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Zero Products of Toeplitz operators on the Hardy and Bergman spaces over an annulus

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Abstract. We study the zero product problem of Toeplitz operators on the Hardy space and Bergman space over an annulus. Assuming a condition on the Fourier expansion of the symbols, we show that there are no zero divisors in the class of Toeplitz operators on the Hardy space of the annulus. Using the reduction theorem due to Abrahamse, we characterize compact Hankel operators on the Hardy space of the annulus, which also leads to a zero product result. Similar results are proved for the Bergman space over the annulus.

Contents

1. Introduction

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $H^2(\mathbb{D})$ be the Hardy space over $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{T})$ (where $\mathbb{T}$ is the unit circle), the Toeplitz operator T_φ with symbol φ is defined to be

$$T_\varphi f = P(\varphi f)$$

where P is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. The algebraic properties of these operators were studied by Brown and Halmos in their seminal paper [4]. Among other results, one of the important results they established was that there are no zero divisors for the class of Toeplitz operators. In other words, if $T_\varphi T_\psi = 0$, then either φ or ψ is identically zero. This result has attracted a lot of attention in the past. In particular, there have been attempts to extend this result to other spaces, like the Bergman space over the $\mathbb{D}$, and to spaces over domains in higher dimensions. See, for example, [5] [6] and [7]. Interestingly, the zero product theorem for the Bergman space in full generality is still open even for the unit disc. In [2], Ahern and Čučković proved that if

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the symbols are bounded harmonic functions on the disc, then the zero product theorem is true. Notice that if φ is a bounded function on $\mathbb{D}$, it admits a polar decomposition

$$\varphi(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \varphi_k(r)\, e^{ik\theta},$$

where $\varphi_k(r)$ are the Fourier coefficients of the function $e^{i\theta} \to \varphi(re^{i\theta})$. A zero product theorem for Toeplitz operators on the Bergman space over the disc was proved in [11] under some assumptions on the polar decomposition of one of the symbols. More precisely, assume that $\psi \in L^\infty(\mathbb{D})$ and $\varphi \in L^\infty(\mathbb{D})$, with the polar decomposition

$$\varphi(re^{i\theta}) = \sum_{k=-\infty}^{N} \varphi_k(r)\, e^{ik\theta}$$

where N is a positive integer. Assume that n_0 is the smallest integer such that $\widehat{\varphi_N}(2n+N+2) \neq 0$ for all $n \geq n_0$, where $\widehat{\varphi_N}$ is the Mellin transform defined by

$$\widehat{\varphi_N}(z) = \int_0^1 \varphi_N(r)\, r^{z-1}\, dr,$$

then $T_\psi T_\varphi = 0$ implies $\psi = 0$.

Our aim in this paper is to prove similar results for the Toeplitz operators defined on the Hardy space and the Bergman space over the annulus

$$A = A_{1,R} = \{z \in \mathbb{C} : \ R < |z| < 1\}.$$

While we follow the methods in [11], we also bring in a powerful theorem, namely the reduction theorem due to Abrahamse [1], to deal with these questions. The reduction theorem allows us to reduce some of the problems for Toeplitz operators on general multi-connected domains to that of the unit disc. Crucially, using this theorem we also provide a characterization of the compactness of Hankel operators on the annulus, thus establishing an analogue of Hartman's theorem.

To state the main results of the paper we begin by recalling the Hardy space over $A = A_{1,R} = \{z \in \mathbb{C} : \ R < |z| < 1\}$, and some necessary details. This space was introduced and studied in detail by Sarason in [13]. We denote by ∂A the boundary of the annulus A. Then $\partial A = C \cup C_0$, where $C = \{z \in \mathbb{C} : |z| = 1\}$ and $C_0 = \{z \in \mathbb{C} : |z| = R\}$. While viewing the unit circle as a boundary component of the annulus, we use C and otherwise we use the standard notation $\mathbb{T}$. To define the Hardy space $H^2(\partial A)$ as a subspace of $L^2(\partial A)$, we need to introduce the measure, norm, and inner product on $L^2(\partial A)$.

