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Generating iterative schemes to locate common fixed points of nonlinear mappings using shrinking projection methods

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ABSTRACT. We introduce iterative scheme generating methods (ISGMs) to find common fixed points of nonlinear mappings through shrinking projection methods, leading to strong convergence theorems. These findings extend a recent study by Kondo [A. Kondo, Math. Ann. **391** (2025), 2007–2028], which only demonstrates weak convergence. Although ISGMs combined with shrinking projection methods were explored in a prior study [A. Kondo, Carpathian J. Math. **40** (2024), 819–840], that work depended on mean-valued sequence properties. This study develops ISGMs without relying on mean-valued sequences, yielding infinitely many strong convergence theorems. An application to a common split feasibility problem is also presented.

CONTENTS

1. Introduction

Let H be a real Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let S be a mapping from C into H, where C is a nonempty subset of H. Denote by

$$F(S) = \{x \in C : Sx = x\}$$

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the set of all fixed points of S. A mapping $S : C \to H$ is termed *nonexpansive* if $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$. Due to its broad applicability, researchers have widely studied constructing a sequence approximating a fixed point of a nonexpansive mapping. Recently, Kondo [27] proved the following theorem:

Theorem 1.1 ([27]). *Let C be a nonempty, closed, and convex subset of H. Let $S, T : C \to C$ be quasi-nonexpansive mappings* (2.8) *such that $I - S$ and $I - T$ are demiclosed* (2.9) *and $F(S) \cap F(T) \neq \emptyset$, where I denotes the identity mapping. Denote by $P_{F(S)\cap F(T)}$ the metric projection from H onto $F(S)\cap F(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval* $[0,1]$ *such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$, where $\mathbb{N} = \{1, 2, \cdots\}$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} &x_1 \in C \text{ is given,} \\ &x_{n+1} = a_n y_n + b_n S z_n + c_n T w_n \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \le \|x_n - q\|, \ \|z_n - q\| \le \|x_n - q\|, \text{ and } \|w_n - q\| \le \|x_n - q\| \tag{1.1}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0, \ x_n - z_n \to 0, \text{ and } x_n - w_n \to 0. \tag{1.2}$$

Then, the sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in F(S) \cap F(T)$, where $\hat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

The class of mappings considered in Theorem 1.1 includes nonexpansive mappings, as well as more general types of mappings; see the Appendix of Kondo [27] for further details. In Theorem 1.1, the sequence $\{x_n\}$ is defined with given sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$, and the required conditions for these sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are explicitly stated in (1.1) and (1.2). Consequently, many iterative schemes can be derived from this theorem. For example, consider the following iterative scheme:

$$\begin{aligned} z_n &= \lambda'_n x_n + \left(1 - \lambda'_n\right) T x_n, \\ y_n &= \lambda_n z_n + (1 - \lambda_n) S z_n, \\ x_{n+1} &= a_n y_n + b_n S y_n + c_n T y_n, \end{aligned} \tag{1.3}$$

where an initial point $x_1 \in C$ is provided. The coefficients of the convex combinations λ_n and λ'_n are required to satisfy $\lambda_n \to 1$ and $\lambda'_n \to 1$, respectively; see Corollary 4.4 in Kondo [27]. It can be confirmed that y_n in (1.3) satisfies the conditions $\|y_n - q\| \le \|x_n - q\|$ and $x_n - y_n \to 0$. Therefore, according to Theorem 1.1, the sequence $\{x_n\}$, defined by the rule (1.3), converges weakly to a common fixed point of S and T. The iterative scheme in (1.3) is a three-step scheme; see Noor [36], Dashputre and Diwan [7], and Phuengrattana and Suantai [37]. By setting $\lambda'_n = 1$ for all $n \in \mathbb{N}$ in (1.3), a two-step iterative scheme is obtained. For more on the two-step iterative methods, see Ishikawa [14], Xu

[45], Tan and Xu [44], and Berinde [2, 3]. This method, which generates infinitely many iterative schemes, is referred to as an *iterative scheme generating method (ISGM)*; see Kondo [22, 24, 25, 26, 28].

In 2003, Nakajo and Takahashi [35] proposed the CQ method and proved a strong convergence theorem for finding a fixed point of nonexpansive mapping. In 2006, Martinez-Yanes and Xu [33] extended the CQ method and proved the following theorem:

Theorem 1.2 ([33]). *Let C be a nonempty, closed, and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $0 \leq \alpha_n \leq \alpha < 1$ and $\beta_n \to 1$, where $\alpha \in [0,1)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ y_n &= \beta_n x_n + (1-\beta_n) S x_n, \\ X_n &= \alpha_n x_n + (1-\alpha_n) S y_n, \\ C_n &= \Big\{ h \in C : \|X_n - h\|^2 \leq \|x_n - h\|^2 \\ &\qquad + (1-\alpha_n)\left(\|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n,\, h\rangle\right)\Big\}, \\ Q_n &= \{h \in C : \langle x - x_n,\, x_n - h\rangle \geq 0\} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to an element $\widehat{x}$ in $F(S)$, where $\widehat{x} = P_{F(S)} x$.

In 2008, Takahashi *et al.* [42] also developed the CQ method and established a strong convergence theorem utilizing metric projections on shrinking sets $\{C_n\}$, where $\{C_n\}$ satisfies the condition $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$. This approach is referred to as *the shrinking projection method*. For other related works, see Kimura and Nakajo [16], as well as Ibaraki and Saejung [13]. In 2023, Kondo [24] applied ISGMs with mean-valued sequences, such as

$$X_n = a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k z_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n, \tag{1.4}$$

to the CQ and shrinking projection methods and obtained various strong convergence theorems. In (1.4), the point x_{n+1} is determined depending on X_n as in Theorem 1.2. For iterative methods involving mean-valued sequences, refer to Shimizu and Takahashi [38], Atsushiba and Takahashi [1], Kondo [23], and articles cited therein.

This study establishes ISGMs using the shrinking projection method incorporating the Martinez-Yanes and Xu method. Through these efforts, we derive numerous strong convergence theorems. These results enhance Theorem 1.1, which previously only provided weak convergence. Although ISGMs utilizing the shrinking projection method and the Martinez-Yanes and Xu method were also explored in Kondo [26], that study's findings depended on the properties of mean-valued sequences. In contrast, as demonstrated in Theorem 1.1, this study develops ISGMs without relying on mean-valued sequences. We explore a broader class of mappings, including nonexpansive mappings as specific instances. As an application, a common split feasibility problem is also presented.

The structure of this article is as follows: Section 2 provides essential preliminary information. In Section 3, we establish an ISGM using the shrinking projection method. Section 4 integrates the Martinez-Yanes and Xu iterative scheme with the shrinking projection method, further extending the ISGM. Section 5 offers a comparison between the present study and previous work [26], highlighting the unique contributions of this research. Section 6 presents iterative schemes derived from the results in Section 3 and 4 to demonstrate the broad applicability of the main findings of this study. Finally, in Section 7, we apply the result of this study to a common split feasibility problem to further support the effectiveness of this study.

2. Preliminaries

This section introduces preliminary concepts and results. Let H represent a real Hilbert space. For $x, y, z \in H$ and $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$, the following holds:

$$\begin{aligned} &\|ax + by + cz\|^2 \\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2; \end{aligned} \tag{2.1}$$

see Maruyama *et al.* [34] and Zegeye and Shahzad [48]. In (2.1), the conditions $a, b, c \in [0, 1]$ are not required. However, if $a, b, c \in [0, 1]$, the following inequality holds:

$$\|ax + by + cz\|^2 \le a\|x\|^2 + b\|y\|^2 + c\|z\|^2. \tag{2.2}$$

Let C be a nonempty, closed, and convex subset of H. We define P_C as the *metric projection* from H onto C, meaning

$$\|x - P_C x\| \le \|x - h\| \quad \text{for all } x \in H \text{ and } h \in C. \tag{2.3}$$

The metric projection P_C is firmly nonexpansive, that is,

$$\|P_C x - P_C y\|^2 \le \langle x - y,\ P_C x - P_C y\rangle \quad \text{for all } x, y \in H. \tag{2.4}$$

Any firmly nonexpansive mapping is nonexpansive. Thus, metric projections are nonexpansive. The metric projection $P_C : H \to C$ satisfies

$$\langle x - P_C x,\ P_C x - h\rangle \ge 0 \quad \text{and} \tag{2.5}$$

$$\|x - P_C x\|^2 + \|P_C x - h\|^2 \le \|x - h\|^2 \tag{2.6}$$

for all $x \in H$ and $h \in C$. Furthermore, note that $F(P_C) = C$.

