_\lambda} k_\lambda, \quad \overline{\mathcal{Y}} := \mathcal{Y} \otimes_{\mathcal{O}_\lambda} k_\lambda.$$

By (3.27), the action of $\mathcal{R}_\lambda$ commutes with the action of $\mathcal{O}_\lambda[G_F]$ on $T_\lambda(A)$. Hence, the equality $u\, e\, u = (1-e)$ yields a $\mathcal{O}_\lambda[G_F]$ - isomorphism between $\mathcal{X}$ and $\mathcal{Y}$, a $E_\lambda[G_F]$ - isomorphism between X and Y, and a $k_\lambda[G_F]$ - isomorphism between $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$.

Lemma 4.1. [4, Lemma 3.22] *Modules $\mathcal{X}$ and $\mathcal{Y}$ are orthogonal with respect to ψ_λ. Moreover, modules X and Y are orthogonal with respect to ψ_λ^0, and $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$ are orthogonal with respect to $\overline{\psi}_\lambda$.*

Proof. Note that $t\, e = e$ and $t(1-e) = -(1-e)$. Then for every $v_1 \in \mathcal{X}$ and $v_2 \in \mathcal{Y}$, we obtain $t\, v_1 = v_1$ and $t\, v_2 = -v_2$. Hence by Proposition 3.7, we obtain

$$\begin{aligned} \psi_\lambda(v_1, v_2) &= \psi_\lambda(t\, v_1, v_2) = \psi_\lambda(v_1, t^*\, v_2) = \psi_\lambda(v_1, \bar{t}^{Tr}\, v_2) \\ &= \psi_\lambda(v_1, t\, v_2) = \psi_\lambda(v_1, -v_2) = -\psi_\lambda(v_1, v_2). \end{aligned}$$

Hence, $\psi_\lambda(v_1, v_2) = 0$ for every $v_1 \in \mathcal{X}$ and for every $v_2 \in \mathcal{Y}$. $\square$

The discussion before Lemma 4.1 gives the following isomorphism of $\mathcal{O}_\lambda[G_F]$-modules

$$T_\lambda(A) \cong \mathcal{X} \oplus \mathcal{X}. \tag{4.1}$$

Then by (3.28) we obtain the following isomorphism of $\mathcal{O}_\lambda$-algebras,

$$M_2(\mathrm{End}_{\mathcal{O}_\lambda[G_F]}(\mathcal{X})) \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_\lambda[G_F]}(T_\lambda(A)) \xrightarrow{\sim} M_2(\mathcal{O}_\lambda). \tag{4.2}$$

Because $\mathcal{O}_\lambda$ is a discrete valuation ring, by rank and dimension comparison we obtain:

$$\mathrm{End}_{\mathcal{O}_\lambda[G_F]}(\mathcal{X}) \xrightarrow{\sim} \mathcal{O}_\lambda, \qquad \mathrm{End}_{E_\lambda[G_F]}(X) \xrightarrow{\sim} E_\lambda, \qquad \mathrm{End}_{k_\lambda[G_F]}(\overline{\mathcal{X}}) \xrightarrow{\sim} k_\lambda. \quad (4.3)$$

Therefore, the representations of G_F on the spaces X and Y (resp. on the spaces $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) are absolutely irreducible over E_λ (resp. k_λ). Then bilinear form ψ_λ^0 (resp. $\overline{\psi}_\lambda$) restricted to the spaces X and Y (resp. to the spaces $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) is non-degenerate or isotropic.

Remark 4.2. It is also possible to obtain another decomposition of the Tate module with the idempotent $f = \frac{1}{2}(1+u) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then one may take $\mathcal{X} = f\, T_\lambda(A)$ and $\mathcal{Y} = (1-f)\, T_\lambda(A)$. Such a decomposition is considered in [4] and in this case it has the same properties as the decomposition in Lemma 4.1 (cf. also [8, p. 91-93]).