Definition 1.1. *A subset E of ∂A is called measurable if $\{a \in [0, 2\pi) : e^{ia} \in E\}$ and $\{b \in [0, 2\pi) : Re^{ib} \in E\}$ are both Borel subsets of $\mathbb{R}$.*

Let σ be the measure defined on ∂A obtained by summing the Lebesgue measure on each component of ∂A and normalised so that $\sigma(\partial A) = 2$. More precisely, for $E \subseteq \partial A$ measurable, we define

$$\sigma(E) = \frac{1}{2\pi}\Big(\big(\mu\{a \in [0,2\pi) : e^{ia} \in E\}\big) + \big(\mu\{b \in [0,2\pi) : Re^{ib} \in E\}\big)\Big),$$

where μ denotes the Lebesgue measure on $\mathbb{R}$.

With this measure σ, we will define the space $L^2(\partial A)$, as the space of all σ-measurable square integrable functions as follows:

$$L^2(\partial A) = \{f : \partial A \longrightarrow \mathbb{C} : \|f\|_{\partial A} < \infty\}$$

where

$$\|f\|_{\partial A}^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^2 dt + \frac{1}{2\pi}\int_0^{2\pi} |f(Re^{it})|^2 dt,$$

and for $f, g \in L^2(\partial A)$, the corresponding inner product is given by

$$\begin{aligned}\langle f, g\rangle_{\partial A} &= \int_{\partial A} f\overline{g}\, d\sigma \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}dt + \frac{1}{2\pi}\int_0^{2\pi} f(Re^{it})\overline{g(Re^{it})}dt.\end{aligned}$$

The Hardy space $H^2(\partial A)$ is defined to be the closure in $L^2(\partial A)$ of rational functions on $\mathbb{C}$, having no poles in $\overline{A}$. Recall that (see [10]), the set $\{e_n(z)\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $H^2(\partial A)$ where

$$e_n(z) = \frac{1}{\sqrt{1+R^{2n}}} z^n, \quad z \in \partial A. \tag{1}$$

The orthogonal complement of $H^2(\partial A)$ in $L^2(\partial A)$ (see [10]) is the closed subspace, $\overline{\text{span}}\{f_n, n \in \mathbb{Z}\}$, where the functions f_n are defined by

$$f_n(z) = \begin{cases} \frac{R^n}{\sqrt{1+R^{2n}}} z^n, & \text{if } |z| = 1 \\ \frac{-1}{R^n\sqrt{1+R^{2n}}} z^n, & \text{if } |z| = R. \end{cases} \tag{2}$$

To study the Toeplitz operator T_f on $H^2(\partial A)$, we need the following definition of Fourier coefficients of f for the outer and inner components C and C_0 respectively of ∂A.

Definition 1.2. *Let $f \in L^2(\partial A)$. For $n \in \mathbb{Z}$, the n-th pair of Fourier coefficients of f denoted by $\widehat{f_C}(n)$ and $\widehat{f_{C_0}}(n)$ respectively and are defined by*

$$\widehat{f_C}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}dt$$

$$\widehat{f_{C_0}}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{it})e^{-int}dt.$$

Let P_R denote the orthogonal projection of $L^2(\partial A)$ onto $H^2(\partial A)$. For $f \in L^\infty(\partial A)$, the Toeplitz operator $T_f : H^2(\partial A) \longrightarrow H^2(\partial A)$ is defined by $T_f h = P_R(fh)$, for all $h \in H^2(\partial A)$. We have the following zero product theorem for Toeplitz operators on $H^2(\partial A)$. See Section 2 for a proof.