Let C be a nonempty, closed, and convex subset of H with $x \in H$ and $d \in \mathbb{R}$. Then, a subset D of C defined by

$$D = \{h \in C : 0 \le \langle x,\ h\rangle + d\} \tag{2.7}$$

is also closed and convex; refer to Martinez-Yanes and Xu [33].

A mapping $S : C \to H$ with $F(S) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Sx - q\| \leq \|x - q\| \quad \text{for all } x \in C \text{ and } q \in F(S). \tag{2.8}$$

The set of all fixed points of a quasi-nonexpansive mapping is closed and convex; see Itoh and Takahashi [15]. Any nonexpansive mapping that has a fixed point is quasi-nonexpansive.

Let $\{x_n\}$ be a sequence in H. Denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong and weak convergence to a point x, respectively. Let C be a nonempty, closed, and convex subset of H and let $S : C \to C$ with $F(S) \neq \emptyset$. The mapping $I - S$ is called *demiclosed* if

$$x_n - Sx_n \to 0 \text{ and } x_n \rightharpoonup u \Longrightarrow u \in F(S), \tag{2.9}$$

where $\{x_n\}$ is a sequence in C and I stands for the identity mapping. Note that it is often said that S is demiclosed when (2.9) holds. The class of quasi-nonexpansive mappings with the condition (2.9) includes nonexpansive mappings and a broader categories of mappings; for further details, see Appendix in Kondo [27].

In what follows, we assume the existence of a common fixed point for nonlinear mappings. The following is a simple version of a classical result demonstrated in 1965 by Browder [4] in a certain class of a Banach space:

Theorem 2.1. *Let C be a nonempty, closed, convex, and bounded subset of H. Let $S, T : C \to C$ be nonexpansive mappings such that $ST = TS$. Then, $F(S) \cap F(T)$ is not empty.*

See also Göhde [10] and Kirk [17]. For more recent developments on fixed point and common fixed point theorems, see [8, 9, 12, 18, 21, 31, 39, 40, 47] and the articles cited therein.

3. Takahashi–Takeuchi–Kubota method

This section presents a strong convergence theorem approximating a common fixed point of two nonlinear mappings. As we show in Section 6, this theorem generates many other iterative schemes. We employ the shrinking projection method by Takahashi *et al.* [42].

For that aim, we can relax a required assumption for mappings in comparison to (2.9). Let C be a nonempty, closed, and convex subset of a real Hilbert space H and let $S : C \to C$ with $F(S) \neq \emptyset$. Let $\{z_n\}$ be a sequence in C. Following Kondo [20], consider the following condition:

$$z_n - Sz_n \to 0 \text{ and } z_n \to u \Longrightarrow u \in F(S). \tag{3.1}$$

If the mapping S is continuous or $I - S$ is demiclosed (2.9), then S satisfies the condition (3.1). Therefore, broad classes of mappings, including nonexpansive mappings, satisfy this condition (3.1). In the remainder of this article, we will focus on quasi-nonexpansive mappings (2.8) that satisfy the condition (3.1).

In the main theorems presented below, we assume the following setting:

($\star$) Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S, T : C \to C$ be quasi-nonexpansive mappings (2.8) that satisfy the condition (3.1). Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$.

Then, we can prove the following theorem:

Theorem 3.1. *Assume the setting ($\star$). Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ C_1 &= C, \\ X_n &= a_n y_n + b_n S z_n + c_n T w_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \le \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \le \|x_n - q\|, \quad \|z_n - q\| \le \|x_n - q\|, \quad \|w_n - q\| \le \|x_n - q\| \tag{3.2}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0, \;\; x_n - z_n \to 0, \;\; x_n - w_n \to 0, \tag{3.3}$$

as $n \to \infty$. Then, $\{x_n\}$ converges strongly to an element $\hat{u}$ in $F(S) \cap F(T)$, where $\hat{u} = P_{F(S)\cap F(T)} u$.

Proof. First, we verify the following: (a) C_n is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{X_n\}$ in C, and $\{C_n\}$ are properly defined. Here, we prove (a) through (c) simultaneously by using an induction method.

(i) We start with the case $n = 1$. Given $x_1 \in C_1\,(= C)$, we can select y_1, z_1, and $w_1 \in C$ that satisfy (3.2) and (3.3) for $n = 1$. For instance, if we set $y_1 = z_1 = w_1 = x_1$, then both (3.2) and (3.3) hold. With $x_1, y_1, z_1, w_1 \in C$, X_1 and C_2 are defined as follows:

$$\begin{aligned} X_1 &= a_1 y_1 + b_1 S z_1 + c_1 T w_1 \in C \;\text{ and} \\ C_2 &= \{h \in C_1 : \|X_1 - h\| \le \|x_1 - h\|\}. \end{aligned}$$

As C_1 is closed and convex, C_2 is also closed and convex. Observe that $F(S) \cap F(T) \subset C_2$. Choose $q \in F(S) \cap F(T)\,(\subset C_1)$ arbitrarily. As the mappings S and

T are quasi-nonexpansive (2.8), from (3.2), we have

$$\begin{aligned}\|X_1 - q\| &= \|a_1 y_1 + b_1 S z_1 + c_1 T w_1 - q\| \\ &\le a_1 \|y_1 - q\| + b_1 \|S z_1 - q\| + c_1 \|T w_1 - q\| \\ &\le a_1 \|y_1 - q\| + b_1 \|z_1 - q\| + c_1 \|w_1 - q\| \\ &\le a_1 \|x_1 - q\| + b_1 \|x_1 - q\| + c_1 \|x_1 - q\| \\ &= \|x_1 - q\|.\end{aligned}$$

This indicates that $q \in C_2$. Therefore, $F(S) \cap F(T) \subset C_2$ as asserted. As $F(S) \cap F(T) \neq \emptyset$ is assumed, it follows that $C_2 \neq \emptyset$. Consequently, the metric projection P_{C_2} exists and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given that $x_2 \in C_2$ (with $C_2 \subset C_1 = C$), we can choose y_2, z_2, and $w_2 \in C$ under the conditions provided in (3.2) and (3.3) for $n = 2$. Then, X_2 and C_3 are defined accordingly:

$$\begin{aligned}&X_2 = a_2 y_2 + b_2 S z_2 + c_2 T w_2 \in C \text{ and} \\ &C_3 = \{h \in C_2 : \|X_2 - h\| \le \|x_2 - h\|\}.\end{aligned}$$

By the same reasoning as in case (i), we can confirm that C_3 is closed and convex and that $F(S) \cap F(T) \subset C_3$. As $F(S) \cap F(T) \neq \emptyset$ is supposed, we conclude that $C_3 \neq \emptyset$. Consequently, the metric projection P_{C_3} exists and $x_3 = P_{C_3} u_3$ is defined.

By repeating the same argument, we can establish (a), (b), and (c) as stated.

Define $\overline{u}_n = P_{C_n} u \, (\in C_n)$. The sequence $\{\overline{u}_n\}$ is contained in C, as $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$. As $\overline{u}_n = P_{C_n} u$ and $F(S) \cap F(T) \subset C_n$, it follows from (2.3) that

$$\left\|u - \overline{u}_n\right\| \le \|u - q\| \tag{3.4}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This implies that the sequence $\{\overline{u}_n\}$ is bounded.