Theorem 4.3. *Let A be an abelian variety of type IV, and let $d \leq 2$. For each $l \in \mathcal{P}$ there exists a free $\mathcal{O}_\lambda$-module $\mathcal{W}_\lambda(A)$ of rank $\frac{2g}{ed}$ with the following properties:*

(i) There exists an isomorphism of $\mathcal{O}_\lambda[G_F]$-modules

$$T_\lambda(A) \cong \mathcal{W}_\lambda(A) \oplus \mathcal{W}_\lambda(A).$$

(ii) There exists a hermitian non-degenerate form

$$\psi_\lambda \;:\; \mathcal{W}_\lambda(A) \times \mathcal{W}_\lambda(A) \to \mathcal{O}_\lambda.$$

(iii) For $W_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$, the induced hermitian form

$$\psi_\lambda^0 \;:\; W_\lambda(A) \times W_\lambda(A) \to E_\lambda$$

is non-degenerate.

(iv) For $\overline{W}_\lambda(A) = \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} k_\lambda$, the induced hermitian form

$$\overline{\psi}_\lambda \;:\; \overline{W}_\lambda(A) \times \overline{W}_\lambda(A) \to k_\lambda$$

is non-degenerate.

(v) The G_F-modules $W_\lambda(A)$ and $\overline{W}_\lambda(A)$ are absolutely irreducible. The hermitian forms ψ_λ, ψ_λ^0 and $\overline{\psi}_\lambda$ are G_F - equivariant.

Proof. For the case $d = 1$, we take $\mathcal{W}_\lambda(A) = T_\lambda(A)$ and the forms (3.19), (3.20) and (3.21). Then statements (i)–(v) hold in this case by Lemma 3.5 and equality $D = E$.

Now consider the case $d = 2$. Part (i) follows by (4.1) taking $\mathcal{W}_\lambda(A) := \mathcal{X}$. By Lemma 3.5, the hermitian forms (3.19), (3.20) and (3.21) are non-degenerate and G_F-equivariant. Restricting forms (3.19), (3.20) and (3.21) to the corresponding forms in (ii), (iii) and (iv) gives again hermitian and G_F-equivariant forms. We denote the restrictions also ψ_λ, ψ_λ^0 and $\overline{\psi}_\lambda$ by abuse of notations. The G_F-modules $W_\lambda(A)$ and $\overline{W}_\lambda(A)$ are absolutely irreducible by (4.3). Hence (v) holds. In addition, the forms in (iii) and (iv) are non-degenerate or isotropic.

By Lemma 4.1 and decompositions $V_\lambda(A) \cong W_\lambda(A) \oplus W_\lambda(A)$ and $A[\lambda] \cong \overline{\mathcal{W}}_\lambda(A) \oplus \overline{\mathcal{W}}_\lambda(A)$, they can not be isotropic because forms (3.20) and (3.21) are non-degenerate. Hence, (*iii*) and (*iv*) holds. Since the form (*iii*) is non-degenerate, the form (*ii*) is non-degenerate so (*ii*) follows. □

5. Main theorem for $d > 2$

Let matrices t, u and e be as in the previous section. For $d > 2$ consider the $d \times d$ matrix $\widehat{e}_i$ consisting of 1 at the i-th place of the diagonal and zeros at all other places for $i = 1, 2, \dots, d$. Observe that $\widehat{e}_i \in \mathrm{End}_{\mathcal{O}_\lambda[\mathrm{G_F}]}(T_\lambda(A))$ by (3.28).

Let $\mathcal{X}_i := \widehat{e}_i\, T_\lambda(A)$ for $i = 1, 2, \dots, d$. Also put $X_i := \mathcal{X}_i \otimes_{\mathcal{O}_\lambda} E_\lambda$ and $\overline{\mathcal{X}}_i := \mathcal{X}_i \otimes_{\mathcal{O}_\lambda} k_\lambda$. Consider the following $d \times d$ matrices:

$$U_{ij} = \begin{array}{cc} \begin{array}{ccccccc} & & i & & j & & \end{array} & \\ \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} & \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \end{array} \quad \text{and } T_{ij} = \begin{array}{cc} \begin{array}{ccccccc} & & i & & j & & \end{array} & \\ \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & 0 & & \\ & & & \ddots & & & \\ & & 0 & & -1 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix} & \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \end{array}$$

U_{ij} is the permutation matrix obtained from the identity matrix $\mathrm{I_d}$ by swapping i-th and j-th rows. T_{ij} is the matrix with only two non-zero entries 1 in the ii-th place and -1 in the jj-th place.