Theorem 1.3. *Let $g(re^{i\theta}) = \sum_{k=-\infty}^{N} g_k(r)e^{ik\theta}$ and $f(re^{i\theta}) = \sum_{k=-\infty}^{N'} f_k(r)e^{ik\theta}$, for $r = R, 1$ and $N, N' \in \mathbb{Z}$, be two functions of $L^\infty(\partial A)$. Then $T_f T_g = 0$ implies $f = 0$ or $g = 0$.*

Our next theorem is a characterization of compact Hankel operators on the annulus, which uses the reduction theorem due to Abrahamse. The reduction theorem is a powerful tool which connects Toeplitz operators on the Hardy space over a multiply connected domain to Toeplitz operators on the direct sum of copies of $H^2(\mathbb{D})$ modulo compact operators. We prove a characterization of compact Hankel operators on the annulus using the reduction theorem and deduce a zero product theorem for Toeplitz operators.

Let the domain $\mathcal{D}$ stand for $\mathbb{D}$ or $A_{1,R}$ and let $\partial\mathcal{D}$ be its boundary. Let $\phi \in L^\infty(\partial\mathcal{D})$ and $P_R : L^2(\partial\mathcal{D}) \to H^2(\partial\mathcal{D})$ be the orthogonal projection. Then the Hankel operator H_ϕ, with symbol ϕ is defined by $H_\phi : H^2(\partial\mathcal{D}) \to H^2(\partial\mathcal{D})^\perp$,

$$H_\phi(f) = (I - P_R)\phi f, \quad \text{for all } f \in H^2(\partial\mathcal{D}).$$

In Section 3, we prove an analogue of Hartman's theorem, characterizing compact Hankel operators. For $\phi \in L^\infty(\partial A)$, let $\phi_C(z) = \phi(z)$ for $z \in \mathbb{T}$, and $\phi_{C_0}(z) = \phi(R/z)$ for $z \in \mathbb{T}$.

Theorem 1.4. *The operator $H_\phi : H^2(\partial A) \longrightarrow H^2(\partial A)^\perp$ with $\phi \in L^\infty(\partial A)$ is compact if and only if the functions $\phi_C, \phi_{C_0} \in H^\infty + C$.*

As an immediate corollary, we deduce a zero product theorem. See Section 3 for the proof.

Theorem 1.5. *Let $\phi, \psi \in L^\infty(\partial A)$ be such that $\overline{\phi}_C$ and $\overline{\phi}_{C_0}$ (or ψ_C and ψ_{C_0}) belong to $H^\infty + C$. Then $T_\phi T_\psi = 0$ on $H^2(\partial A)$, if and only if $\phi = 0$ or $\psi = 0$.*

Next, we move on to a similar problem on the Bergman space on the annulus. Recall that, the Bergman space $B^2(A_{1,R})$ is the space of all square integrable holomorphic functions on $A_{1,R}$ i.e.,

$$B^2(A_{1,R}) = \{f : A_{1,R} \to \mathbb{C}, \text{ holomorphic and } \int_{A_{1,R}} |f(z)|^2\, dA(z) < \infty\},$$

where $dA(z) = dxdy$ is the area measure. For $f, g \in B^2(A_{1,R})$, the norm and inner product of the space are given by

$$\|f\|^2_{B^2(A_{1,R})} = \frac{1}{2\pi} \int_{A_{1,R}} |f(z)|^2 dA(z),$$

$$\langle f, g\rangle_{B^2(A_{1,R})} = \frac{1}{2\pi}\int_{A_{1,R}} f(g)\overline{g(z)}dA(z).$$

It is easy to see that $B^2(A_{1,R})$ is a closed subspace of $L^2(A_{1,R}, dA)$. Let $L^\infty(A_{1,R})$ denote the algebra of all essentially bounded functions on $A_{1,R}$ and $P_{B^2(A_{1,R})}$ be the orthogonal projection from $L^2(A_{1,R})$ onto $B^2(A_{1,R})$. Corresponding to $f \in L^\infty(A_{1,R})$, the Toeplitz operator $T_f : B^2(A_{1,R}) \to B^2(A_{1,R})$ is defined by $T_f\varphi = P_{B^2(A_{1,R})}(f\varphi)$. For Toeplitz operators on $B^2(A_{1,R})$, we prove the following zero product theorem in Section 4.