From $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, it follows that

$$\left\|u - \overline{u}_n\right\| \le \left\|u - \overline{u}_{n+1}\right\|$$

for all $n \in \mathbb{N}$. In other words, the sequence $\{\|u - \overline{u}_n\|\} (\subset \mathbb{R})$ is monotone increasing. As $\{\overline{u}_n\}$ is bounded, $\{\|u - \overline{u}_n\|\}$ is also bounded. Therefore, the sequence $\{\|u - \overline{u}_n\|\}$ of real numbers converges.

We now show that $\{\overline{u}_n\}$ converges in C, meaning that there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{3.5}$$

Choose $m, n \in \mathbb{N}$ such that $m \ge n$. As the sequence of sets $\{C_n\}$ is shrinking, it follows from $m \ge n$ that $C_m \subset C_n$. Given that $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, we obtain from (2.6) that

$$\left\|u - \overline{u}_n\right\|^2 + \left\|\overline{u}_n - \overline{u}_m\right\|^2 \le \left\|u - \overline{u}_m\right\|^2.$$

As $\{\|u-\overline{u}_n\|\}$ converges, it holds that $\overline{u}_n-\overline{u}_m \to 0$ as $m,n\to\infty$, meaning that $\{\overline{u}_n\}$ is a Cauchy sequence in C. As C is closed in the real Hilbert space H, it is complete. Thus, there exists $\overline{u}\in C$ such that $\overline{u}_n\to\overline{u}$ as claimed.

We now prove that

$$x_n \to \overline{u}. \tag{3.6}$$

As the metric projection P_{C_n} is nonexpansive, it follows from the assumption $u_n\to u$ and (3.5) that

$$\begin{aligned}\|x_n-\overline{u}\| &\le \|x_n-\overline{u}_n\|+\|\overline{u}_n-\overline{u}\|\\ &= \|P_{C_n}u_n-P_{C_n}u\|+\|\overline{u}_n-\overline{u}\|\\ &\le \|u_n-u\|+\|\overline{u}_n-\overline{u}\|\to 0\end{aligned}$$

as claimed. As $\{x_n\}$ converges, it is bounded. Moreover, from (3.3), we obtain

$$z_n\to\overline{u} \text{ and } w_n\to\overline{u}. \tag{3.7}$$

Next, observe that

$$x_n-X_n\to 0. \tag{3.8}$$

Indeed, as $\{x_n\}$ is convergent, it holds that $x_n-x_{n+1}\to 0$ as $n\to\infty$. Given that $x_{n+1}=P_{C_{n+1}}u_{n+1}\in C_{n+1}$, it follows that

$$\|X_n-x_{n+1}\|\le\|x_n-x_{n+1}\|\to 0.$$

Consequently, we have

$$\|x_n-X_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-X_n\|\to 0$$

as claimed. As $\{x_n\}$ is bounded, $\{X_n\}$ is also bounded, according to (3.8).

We show that

$$y_n-Sz_n\to 0 \text{ and } y_n-Tw_n\to 0. \tag{3.9}$$

Select $q\in F(S)\cap F(T)$ arbitrarily. As S and T are quasi-nonexpansive (2.8), from (2.1) and (3.2), the following holds:

$$\begin{aligned}&\|X_n-q\|^2\\ =\;& \|a_n(y_n-q)+b_n(Sz_n-q)+c_n(Tw_n-q)\|^2\\ =\;& a_n\|y_n-q\|^2+b_n\|Sz_n-q\|^2+c_n\|Tw_n-q\|^2\\ &-a_nb_n\|y_n-Sz_n\|^2-b_nc_n\|Sz_n-Tw_n\|^2-c_na_n\|Tw_n-y_n\|^2\\ \le\;& a_n\|y_n-q\|^2+b_n\|z_n-q\|^2+c_n\|w_n-q\|^2\\ &-a_nb_n\|y_n-Sz_n\|^2-b_nc_n\|Sz_n-Tw_n\|^2-c_na_n\|Tw_n-y_n\|^2\\ \le\;& a_n\|x_n-q\|^2+b_n\|x_n-q\|^2+c_n\|x_n-q\|^2\\ &-a_nb_n\|y_n-Sz_n\|^2-b_nc_n\|Sz_n-Tw_n\|^2-c_na_n\|Tw_n-y_n\|^2\\ =\;& \|x_n-q\|^2\\ &-a_nb_n\|y_n-Sz_n\|^2-b_nc_n\|Sz_n-Tw_n\|^2-c_na_n\|Tw_n-y_n\|^2.\end{aligned}$$

As $b_n c_n \|Sz_n - Tw_n\|^2 \geq 0$, we have

$$\begin{aligned} & a_n b_n \|y_n - Sz_n\|^2 + a_n c_n \|y_n - Tw_n\|^2 \\ & \leq \|x_n - q\|^2 - \|X_n - q\|^2 \\ & \leq (\|x_n - q\| + \|X_n - q\|) \left| \|x_n - q\| - \|X_n - q\| \right| \\ & \leq (\|x_n - q\| + \|X_n - q\|) \|x_n - X_n\| . \end{aligned}$$

As both $\{x_n\}$ and $\{X_n\}$ are bounded, and from (3.8), along with the assumptions $\underline{\lim}_{n\to\infty} a_n b_n > 0$ and $\underline{\lim}_{n\to\infty} a_n c_n > 0$, we obtain (3.9) as asserted.

Next, we aim to demonstrate that

$$z_n - Sz_n \to 0 \ \text{ and } \ w_n - Tw_n \to 0. \tag{3.10}$$

Using (3.3) and (3.9), we have

$$\|z_n - Sz_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Sz_n\| \to 0.$$

The second part of (3.10) can be verified in a similar way. As S and T satisfy the condition (3.1), according to (3.7) and (3.10), it holds that $\overline{u} \in F(S) \cap F(T)$.

Finally, we verify that

$$\overline{u} \left(= \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n \right) = \widehat{u} \left(= P_{F(S)\cap F(T)} u \right).$$

As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)} u$, it is sufficient to prove that $\left\| u - \overline{u} \right\| \leq \|u - \widehat{u}\|$. From $\widehat{u} \in F(S) \cap F(T)$ and (3.4), it holds that $\left\| u - \overline{u}_n \right\| \leq \|u - \widehat{u}\|$. From (3.5), we obtain $\left\| u - \overline{u} \right\| \leq \|u - \widehat{u}\|$. Therefore, $\overline{u} = \widehat{u}$. Given (3.6), we can conclude that $x_n \to \widehat{u} \left(= \overline{u} \right)$. This completes the proof. □

For the convergent sequence $\{u_n\} (\subset H)$ in Theorem 3.1, see Yao *et al.* [46] and Hojo *et al.* [11]. Setting $y_n = z_n = w_n = x_n$ in Theorem 3.1 yields the following corollary, which corresponds to Theorem 4.1 in Kondo [20]:

Corollary 3.2 ([20]). *Assume the setting* $(\star)$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given}, \\ C_1 &= C, \\ X_n &= a_n x_n + b_n S x_n + c_n T x_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \leq \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to an element* $\widehat{u}$ *in* $F(S) \cap F(T)$*, where* $\widehat{u} = P_{F(S)\cap F(T)} u$.