By (3.28), $U_{ij}, T_{ij} \in \mathrm{End}_{\mathcal{O}_\lambda[\mathrm{G_F}]}(T_\lambda(A))$ for all $i, j = 1, \dots, d$. The equality $U_{ij}\widehat{e}_i U_{ij} = \widehat{e}_j$ gives the following $\mathcal{O}_\lambda[G_F]$-isomorphism:

$$\begin{aligned} \mathcal{X}_i &\to \mathcal{X}_j \\ \widehat{e}_i v &\mapsto \widehat{e}_j v \end{aligned} \tag{5.1}$$

for every $v \in T_\lambda(A)$. Hence all $\mathcal{O}_\lambda[G_F]$-modules $\mathcal{X}_1, \dots, \mathcal{X}_d$ are pairwise isomorphic. It follows that all $E_\lambda[G_F]$-modules $X_1, \dots, X_d$ are pairwise isomorphic and all $k_\lambda[G_F]$-modules $\overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_d$ are pairwise isomorphic.

Lemma 5.1. *The modules $\mathcal{X}_1, \dots, \mathcal{X}_d$ are orthogonal with respect to ψ_λ. Moreover $X_1, \dots, X_d$ are orthogonal with respect to ψ_λ^0 and $\overline{\mathcal{X}}_1, \dots, \overline{\mathcal{X}}_d$ are orthogonal with respect to $\overline{\psi}_\lambda$.*

Proof. Note that $T_{ij}\, \widehat{e}_i = \widehat{e}_i$ and $T_{ij}\widehat{e}_j = -\widehat{e}_j$. Then, for every $v_1 \in \mathcal{X}_i$ and $v_2 \in \mathcal{X}_j$, we obtain $T_{ij}\, v_1 = v_1$ and $T_{ij}\, v_2 = -v_2$. Hence

$$\begin{aligned} \psi_\lambda(v_1, v_2) = \psi_\lambda(T_{ij}\, v_1, v_2) = \psi_\lambda(v_1, T_{ij}^*\, v_2) &= \psi_\lambda(v_1, \overline{T}_{ij}^{Tr}\, v_2) \\ = \psi_\lambda(v_1, T_{ij}\, v_2) = \psi_\lambda(v_1, -v_2) &= -\psi_\lambda(v_1, v_2). \end{aligned}$$

Thus $\psi_\lambda(v_1, v_2) = 0$ for every $v_1 \in \mathcal{X}_i$ and for every $v_2 \in \mathcal{X}_j$. □

Let $\mathcal{X} := \mathcal{X}_1$. We obtain the following isomorphism of $\mathcal{O}_\lambda[G_F]$-modules:

$$T_\lambda(A) \cong \bigoplus_{i=1}^{d} \mathcal{X} = \mathcal{X}^d. \tag{5.2}$$

By (3.28), there is a natural isomorphism of $\mathcal{O}_\lambda$-algebras

$$M_d(\mathrm{End}_{\mathcal{O}_\lambda[\mathrm{G_F}]}(\mathcal{X})) \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_\lambda[\mathrm{G_F}]}(T_\lambda(A)) \xrightarrow{\sim} M_d(\mathcal{O}_\lambda). \tag{5.3}$$