Theorem 1.6. *Let $f, g \in L^\infty(A_{1,R})$ such that $f(re^{i\theta}) = \sum_{k=-\infty}^{M} f_k(r)e^{ik\theta}$ and $g(re^{i\theta}) = \sum_{k=-\infty}^{N} g_k(r)e^{ik\theta}$ for some $M, N \in \mathbb{Z}$. Assume $n_0 \in \mathbb{Z}$ to be the smallest integer such that $\widehat{g_N}(2n+N+2) \neq 0$ (where $\widehat{g_N}$ is the Mellin transform of g_N) for all $n \geq n_0$. If $T_fT_g = 0$ then $f = 0$.*

2. Toeplitz operators on $H^2(\partial A)$

In this section, we prove Theorem 1.3. Recall that, for $f \in L^\infty(\partial A)$, the Toeplitz operator T_f is defined on $H^2(\partial A)$ by $T_fh = P_R(fh)$, for all $h \in H^2(\partial A)$ where P_R is the orthogonal projection from $L^2(\partial A)$ onto $H^2(\partial A)$.

For $f \in L^\infty(\partial A)$, T_f is always bounded ([10]). A simple computation reveals ([10], page 51),

$$\langle T_fe_k, e_j\rangle_{\partial A} = \frac{1}{\sqrt{1+R^{2j}}\sqrt{1+R^{2k}}}\left(\widehat{f_C}(j-k) + R^{j+k}\widehat{f_{C_0}}(j-k)\right). \tag{3}$$

Equation (3) helps us to write the matrix representation $[T_f]$ of the Toeplitz operator T_f with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ on $H^2(\partial A)$. Indeed, $[T_f] = [a_{j,k}]_{j,k=-\infty}^{\infty}$ is the following matrx:

$$\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \frac{\widehat{f_C}(0)+R^{-4}\widehat{f_{C_0}}(0)}{1+R^{-4}} & a_{-2,-1} & a_{-2,0} & a_{-2,1} & a_{-2,2} & \vdots \\
\vdots & a_{-1,-2} & \frac{\widehat{f_C}(0)+R^{-2}\widehat{f_{C_0}}(0)}{1+R^{-2}} & a_{-1,0} & a_{-1,1} & a_{-1,2} & \vdots \\
\vdots & a_{0,-2} & a_{0,-1} & \frac{\widehat{f_C}(0)+\widehat{f_{C_0}}(0)}{2} & a_{0,1} & a_{0,2} & \vdots \\
\vdots & a_{1,-2} & a_{1,-1} & a_{1,0} & \frac{\widehat{f_C}(0)+R^{2}\widehat{f_{C_0}}(0)}{1+R^{2}} & a_{1,2} & \vdots \\
\vdots & a_{2,-2} & a_{2,-1} & a_{2,0} & a_{2,1} & \frac{\widehat{f_C}(0)+R^{4}\widehat{f_{C_0}}(0)}{1+R^{4}} & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

where

$$a_{j,k} = \frac{1}{\sqrt{1+R^{2j}}\sqrt{1+R^{2k}}}\left(\widehat{f_C}(j-k) + R^{j+k}\widehat{f_{C_0}}(j-k)\right). \tag{4}$$

For $n \in \mathbb{Z}$, we refer to the subdiagonal containing the entries $a_{n,n} = \frac{\widehat{f_C}(0)+R^{2n}\widehat{f_{C_0}}(0)}{1+R^{2n}}$ as the main diagonal of $[T_f]$. Based on this matrix representation, we now prove the following lemma:

Lemma 2.1. *T_f is zero if and only if any two columns of $[T_f]$ are zero.*

Proof. It suffices to prove the "if" part. Let $p \in \mathbb{Z}$, and C_p denote p-th column (that is the column whose entries are $a_{n,p}, n \in \mathbb{Z}$). For $p \in \mathbb{Z}$, let D_p denote the p-th subdiagonal (that is the subdiagonal whose entries are $a_{n,n+p}$) of $[T_f]$.