4. Martinez–Yanes and Xu method

In this section, we incorporate the method of Martinez–Yanes and Xu [33] into the usual shrinking projection method presented in the previous section (Section 3). The following is the main theorem of this section, where the setting $(\star)$ is given at the beginning of Section 3:

Theorem 4.1. *Assume the setting $(\star)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ C_1 &= C, \\ X_n &= a_n y_n + b_n S z_n + c_n T w_n, \\ C_{n+1} &= \left\{ h \in C_n : \|X_n - h\|^2 \le a_n \|y_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2 \right\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \le \|x_n - q\|, \quad \|z_n - q\| \le \|x_n - q\|, \quad \|w_n - q\| \le \|x_n - q\| \tag{4.1}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0, \quad x_n - z_n \to 0, \quad x_n - w_n \to 0 \tag{4.2}$$

as $n \to \infty$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u}$ in $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

Remark 4.2. *See the definition of C_{n+1}. It follows that*

$$\|X_n - h\|^2 \le a_n \|y_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2$$

$$\begin{aligned} \Leftrightarrow \quad & 0 \le a_n \|y_n\|^2 + b_n \|z_n\|^2 + c_n \|w_n\|^2 - \|X_n\|^2 \\ & -2 \langle a y_n + b z_n + c w_n - X_n,\ h \rangle \end{aligned} \tag{4.3}$$

$$\begin{aligned} \Leftrightarrow \quad & \|X_n - h\|^2 \le \|y_n - h\|^2 + b_n \left(\|z_n\|^2 - \|y_n\|^2 - 2 \langle z_n - y_n,\ h \rangle \right) \\ & + c_n \left(\|w_n\|^2 - \|y_n\|^2 - 2 \langle w_n - y_n,\ h \rangle \right). \end{aligned} \tag{4.4}$$

From (4.4), Theorem 4.1 corresponds to the Martinez–Yanes and Xu type; see Theorem 1.2 in Section 1. Suppose that $X_n, y_n, z_n, w_n \in C$ and $a_n, b_n, c_n \in \mathbb{R}$ are given. From (2.7) and (4.3), the set C_{n+1} is closed and convex if C_n is closed and convex.

Proof. At the outset, observe that (a) C_n is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{X_n\}$ $(\subset C)$, and $\{C_n\}$ are defined properly. We demonstrate (a)–(c) by induction.

(i) Given $x_1 \in C_1 (= C)$, we can select y_1, z_1, and $w_1 \in C$ to satisfy (4.1) and (4.2) for $n = 1$. For instance, by letting $y_1 = z_1 = w_1 = x_1$, those conditions are fulfilled. With $x_1, y_1, z_1, w_1 \in C$, X_1 and C_2 are defined as follows:

$$X_1 = a_1 y_1 + b_1 S z_1 + c_1 T w_1 \in C \text{ and}$$
$$C_2 = \left\{ h \in C_1 : \|X_1 - h\|^2 \leq a_1 \|y_1 - h\|^2 + b_1 \|z_1 - h\|^2 + c_1 \|w_1 - h\|^2 \right\}.$$

From (2.7) and (4.3), C_2 is closed and convex as C_1 $(= C)$ is closed and convex. We verify that $F(S) \cap F(T) \subset C_2$. Let $q \in F(S) \cap F(T)$ $(\subset C_1)$. As S and T are quasi-nonexpansive (2.8), from (2.2), it follows that

$$\begin{aligned} \|X_1 - q\|^2 &= \|a_1 y_1 + b_1 S z_1 + c_1 T w_1 - q\|^2 \\ &= \|a_1 (y_1 - q) + b_1 (S z_1 - q) + c_1 (T w_1 - q)\|^2 \\ &\leq a_1 \|y_1 - q\|^2 + b_1 \|S z_1 - q\|^2 + c_1 \|T w_1 - q\|^2 \\ &\leq a_1 \|y_1 - q\|^2 + b_1 \|z_1 - q\|^2 + c_1 \|w_1 - q\|^2, \end{aligned}$$

which implies that $q \in C_2$. Therefore, $F(S) \cap F(T) \subset C_2$ as asserted. Given the assumption that $F(S) \cap F(T) \neq \emptyset$, C_2 is nonempty. Thus, the metric projection P_{C_2} exists and $x_2 = P_{C_2} u_2$ is defined.

(ii) Given $x_2 \in C_2$ $(\subset C_1 = C)$, we can select y_2, z_2, and $w_2 \in C$ such that (4.1) and (4.2) are satisfied for $n = 2$. With these elements, X_2 and C_3 are defined as follows:

$$X_2 = a_2 y_2 + b_2 S z_2 + c_2 T w_2 \in C \text{ and}$$
$$C_3 = \left\{ h \in C_2 : \|X_2 - h\|^2 \leq a_2 \|y_2 - h\|^2 + b_2 \|z_2 - h\|^2 + c_2 \|w_2 - h\|^2 \right\}.$$

Using the same argument as in case (i), we can demonstrate that C_3 is closed and convex and that $F(S) \cap F(T) \subset C_3$. From the assumption $F(S) \cap F(T) \neq \emptyset$, we conclude that $C_3 \neq \emptyset$. Consequently, the metric projection P_{C_3} exists and $x_3 = P_{C_3} u_3$ is defined.

Repeating the same analysis guarantees that (a), (b), and (c) are true.

Define $\overline{u}_n = P_{C_n} u \in C_n$. As $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$, $\{\overline{u}_n\}$ is a sequence contained in C. We claim that

$$\left\| u - \overline{u}_n \right\| \leq \|u - q\| \tag{4.5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This follows from the definition $\overline{u}_n = P_{C_n} u$ and the fact that $q \in F(S) \cap F(T) \subset C_n$; see (2.3). From (4.5), we can conclude that $\{\overline{u}_n\}$ is bounded.

Note that

$$\left\| u - \overline{u}_n \right\| \leq \left\| u - \overline{u}_{n+1} \right\| \tag{4.6}$$

for all $n \in \mathbb{N}$. Indeed, as $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, the inequality (4.6) follows. This implies that $\left\{ \left\| u - \overline{u}_n \right\| \right\}$ is monotone increasing. As $\left\{ \left\| u - \overline{u}_n \right\| \right\}$ is bounded, it is convergent.

We now prove that $\{\overline{u}_n\}$ converges in C; that is, there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{4.7}$$

Let $m, n \in \mathbb{N}$ with $m \geq n$. As $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, from (2.6), it holds that

$$\left\|u - \overline{u}_n\right\|^2 + \left\|\overline{u}_n - \overline{u}_m\right\|^2 \leq \left\|u - \overline{u}_m\right\|^2.$$

As $\left\{\left\|u - \overline{u}_n\right\|\right\}$ is convergent, it follows that $\overline{u}_n - \overline{u}_m \to 0$ as $m, n \to \infty$. Thus, $\left\{\overline{u}_n\right\}$ is a Cauchy sequence in C. As C is complete, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ as claimed.

Our next aim is to demonstrate that $\{x_n\}$ has the same limit point, that is,

$$x_n \to \overline{u}. \tag{4.8}$$

As the metric projection is nonexpansive, using (4.7) and the hypothesis $u_n \to u$, we obtain

$$\begin{aligned}\left\|x_n - \overline{u}\right\| &\leq \left\|x_n - \overline{u}_n\right\| + \left\|\overline{u}_n - \overline{u}\right\| \\ &= \left\|P_{C_n} u_n - P_{C_n} u\right\| + \left\|\overline{u}_n - \overline{u}\right\| \\ &\leq \|u_n - u\| + \left\|\overline{u}_n - \overline{u}\right\| \to 0\end{aligned}$$

as n tends to infinity. This shows that (4.8) holds true as claimed. Consequently, $\{x_n\}$ is bounded. From (4.1), $\{y_n\}, \{z_n\}$, and $\{w_n\}$ are also bounded. Furthermore, according to (4.2) and (4.8), we have

$$z_n \to \overline{u} \text{ and } w_n \to \overline{u}. \tag{4.9}$$

As $\{x_n\}$ converges, it holds that

$$x_n - x_{n+1} \to 0. \tag{4.10}$$

Next, let us show that

$$X_n - x_{n+1} \to 0. \tag{4.11}$$

As $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, it follows from the definition of C_{n+1} that

$$\begin{aligned} &\|X_n - x_{n+1}\|^2 \qquad (4.12)\\ \leq\ & a_n \|y_n - x_{n+1}\|^2 + b_n \|z_n - x_{n+1}\|^2 + c_n \|w_n - x_{n+1}\|^2 \\ \leq\ & a_n (\|y_n - x_n\| + \|x_n - x_{n+1}\|)^2 + b_n (\|z_n - x_n\| + \|x_n - x_{n+1}\|)^2 \\ & + c_n (\|w_n - x_n\| + \|x_n - x_{n+1}\|)^2 .\end{aligned}$$

From (4.2) and (4.10), we can conclude that $X_n - x_{n+1} \to 0$ as stated. From (4.10) and (4.11), we have $x_n - X_n \to 0$. As $\{x_n\}$ is bounded, $\{X_n\}$ is also bounded.