Again by rank and dimension comparison, we obtain:

$$\mathrm{End}_{\mathcal{O}_\lambda[G_F]}(\mathcal{X}) \xrightarrow{\sim} \mathcal{O}_\lambda, \qquad \mathrm{End}_{E_\lambda[G_F]}(X) \xrightarrow{\sim} E_\lambda, \qquad \mathrm{End}_{\mathrm{k}_\lambda[\mathrm{G_F}]}(\overline{\mathcal{X}}) \xrightarrow{\sim} k_\lambda. \tag{5.4}$$

Therefore, the representation of G_F on the space X (resp. on the space $\overline{\mathcal{X}}$) is absolutely irreducible over E_λ (resp. k_λ). Then, the hermitian form ψ_λ^0 (resp. $\overline{\psi}_\lambda$) restricted to the space X (resp. to the space $\overline{\mathcal{X}}$) is either non-degenerate or isotropic.

Theorem 5.2. *Let A be an abelian variety of type IV. Let $d > 2$ and $l \in \mathcal{P}$. Then there exists a free $\mathcal{O}_\lambda$-module $\mathcal{W}_\lambda(A)$ of rank $\frac{2g}{ed}$ with the following properties:*

(i) There exists an isomorphism of $\mathcal{O}_\lambda[G_F]$-modules $T_\lambda(A) \cong \mathcal{W}_\lambda(A)^d$.

(ii) There exists a hermitian, non-degenerate form

$$\psi_\lambda \ : \ \mathcal{W}_\lambda(A) \times \mathcal{W}_\lambda(A) \to \mathcal{O}_\lambda.$$

(iii) For $W_\lambda(A) := \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} E_\lambda$, the induced hermitian form

$$\psi_\lambda^0 \ : \ W_\lambda(A) \times W_\lambda(A) \to E_\lambda$$

is non-degenerate.

(iv) For $\overline{\mathcal{W}}_\lambda(A) := \mathcal{W}_\lambda(A) \otimes_{\mathcal{O}_\lambda} k_\lambda$, the induced hermitian form

$$\overline{\psi}_\lambda \ : \ \overline{\mathcal{W}}_\lambda(A) \times \overline{\mathcal{W}}_\lambda(A) \to k_\lambda$$

is non-degenerate.

(v) The G_F modules $W_\lambda(A)$ and $\overline{\mathcal{W}}_\lambda(A)$ are absolutely irreducible. The hermitian forms ψ_λ, ψ_λ^0 and $\overline{\psi}_\lambda$ are G_F-equivariant.

Proof. The proof for $d > 2$ is very similar to the proof of Theorem 4.3 for $d = 2$. Indeed, part (i) follows from (5.2) by taking $\mathcal{W}_\lambda(A) := \mathcal{X}$. By Lemma 3.5, the hermitian forms (3.19), (3.20) and (3.21) are non-degenerate and G_F-equivariant. Restricting forms (3.19), (3.20) and (3.21) to the corresponding forms in (ii), (iii) and (iv) gives again hermitian and G_F-equivariant forms. We denote the restrictions also ψ_λ, ψ_λ^0 and $\overline{\psi}_\lambda$ by abuse of notation. The G_F-modules $W_\lambda(A)$ and $\overline{\mathcal{W}}_\lambda(A)$ are absolutely irreducible by (5.4). Hence, (v) holds. In addition the forms in (iii) and (iv) are non-degenerate or isotropic. By Lemma 5.1 and decompositions $V_\lambda(A) \cong W_\lambda(A)^d$ and $A[\lambda] \cong \overline{\mathcal{W}}_\lambda(A)^d$, they can not be isotropic because forms (3.20) and (3.21) are non-degenerate. Hence, (iii) and (iv) holds. Since the form (iii) is non-degenerate, the form (ii) is non-degenerate so (ii) follows. $\square$

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(Grzegorz Banaszak) FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, ADAM MICKIEWICZ UNIVERSITY, 61-614 POZNAŃ, POLAND
banaszak@amu.edu.pl

(Aleksandra Kaim-Garnek) FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, ADAM MICKIEWICZ UNIVERSITY, 61-614 POZNAŃ, POLAND
akaim@amu.edu.pl

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