Now, consider two columns C_r and C_s of $[T_f]$ for $r \neq s$. Then, for each $n \in \mathbb{Z}$, there exist $m, t \in \mathbb{Z}$ such that $a_{m,r} \in D_n \cap C_r$ and $a_{t,s} \in D_n \cap C_s$ such that

$$m - r = t - s = n$$

and by (4), we can write

$$a_{m,r} = \frac{1}{\sqrt{1+R^{2m}}\sqrt{1+R^{2r}}}\left(\widehat{f_C}(n) + R^{m+r}\widehat{f_{C_0}}(n)\right),$$

and

$$a_{t,s} = \frac{1}{\sqrt{1+R^{2t}}\sqrt{1+R^{2s}}}\left(\widehat{f_C}(n) + R^{t+s}\widehat{f_{C_0}}(n)\right).$$

Since $r \neq s$, we have $m + r \neq t + s$, and since $a_{m,r} = a_{t,s} = 0$, we get from the above that $\widehat{f_C}(n) = \widehat{f_{C_0}}(n) = 0$ for every n and so $f = 0$. □

Remark 2.2. Clearly, similar proof works if any two rows are zero.

Now, we are ready to prove Theorem 1.3.

Proof. Let $g \neq 0$ and assume without loss of generality, at least one of the Fourier coefficients $\widehat{g_C}(N)$ or $\widehat{g_{C_0}}(N)$ is nonzero. Then, with respect to $\{e_n\}_{n\in\mathbb{Z}}$, the matrix $[T_g]$ of T_g has an upper triangular form, as the (j,k)-th entry a_{jk} is a combination of $\widehat{g_C}(j-k)$ and $\widehat{g_{C_0}}(j-k)$ which is zero provided $j-k > N$. Notice that the first nonzero subdiagonal from the bottom left corner has entries

$$a_{m,n} = \frac{\widehat{g_C}(N) + R^{m+n}\widehat{g_{C_0}}(N)}{\sqrt{1+R^{2m}}\sqrt{1+R^{2n}}},$$

with $N = m - n$. Moreover, in this subdiagonal, $a_{m,n}$ can vanish at most at one position. Because, if there exist distinct (m_1, n_1), (m_2, n_2) such that $a_{m_1,n_1} = a_{m_2,n_2} = 0$, where $m_2 = m_1 + k_1$ and $n_2 = n_1 + k_1$ for some $k_1 (\neq 0) \in \mathbb{Z}$, then

$$\widehat{g_C}(N) + R^{m_1+n_1}\widehat{g_{C_0}}(N) = 0 \tag{5}$$

$$\widehat{g_C}(N) + R^{m_2+n_2}\widehat{g_{C_0}}(N) = 0. \tag{6}$$

Since $m_2+n_2 = m_1+n_1+2k_1 \neq m_1+n_1$, it follows that $\widehat{g_C}(N) = \widehat{g_{C_0}}(N) = 0$, which contradicts our assumption. Hence, we can choose $n_0 \in \mathbb{N}$ such that,

$$\widehat{g_C}(N) + R^{2n+N}\widehat{g_{C_0}}(N) \neq 0 \quad \text{for all } n \geq n_0. \tag{7}$$

Now, for any $n \in \mathbb{Z}$ (using the equation (4) for g),

$$T_g(z^n) = \frac{\widehat{g_C}(N) + R^{2n+N}\widehat{g_{C_0}}(N)}{(1+R^{2(n+N)})} z^{n+N} + \sum_{k=-\infty}^{N-1} \frac{\widehat{g_C}(k) + R^{2n+k}\widehat{g_{C_0}}(k)}{(1+R^{2(n+k)})} z^{k+n}. \tag{8}$$