Observe that

$$y_n - Sz_n \to 0 \text{ and } y_n - Tw_n \to 0. \tag{4.13}$$

Choose $q \in F(S) \cap F(T)$ arbitrarily. Using (2.1), (2.8), and (4.1) yields

$$\begin{aligned}
& \|X_n - q\|^2 \\
= \ & \|a_n (y_n - q) + b_n (Sz_n - q) + c_n (Tw_n - q)\|^2 \\
= \ & a_n \|y_n - q\|^2 + b_n \|Sz_n - q\|^2 + c_n \|Tw_n - q\|^2 \\
& -a_n b_n \|y_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 \\
\leq \ & a_n \|y_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\
& -a_n b_n \|y_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 \\
\leq \ & a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\
& -a_n b_n \|xy_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 \\
= \ & \|x_n - q\|^2 \\
& -a_n b_n \|y_n - Sz_n\|^2 - b_n c_n \|Sz_n - Tw_n\|^2 - c_n a_n \|Tw_n - y_n\|^2 .
\end{aligned}$$

As $b_n c_n \|Sz_n - Tw_n\|^2 \geq 0$, we have

$$\begin{aligned}
& a_n b_n \|y_n - Sz_n\|^2 + a_n c_n \|y_n - Tw_n\|^2 \\
& \leq \|x_n - q\|^2 - \|X_n - q\|^2 \\
& \leq (\|x_n - q\| + \|X_n - q\|) \left| \|x_n - q\| - \|X_n - q\| \right| \\
& \leq (\|x_n - q\| + \|X_n - q\|) \|x_n - X_n\| .
\end{aligned}$$

Recall that $\{x_n\}$ and $\{X_n\}$ are bounded and $x_n - X_n \to 0$. Thus, we obtain $y_n - Sz_n \to 0$ and $y_n - Tw_n \to 0$ as asserted.

From (4.2) and (4.13), it follows that

$$z_n - Sz_n \to 0 \text{ and } w_n - Tw_n \to 0. \tag{4.14}$$

As S and T satisfy the condition (3.1), from (4.9) and (4.14), we obtain $\overline{u} \in F(S) \cap F(T)$.

Our objective is to demonstrate that $x_n \to \widehat{u}$. From (4.8), it is sufficient to show that

$$\overline{u} \left(= \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n \right) = \widehat{u} \left(= P_{F(S)\cap F(T)} u \right).$$

Applying (4.5) for $q = \widehat{u} \in F(S) \cap F(T)$, we have $\left\| u - \overline{u}_n \right\| \leq \|u - \widehat{u}\|$ for all $n \in \mathbb{N}$. From (4.7), it holds that $\left\| u - \overline{u} \right\| \leq \|u - \widehat{u}\|$. As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)} u$, we obtain $\overline{u} = \widehat{u}$. This concludes the proof. □

Setting $y_n = z_n = w_n = x_n$ in Theorem 4.1, we again obtain Corollary 3.2.

5. Remarks

This section provides brief notes regarding the main theorems of this study in comparison with previous results. Let $S : C \to C$ with $F(S) \neq \emptyset$, where C is a nonempty, closed, and convex subset of a real Hilbert space H. For a bounded sequence $\{z_n\}$ in C, define $Z_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k z_n\,(\in C)$. Kondo [19] called a mapping $S : C \to C$ *mean-demiclosed* if

$$Z_{n_j} \rightharpoonup u \text{ (weak convergence)} \implies u \in F(S), \tag{5.1}$$

where $\left\{Z_{n_j}\right\}$ is a subsequence of $\{Z_n\}$. According to Kondo and Takahashi [29], a nonexpansive mapping is mean-demiclosed; see also Claim 1 in Kondo [24] or Proposition 2.1 in Kondo [26]. Furthermore, consider the following condition:

$$Z_{n_j} \to u \text{ (strong convergence)} \implies u \in F(S). \tag{5.2}$$

A mean-demiclosed mapping (5.1) satisfies the condition (5.2) and therefore, broad classes of mappings, including nonexpansive mappings, satisfy this condition (5.2); see Appendix in Kondo [24]. Consider the following setting:

($\star\star$) Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $S, T : C \to C$ be quasi-nonexpansive mappings (2.8) that satisfy the condition (5.2). Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\varliminf_{n\to\infty} a_n b_n > 0$, and $\varliminf_{n\to\infty} a_n c_n > 0$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$.

The only difference between the settings ($\star$) and ($\star\star$) is with regard to the mapping conditions (3.1) and (5.2), where the setting ($\star$) is stated at the beginning of Section 3. The following two theorems are contained in Kondo [26]:

Theorem 5.1 ([26]). *Assume the setting ($\star\star$). Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned}
x_1 &= x \in C \text{ is given,}\\
C_1 &= C,\\
X_n &= a_n y_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n,\\
C_{n+1} &= \{h \in C_n : \|X_n - h\| \le \|x_n - h\|\},\\
x_{n+1} &= P_{C_{n+1}} u_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$, where $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are sequences in C that satisfy the following conditions:

$$\|y_n - q\| \le \|x_n - q\|, \quad \|z_n - q\| \le \|x_n - q\|, \quad \|w_n - q\| \le \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$ and

$$x_n - y_n \to 0 \tag{5.3}$$

as $n \to \infty$. Then, $\{x_n\}$ converges strongly to an element $\widehat{u} \in F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

Theorem 5.2 ([26]). *Assume the setting* ($\star\star$). *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given}, \\ C_1 &= C, \\ X_n &= a_n y_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n, \\ C_{n+1} &= \left\{h \in C_n : \|X_n - h\|^2 \le a_n \|y_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2\right\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$, *where* $\{y_n\}$, $\{z_n\}$, *and* $\{w_n\}$ *are sequences in* C *that satisfy the following conditions:*

$$\|y_n - q\| \le \|x_n - q\|, \quad \|z_n - q\| \le \|x_n - q\|, \quad \|w_n - q\| \le \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ *and* $n \in \mathbb{N}$ *and*

$$x_n - y_n \to 0, \;\; x_n - z_n \to 0, \;\; x_n - w_n \to 0 \tag{5.4}$$

as $n \to \infty$. *Then,* $\{x_n\}$ *converges strongly to an element* $\hat{u} \in F(S) \cap F(T)$, *where* $\hat{u} = P_{F(S)\cap F(T)} u$.

First, we compare Theorem 3.1 with 5.1. As can be seen in (3.3) and (5.3), Theorem 3.1 requires additional assumptions $x_n - z_n \to 0$ and $x_n - w_n \to 0$, although it can be established without relying on mean-valued sequences. Furthermore, the conditions for mappings S and T in Theorem 3.1 differ from those in Theorem 5.1. For quasi-nonexpansive mappings with (3.1), see Appendix in Kondo [27] and for quasi-nonexpansive mappings with (5.2), see Appendix in Kondo [24].

Next, we compare Theorem 4.1 with 5.2. In these two theorems, the required conditions on the sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ are the same as those in Theorem 3.1. For this point, see Remark 5.2 in Kondo [26]. In other words, Theorem 4.1 can be proved without using mean-valued sequences and without any additional conditions on the sequences $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$. However, as in the cases of Theorems 3.1 and 5.1, the conditions on the mappings are different.