Let $T_f T_g = 0$. Then for all $n \in \mathbb{Z}$, the equation (8) reduces to

$$\frac{\widehat{g_C}(N) + R^{2n+N}\widehat{g_{C_0}}(N)}{(1 + R^{2(n+N)})} T_f(z^{n+N}) + \sum_{k=-\infty}^{N-1} \frac{\widehat{g_C}(k) + R^{2n+k}\widehat{g_{C_0}}(k)}{(1 + R^{2(n+k)})} T_f(z^{k+n}) = 0 \quad (9)$$

Then for $n = n_0$, the relation (7) and equation (9) together yield

$$T_f(z^{n_0+N}) \in \overline{\text{span}}\{T_f(z^{n_0+N-1}), T_f(z^{n_0+N-2}), \ldots\}. \quad (10)$$

Similarly, for $n = n_0 + 1$ it follows by (7) and (9)

$$T_f(z^{n_0+N+1}) \in \overline{\text{span}}\{T_f(z^{n_0+N}), T_f(z^{n_0+N-1}), T_f(z^{n_0+N-2}), \ldots\}, \quad (11)$$

and further (10) and (11) together imply

$$T_f(z^{n_0+N+1}) \in \overline{\text{span}}\{T_f(z^{n_0+N-1}), T_f(z^{n_0+N-2}), \ldots\}. \quad (12)$$

Claim: For $l \geq 0$,

$$T_f(z^{n_0+N+l}) \in \overline{\text{span}}\{T_f(z^{n_0+N-1}), T_f(z^{n_0+N-2}), \ldots\}. \quad (13)$$

We prove the claim by induction on $l \geq 0$. The proof when $l = 0, 1$ follows by the equations (10) and (12). For the induction step, assume the claim to be true for all $0 \leq l < m$, for some $m \geq 2$. Then for $l = m$, it follows by (7), and (9)

$$T_f(z^{n_0+N+m}) \in \overline{\text{span}}\{T_f(z^{n_0+N+m-1}), \ldots, T_f(z^{n_0+N}), T_f(z^{n_0+N-1}), \ldots\}, \quad (14)$$

and hence the claim follows as by the induction hypothesis,

$$T_f(z^{n_0+N+m-1}), \ldots, T_f(z^{n_0+N}) \in \overline{\text{span}}\{T_f(z^{n_0+N-1}), T_f(z^{n_0+N-2}), \ldots\}.$$

Suppose $f \neq 0$ (equivalently, $T_f \neq 0$). Then

$$\overline{\text{span}}\{T_f(z^{n_0+N-1}), T_f(z^{n_0+N-2}), \ldots\} \neq 0,$$

otherwise the matrix of T_f will have two columns equal to zero implying $f = 0$ (see Lemma 2.1). Since $f \neq 0$, there exists an integer $k_0 \leq N'$ such that at least one of $\widehat{f_C}(k_0)$ or $\widehat{f_{C_0}}(k_0)$ is nonzero. Note that the matrix of T_f also has an upper triangular form, and the subdiagonal involving $\widehat{f_C}(k_0), \widehat{f_{C_0}}(k_0)$ can vanish at most at one position. Since

$$T_f(z^n) = \sum_{k=-\infty}^{N'} \frac{\widehat{f_C}(k) + R^{2n+k}\widehat{f_{C_0}}(k)}{1 + R^{2(n+k)}} z^{k+n}, \quad (15)$$

we have

$$\overline{\text{span}}\{T_f(z^{n_0+N-1}), T_f(z^{n_0+N-2}), \ldots\} = \overline{\text{span}}\{z^{n_0+N+N'-1}, z^{n_0+N+N'-2}, \ldots\}. \quad (16)$$

Now corresponding to k_0, there exists n_{k_0} such that

$$n_{k_0} > n_0 + N \quad \text{and} \quad n_{k_0} + k_0 > n_0 + N + N' - 1.$$

By (15), we can write

$$T_f(z^{n_{k_0}}) = \cdots\cdots + \frac{\widehat{f_C}(k_0) + R^{2n_{k_0}+k_0}\widehat{f_{C_0}}(k_0)}{1 + R^{2(n_{k_0}+k_0)}} z^{n_{k_0}+k_0} + \cdots\cdots. \quad (17)$$

More generally (again by (15)), for all $l' \geq 1$

$$T_f(z^{n_{k_0}+l'}) = \cdots\cdots + \frac{\widehat{f_C}(k_0) + R^{2(n_{k_0}+l')+k_0}\widehat{f_{C_0}}(k_0)}{1 + R^{2(n_{k_0}+l')+2k_0}} z^{n_{k_0}+l'+k_0} + \cdots\cdots. \tag{18}$$

Clearly, for any $l' \geq 1$

$$n_{k_0} + k_0 + l' > n_{k_0} + k_0 > n_0 + N + N' - 1. \tag{19}$$

Now (13) and (15)—(19) altogether imply

$$\widehat{f_C}(k_0) + R^{2n_{k_0}+k_0}\widehat{f_{C_0}}(k_0) = 0 \tag{20}$$

$$\widehat{f_C}(k_0) + R^{2(n_{k_0}+l')+k_0}\widehat{f_{C_0}}(k_0) = 0, \tag{21}$$

which yield $\widehat{f_C}(k_0) = \widehat{f}_{C_0}(k_0) = 0$, contradicting our assumption. Hence, we must have $f = 0$.

For the other case, assume $T_f T_g = 0$ and f is nonzero. If $g \neq 0$, then as we have just shown, f must be zero—which is a contradiction. Hence, g must be zero. □

3. Compactness of Hankel operators

In this section, we prove Theorem 1.4 and Theorem 1.5. We begin with the following lemma (see Lemma 2.18 in [1]).

Lemma 3.1. *If $\varphi \in L^\infty(\partial A)$ vanishes on a set of positive measure, but is not identically zero, then Ker $T_\varphi = 0$.*

With the help of above lemma, we now prove the following result which will be used later.

Lemma 3.2. *Let $\phi, \psi \in L^\infty(\partial A)$ and $T_\phi T_\psi = 0$. If $\phi\psi = 0$ on a set $B \subseteq \partial A$ of positive measure, then either ϕ or ψ is identically zero.*

Proof. If $\phi\psi = 0$ on $B \subseteq \partial A$ with $\sigma(B) > 0$, then there exists $B' \subseteq B$ with $\sigma(B') > 0$ such that at least one of ϕ or ψ vanishes on B'. Two cases can arise:

Case 1: $\phi = 0$ on B'. If $\phi \neq 0$ on ∂A, then by Lemma 3.1, $\ker T_\phi = \{0\}$. Now $T_\phi T_\psi = 0$ implies $\mathrm{Ran} T_\psi \subseteq \ker T_\phi = \{0\}$. Hence, $T_\psi = 0$ and consequently $\psi = 0$.

Case 2: $\psi = 0$ on B'. Since $(T_\phi T_\psi)^* = T_{\bar{\psi}} T_{\bar{\phi}}$, it follows by case 1, $\bar{\phi} = \phi = 0$. □

For the domain $\mathcal{D}$ either $\mathbb{D}$ or A, we know that for $\phi \in L^\infty(\partial\mathcal{D})$, the Hankel operator H_ϕ is bounded. We need the following lemma from [14] (see Lemma 1 in [14]) and [10] (see the proof of Lemma 3.2.1 in [10]).

Lemma 3.3. *For $\phi, \psi \in L^\infty(\partial\mathcal{D})$, $T_{\phi\psi} = T_\phi T_\psi + H^*_{\bar{\phi}} H_\psi$.*

We also need the following theorem of Hartman (see [9] and [3]) on compact Hankel operators for the disc, which will be used.