6. Corollaries

In this section, we provide strong convergence results deduced from Theorems 3.1 and 4.1 to demonstrate the effectiveness of the main theorems of this study. Recall that the setting ($\star$) is described at the beginning of Section 3. The following is a four-step iterative method to approximate a common fixed point of quasi-nonexpansive mappings with the condition (3.1):

Corollary 6.1. *Assume the setting* ($\star$). *Let* $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\lambda'_n\}$, $\{\mu'_n\}$, $\{\nu'_n\}$, $\{\lambda''_n\}$, $\{\mu''_n\}$, *and* $\{\nu''_n\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $\lambda_n +$

$\mu_n + \nu_n = 1$, $\lambda'_n + \mu'_n + \nu'_n = 1$, $\lambda''_n + \mu''_n + \nu''_n = 1$ for all $n \in \mathbb{N}$, $\lambda_n \to 1$, $\lambda'_n \to 1$, and $\lambda''_n \to 1$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} x_1 &= x \in C \text{ is given}, \\ C_1 &= C, \\ w_n &= \lambda''_n x_n + \mu''_n S x_n + \nu''_n T x_n, \\ z_n &= \lambda'_n w_n + \mu'_n S w_n + \nu'_n T w_n, \\ y_n &= \lambda_n z_n + \mu_n S z_n + \nu_n T z_n, \\ X_n &= a_n y_n + b_n S y_n + c_n T y_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \le \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned} \tag{6.1}$$

for all $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ converges strongly to an element $\widehat{u} \in F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

Proof. From Theorem 3.1, it is sufficient to demonstrate that

$$\|y_n - q\| \le \|x_n - q\| \quad \text{for all } q \in F(S) \cap F(T) \text{ and } n \in \mathbb{N}, \tag{6.2}$$

$$x_n - y_n \to 0 \text{ as } n \to \infty. \tag{6.3}$$

Before that, we shall verify that (a) C_n is closed and convex, (b) $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$, and (c) the sequences $\{x_n\}, \{w_n\}, \{z_n\}, \{y_n\}, \{X_n\}$, and $\{C_n\}$ are properly defined. These parts can be shown in a similar manner to the proof of Theorem 3.1 and thus, we omit them here.

Observe that

$$\|w_n - q\| \le \|x_n - q\| \quad \text{for all } q \in F(S) \cap F(T) \text{ and } n \in \mathbb{N}. \tag{6.4}$$

As S and T are quasi-nonexpansive (2.8), we can verify (6.4) as follows:

$$\begin{aligned} \|w_n - q\| &= \|\lambda''_n x_n + \mu''_n S x_n + \nu''_n T x_n - q\| \\ &= \|\lambda''_n (x_n - q) + \mu''_n (S x_n - q) + \nu''_n (T x_n - q)\| \\ &\le \lambda''_n \|x_n - q\| + \mu''_n \|S x_n - q\| + \nu''_n \|T x_n - q\| \\ &\le \lambda''_n \|x_n - q\| + \mu''_n \|x_n - q\| + \nu''_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Similarly, we can prove

$$\|z_n - q\| \le \|w_n - q\| \quad \text{and} \quad \|y_n - q\| \le \|z_n - q\| \tag{6.5}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. From (6.4) and (6.5), we obtain (6.2).

Define $\overline{u}_n = P_{C_n} u \in C_n$. In a similar manner to the proof of Theorem 3.1, we can show that there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ and $x_n \to \overline{u}$. As $\{x_n\}$ is convergent, it is bounded. Moreover, as S and T are quasi-nonexpansive, $\{S x_n\}$ and $\{T x_n\}$ are also bounded. In fact, for $q \in F(S) \cap F(T)$, it holds that

$$\begin{aligned} \|S x_n\| &\le \|S x_n - q\| + \|q\| \\ &\le \|x_n - q\| + \|q\| \end{aligned}$$

for all $n \in \mathbb{N}$. As $\{x_n\}$ is bounded, $\{Sx_n\}$ is also bounded. Similarly, we can verify that $\{Tx_n\}$ is also bounded as claimed.

As $\{x_n\}$ is bounded, from (6.4) and (6.5), $\{w_n\}, \{z_n\}$, and $\{y_n\}$ are also bounded. Consequently, $\{Sw_n\}, \{Tw_n\}, \cdots$ are also bounded.

We show that $x_n - w_n \to 0$. As $\lambda_n'' \to 1$, it follows that $\mu_n'' \to 0$ and $\nu_n'' \to 0$. Thus, we have

$$\begin{aligned}\|x_n - w_n\| &= \left\|x_n - \left(\lambda_n'' x_n + \mu_n'' S x_n + \nu_n'' T x_n\right)\right\| \\ &= \left\|\left(1-\lambda_n''\right) x_n - \mu_n'' S x_n - \nu_n'' T x_n\right\| \\ &\leq \left(1-\lambda_n''\right)\|x_n\| + \mu_n'' \|Sx_n\| + \nu_n'' \|Tx_n\| \to 0\end{aligned}$$

as asserted. Similarly, as $\lambda_n' \to 1$, we have $w_n - z_n \to 0$. Furthermore, as $\lambda_n \to 1$, it follows that $z_n - y_n \to 0$. Therefore, we obtain (6.3). From the above, the desired result follows from Theorem 3.1. □

The iterative scheme (6.1) is a four-step type. Setting $\lambda_n'' = 1$ in (6.1) derives a three-step iterative method and setting $\lambda_n'' = \lambda_n' = 1$ yields a two-step type. When $\lambda_n'' = \lambda_n' = \lambda_n = 1$, Corollary 6.1 coincides with Corollary 3.2. Setting $\lambda_n'' = 1$ and $\mu_n' = \nu_n = 0$, we obtain the (1.3)-type three-step iterative scheme:

$$\begin{aligned} z_n &= \lambda_n' x_n + \left(1-\lambda_n'\right) T x_n, \\ y_n &= \lambda_n z_n + (1-\lambda_n) S z_n, \\ X_n &= a_n y_n + b_n S y_n + c_n T y_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \leq \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1}, \end{aligned} \tag{6.6}$$

where $x_1 = x \in C$ is given and $C_1 = C$. Furthermore, the iterative scheme (6.1) can be replaced by

$$\begin{aligned} y_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n T^2 x_n, \\ X_n &= a_n y_n + b_n S y_n + c_n T y_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \leq \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1}, \end{aligned} \tag{6.7}$$

where $x_1 = x \in C$ is given and $C_1 = C$. In (6.7), the parameters $\lambda_n, \mu_n, \nu_n, \xi_n \in [0,1]$ are required to satisfy $\lambda_n + \mu_n + \nu_n + \xi_n = 1$ and $\lambda_n \to 1$. This type of iterative scheme, which includes the term $T^2 x_n$, was utilized by Maruyama *et al.* [34] to address more general class of mappings than nonexpansive mappings; see also Kondo and Takahashi [30], Kondo [20], and the articles cited therein. Hence, it is effective for nonexpansive mappings.

We also obtain the following result from Theorem 4.1:

Corollary 6.2. *Assume the setting* $(\star)$. *Let* $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\lambda_n'\}$, $\{\mu_n'\}$, *and* $\{\nu_n'\}$ *be sequences of real numbers in the interval* $[0,1]$ *such that* $\lambda_n + \mu_n + \nu_n = 1$, $\lambda_n' + \mu_n' + \nu_n' = 1$ *for all* $n \in \mathbb{N}$, $\lambda_n \to 1$ *and* $\lambda_n' \to 1$. *Define a sequence* $\{x_n\}$ *in* C

as follows:

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ C_1 &= C, \\ z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n, \\ w_n &= \lambda'_n x_n + \mu'_n S x_n + \nu'_n T x_n, \\ X_n &= a_n x_n + b_n S z_n + c_n T w_n, \\ C_{n+1} &= \left\{ h \in C_n : \|X_n - h\|^2 \le a_n \|x_n - h\|^2 + b_n \|z_n - h\|^2 + c_n \|w_n - h\|^2 \right\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. *Then,* $\{x_n\}$ *converges strongly to an element* $\widehat{u}$ *in* $F(S) \cap F(T)$*, where* $\widehat{u} = P_{F(S)\cap F(T)} u$.

The proof is omitted here. To the author's best knowledge, even Corollary 6.2 is a new result in the literature. Apart from the iterative methods explicitly discussed in this section, Theorems 3.1 and 4.1 give rise to an infinite number of iterative methods for finding common fixed points of nonlinear mappings; see also Kondo [22, 24, 25, 26, 27, 28].

7. Application

In this section, we show how to apply the results established in this article to split feasibility problems (SFPs). For SFPs, see Censor and Elfving [6] and Takahashi [41]. A common solution of two SFPs is strongly approximated by applying a three-step version of Corollary 6.1.

We start with an explanation of a SFP. Let H and H' be real Hilbert spaces and let $C\,(\subset H)$ and $Q\,(\subset H')$ be nonempty, closed, and convex subsets of H and H', respectively. Let $A : H \to H'$ be a linear and continuous mapping and let $A^* : H' \to H$ be the adjoint operator of A. The SFP is as follows:

$$\text{(SFP)} \quad \text{Find an element } \widehat{x} \in C \cap A^{-1} Q.$$

The next classes of mappings are frequently used in the literature:

Definition 7.1. *Let* C *be a nonempty subset of a real Hilbert space* H.

(a) *A mapping* $S : C \to H$ *is called* K*-Lipschitz continuous if there exists* $K > 0$ *such that* $\|Sx - Sy\| \le K \|x - y\|$ *for all* $x, y \in C$.

(b) *A mapping* $S : C \to H$ *is called monotone if* $0 \le \langle x - y,\ Sx - Sy\rangle$ *for all* $x, y \in C$.

(c) *A mapping* $S : C \to H$ *is called* α*-inverse strongly monotone if there exists* $\alpha > 0$ *such that*

$$\alpha \|Sx - Sy\|^2 \le \langle x - y,\ Sx - Sy\rangle \quad \text{for all } x, y \in C.$$

Clearly, a mapping is 1-Lipschitz continuous iff it is nonexpansive. For inverse strongly monotone mappings, refer to Browder and Petryshyn [5] and Liu and Nashed [32]. A mapping is 1-inverse strongly monotone iff it is firmly nonexpansive. The metric projection is firmly nonexpansive; see (2.4) in Section

2. Therefore, the metric projection is 1-inverse strongly monotone. It can be demonstrated that an α-inverse strongly monotone mapping is $(1/\alpha)$-Lipschitz continuous and monotone.

To apply fixed point theory to SFPs, the following lemma is crucial:

Lemma 7.2. *Let H and H' be real Hilbert spaces and let $C\ (\subset H)$ and $Q\ (\subset H')$ be nonempty, closed, and convex subsets of H and H', respectively. Let $A : H \to H'$ be a linear and continuous mapping and let $A^* : H' \to H$ be the adjoint operator of A. Assume that $C \cap A^{-1}Q \neq \emptyset$. Then,*

$$C \cap A^{-1}Q = F\left(P_C\left(I - \eta A^*\left(I - P_Q\right)A\right)\right)$$

for all $\eta > 0$.

Proof. First, we verify the inclusion $(\subset)$. Let $x \in C \cap A^{-1}Q$, that is, $x \in C$ and $Ax \in Q$. From $x \in C$, it holds that $P_C x = x$. Furthermore, as $Ax \in Q$, we have that $P_Q Ax = Ax$. Therefore,

$$\begin{aligned} P_C\left(I - \eta A^*\left(I - P_Q\right)A\right)x &= P_C\left(x - \eta A^*\left(Ax - P_Q Ax\right)\right) \\ &= P_C\left(x - \eta A^* 0\right) = P_C x = x. \end{aligned}$$

This indicates that $x \in F\left(P_C\left(I - \eta A^*\left(I - P_Q\right)A\right)\right)$.

Next, we show the part $(\supset)$. Let $\eta > 0$ and $x \in F\left(P_C\left(I - \eta A^*\left(I - P_Q\right)A\right)\right)$. Then,

$$x = P_C\left(x - \eta A^*\left(Ax - P_Q Ax\right)\right). \tag{7.1}$$

This implies that $x \in C$. Our aim is to prove that $Ax \in Q$. From (2.5) and (7.1),

$$\left\langle \left(x - \eta A^*\left(Ax - P_Q Ax\right)\right) - x,\ x - h\right\rangle \geq 0 \ \text{ for all } h \in C,$$

which results in

$$\left\langle -\eta A^*\left(Ax - P_Q Ax\right),\ x - h\right\rangle \geq 0 \ \text{ for all } h \in C.$$

As $\eta > 0$,

$$\left\langle A^*\left(Ax - P_Q Ax\right),\ h - x\right\rangle \geq 0 \ \text{ for all } h \in C.$$

As A is linear and A^* is the adjoint operator of A,

$$\left\langle Ax - P_Q Ax,\ Ah - Ax\right\rangle \geq 0 \ \text{ for all } h \in C. \tag{7.2}$$

As $C \cap A^{-1}Q \neq \emptyset$ is assumed, we can choose $x_0 \in C \cap A^{-1}Q$. This means that $x_0 \in C$ and $Ax_0 \in Q$. As $x_0 \in C$, substituting $h = x_0$ in (7.2), we obtain

$$\left\langle Ax - P_Q Ax,\ Ax_0 - Ax\right\rangle \geq 0. \tag{7.3}$$

On the other hand, from $Ax_0 \in Q$ and (2.5),

$$\left\langle Ax - P_Q Ax,\ P_Q Ax - Ax_0\right\rangle \geq 0. \tag{7.4}$$

Summing (7.3) and (7.4) yields

$$\left\langle Ax - P_Q Ax,\ P_Q Ax - Ax\right\rangle \geq 0,$$

which implies that $Ax = P_Q Ax$. Thus, $Ax \in Q$. This completes the proof. □

We aim to demonstrate that the mapping $P_C\left(I-\eta A^*\left(I-P_Q\right)A\right)$ becomes nonexpansive if η is sufficiently close to 0 (Lemma 7.6). For that aim, we will prove the following:

Sublemma 7.3. *Let C be a nonempty subset of H and let $V : C \to H$. Then, V is firmly nonexpansive if and only if $I-V$ is firmly nonexpansive.*

Proof. Let $x, y \in C$. It follows that

$$\begin{aligned}
& I-V \text{ is firmly nonexpansive.} \\
\Leftrightarrow\ & \|(I-V)(x)-(I-V)(y)\|^2 \le \langle x-y,\ (I-V)(x)-(I-V)(y)\rangle \\
\Leftrightarrow\ & \|x-y-(Vx-Vy)\|^2 \le \langle x-y,\ x-y-(Vx-Vy)\rangle \\
\Leftrightarrow\ & \begin{array}{l} \|x-y\|^2-2\langle x-y,\ Vx-Vy\rangle+\|Vx-Vy\|^2 \\ \qquad \le \|x-y\|^2-\langle x-y,\ Vx-Vy\rangle \end{array} \\
\Leftrightarrow\ & \|Vx-Vy\|^2 \le \langle x-y,\ Vx-Vy\rangle \\
\Leftrightarrow\ & V \text{ is firmly nonexpansive.}
\end{aligned}$$

This concludes the proof. □

Furthermore, we have the following result:

Sublemma 7.4. *Let $A : H \to H'$ be a linear and continuous mapping and let $A^* : H' \to H$ be the adjoint operator of A, where H and H' are real Hilbert spaces. Let $G : H' \to H'$ be an α-inverse strongly monotone mapping, where $\alpha > 0$. Then, A^*GA is an $\frac{\alpha}{\|AA^*\|}$-inverse strongly monotone, where $\|AA^*\|$ is an operator norm of AA^*.*

Proof. Our goal is to demonstrate that

$$\frac{\alpha}{\|AA^*\|}\|A^*GAx-A^*GAy\|^2 \le \langle x-y,\ A^*GAx-A^*GAy\rangle$$

for all $x, y \in C$. We can show this as follows:

$$\begin{aligned}
LHS &= \frac{\alpha}{\|AA^*\|}\langle A^*GAx-A^*GAy,\ A^*GAx-A^*GAy\rangle \\
&= \frac{\alpha}{\|AA^*\|}\langle AA^*GAx-AA^*GAy,\ GAx-GAy\rangle \\
&\le \frac{\alpha}{\|AA^*\|}\|AA^*GAx-AA^*GAy\|\,\|GAx-GAy\| \\
&\le \frac{\alpha}{\|AA^*\|}\|AA^*\|\,\|GAx-GAy\|^2 \\
&= \alpha\|GAx-GAy\|^2 \\
&\le \langle Ax-Ay,\ GAx-GAy\rangle \\
&= \langle x-y,\ A^*GAx-A^*GAy\rangle
\end{aligned}$$

as G is α-inverse strongly monotone. This completes the proof. □

We know the following Sublemma:

Sublemma 7.5. *Let A' be an α-inverse strongly monotone mapping from C into H, where C is a nonempty subset of H and $\alpha > 0$. Then, $I - \eta A'$ is a nonexpansive mapping from C into H if $\eta \in [0, 2\alpha]$.*

Proof. See page 419 in Toyoda and Takahashi [43] or Proposition 5.2 in Kondo [27]. □

Based on Sublemmas 7.3–7.5, we obtain the following:

Lemma 7.6. *Let H and H' be real Hilbert spaces. Let $A : H \to H'$ be a linear and continuous mapping and let $A^* : H' \to H$ be the adjoint operator of A. Let C be a nonempty subset of H and $U : H \to C$ be a K-Lipschitz continuous mapping. Let $V : H' \to H'$ be a firmly nonexpansive mapping. Then, the mapping $U(I - \eta A^*(I - V)A) : H \to C$ is K-Lipschitz continuous if $\eta \in \left[0, \frac{2}{\|AA^*\|}\right]$. In particular, if $U : H \to C$ is nonexpansive, the mapping $U(I - \eta A^*(I - V)A) : H \to C$ is also nonexpansive under the assumption $\eta \in \left[0, \frac{2}{\|AA^*\|}\right]$.*

Proof. Select $\eta \in \left[0, \frac{2}{\|AA^*\|}\right]$ arbitrarily. As V is firmly nonexpansive, from Sublemma 7.3, $I - V$ is also firmly nonexpansive. A firmly nonexpansive mapping is 1-inverse strongly monotone. Thus, from Sublemma 7.4, the self-mapping $A^*(I - V)A$ defined on H is $\frac{1}{\|AA^*\|}$-inverse strongly monotone.

Since $\eta \in \left[0, \frac{2}{\|AA^*\|}\right]$, from Sublemma 7.5, $I - \eta A^*(I - V)A$ is a nonexpansive mapping from H into itself. As $U : H \to C$ is K-Lipschitz continuous, $U(I - \eta A^*(I - V)A) : H \to C$ is also K-Lipschitz continuous. In particular, suppose that $U : H \to C$ is nonexpansive, which means that it is 1-Lipschitz continuous. Then, $U(I - \eta A^*(I - V)A) : H \to C$ also becomes nonexpansive under the same setting. This concludes the proof. □

Using these lemmas and Corollary 6.1 with (6.6), we obtain the following theorem:

Theorem 7.7. *Let H, H_1, and H_2 be real Hilbert spaces. Let $C\,(\subset H), Q_1\,(\subset H_1)$, and $Q_2\,(\subset H_2)$ be nonempty, closed, and convex subsets. Let $P_C : H \to C$, $P_{Q_1} : H_1 \to Q_1$, and $P_{Q_2} : H_2 \to Q_2$ be the metric projections. Let $A_1 : H \to H_1$ and $A_2 : H \to H_2$ be linear continuous mappings and let A_1^* and A_2^* be the adjoint operators of A_1 and A_2, respectively. Assume that*

$$\Omega \equiv C \cap A_1^{-1}Q_1 \cap A_2^{-1}Q_2 \neq \emptyset.$$

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$, $\underline{\lim}_{n\to\infty} a_n b_n > 0$, and $\underline{\lim}_{n\to\infty} a_n c_n > 0$. Let $\{\lambda_n\}$ and $\{\lambda'_n\}$ be sequences of real numbers in the interval $[0, 1]$ such that $\lambda_n \to 1$ and

$\lambda'_n \to 1$. *Let* $\eta_1 \in \left(0, \frac{2}{\|A_1 A_1^*\|}\right]$ *and* $\eta_2 \in \left(0, \frac{2}{\|A_2 A_2^*\|}\right]$. *Let* $\{u_n\}$ *be a sequence in* H *such that* $u_n \to u\ (\in H)$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given}, \\ C_1 &= C, \\ z_n &= \lambda'_n x_n + (1-\lambda'_n) P_C \left(I - \eta_2 A_2^* \left(I - P_{Q_2}\right) A_2\right) x_n, \\ y_n &= \lambda_n z_n + (1-\lambda_n) P_C \left(I - \eta_1 A_1^* \left(I - P_{Q_1}\right) A_1\right) z_n, \\ X_n &= a_n y_n + b_n P_C \left(I - \eta_1 A_1^* \left(I - P_{Q_1}\right) A_1\right) y_n \\ &\quad + c_n P_C \left(I - \eta_2 A_2^* \left(I - P_{Q_2}\right) A_2\right) y_n, \\ C_{n+1} &= \{h \in C_n : \|X_n - h\| \le \|x_n - h\|\}, \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. *Then, the sequence* $\{x_n\}$ *converges strongly to an element* $\hat{u} \in \Omega$, *where* $\hat{u} = P_\Omega u$.

Proof. As the metric projections are firmly nonexpansive, they are nonexpansive. Furthermore, as $\eta_1 \in \left(0, \frac{2}{\|A_1 A_1^*\|}\right] \subset \left[0, \frac{2}{\|A_1 A_1^*\|}\right]$, from Lemma 7.6, the mapping $P_C \left(I - \eta_1 A_1^* \left(I - P_{Q_1}\right) A_1\right)$ from H_1 into C is nonexpansive. Similarly, $P_C \left(I - \eta_2 A_2^* \left(I - P_{Q_2}\right) A_2\right) : H_2 \to C$ is also nonexpansive. As the range of these mappings is C, the sequences $\{z_n\}$, $\{y_n\}$, $\{X_n\}$, and $\{x_n\}$ are properly defined as sequences in C.

As $\eta_1, \eta_2 > 0$, from Lemma 7.2, it holds that

$$F\left(P_C \left(I - \eta_1 A_1^* \left(I - P_{Q_1}\right) A_1\right)\right) = C \cap A_1^{-1} Q_1 \text{ and}$$
$$F\left(P_C \left(I - \eta_2 A_2^* \left(I - P_{Q_2}\right) A_2\right)\right) = C \cap A_2^{-1} Q_2.$$

Therefore,

$$\begin{aligned} &F\left(P_C \left(I - \eta_1 A_1^* \left(I - P_{Q_1}\right) A_1\right)\right) \cap F\left(P_C \left(I - \eta_2 A_2^* \left(I - P_{Q_2}\right) A_2\right)\right) \\ &= C \cap A_1^{-1} Q_1 \cap A_2^{-1} Q_2 = \Omega. \end{aligned}$$

From the above, we can apply Corollary 6.1 with (6.6) by regarding as

$$S = P_C \left(I - \eta_1 A_1^* \left(I - P_{Q_1}\right) A_1\right) \text{ and}$$
$$T = P_C \left(I - \eta_2 A_2^* \left(I - P_{Q_2}\right) A_2\right),$$

and obtain the desired result. □

As a final remark of this article, we can prove similar results using the CQ method by Nakajo and Takahashi [35].

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