Theorem 3.4. *$H_\phi : H^2(\mathbb{D}) \to H^2(\mathbb{D})^\perp$ is compact if and only if $\phi \in H^\infty + C$ on $\mathbb{T}$, where*

$$H^\infty + C = \{f + g : f \in H^\infty(\mathbb{T}), g \in C(\mathbb{T})\},$$

$C(\mathbb{T})$ being the set of all continuous functions on $\mathbb{T}$.

Now, we move towards proving the main results using the reduction theorem. To state the reduction theorem, we briefly recall the settings in [1] (see Part III). As earlier, let $A = A_{1,R}$ stand for the annulus $\{z : R < |z| < 1\}$. Boundary, ∂A consist of the two circles $C = \{z : |z| = 1\}$ and $C_0 = \{z : |z| = R\}$. Interior of C is the unit disc $\mathbb{D}$, and let us denote the exterior of C_0 including the point ∞ by D_0. Thus $D_0 = \{z : |z| > R\} \cup \{\infty\}$. Then, by the Caratheodory extension of the Riemann mapping theorem, we get two homeomorphisms π and π_0, mapping $\mathbb{D} \cup \mathbb{T}$ onto $\mathbb{D} \cup C$ and $D_0 \cup C_0$ respectively, which are conformal equivalences between the interiors. Clearly, we can take

$$\pi(z) = z, \text{ and } \pi_0(z) = R/z.$$

Associated with the function $\phi \in L^\infty(\partial A)$ are the functions $\phi_C(z) = \phi \circ \pi(z) = \phi(z)$ and $\phi_{C_0}(z) = \phi \circ \pi_0(z) = \phi(R/z)$, in $L^\infty(\mathbb{T})$. The reduction theorem relates the Toeplitz operator T_ϕ with the Toeplitz operators T_{ϕ_C} and $T_{\phi_{C_0}}$ on $H^2(\mathbb{D})$. Let $\mathcal{I}(H^2(\partial A))$ be the C^*-algebra generated by $\{T_\phi : \phi \in L^\infty(\partial A)\}$, and let $\mathcal{I}(H^2(\mathbb{D}))$ be the C^*-algebra on $H^2(\mathbb{D})$ generated by $\{T_f : f \in L^\infty(\mathbb{T})\}$. For any Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ be the Banach algebra of all bounded operators, $\mathcal{K}(\mathcal{H})$ be the closed ideal of compact operators, and for $T \in \mathcal{B}(\mathcal{H})$, $[T]$ be the coset $T + \mathcal{K}(\mathcal{H})$. Two operators S and T in the same coset are said to be equivalent modulo the compact operators, denoted $S \equiv T$. We now state the reduction theorem for Toeplitz operators on annulus as follows ([1], see Theorem 3.1):

Theorem 3.5. *There is a $*$-isometric isomorphism between the C^*-algebras*

$$\mathcal{I}(H^2(\partial A))/\mathcal{K}(H^2(\partial A)) \text{ and } \mathcal{I}(H^2(\mathbb{D}))/\mathcal{K}(H^2(\mathbb{D})) \oplus \mathcal{I}(H^2(\mathbb{D}))/\mathcal{K}(H^2(\mathbb{D}))$$

which takes $[T_\phi]$ to $[T_{\phi_C}] \oplus [T_{\phi_{C_0}}]$.

To prove the main results of this section, we need some lemmas.

Lemma 3.6. *$\overline{e_n} = \frac{2R^n}{1+R^{2n}} e_{-n} + \frac{1-R^{2n}}{1+R^{2n}} f_{-n}$ on $L^2(\partial A)$.*

Proof. Since the set $\{e_n, f_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\partial A)$, it follows for any $n \in \mathbb{Z}$,

$$\overline{e_n} = \sum_{m\in\mathbb{Z}} \langle \overline{e_n}, e_m \rangle_{\partial A}\, e_m + \sum_{m\in\mathbb{Z}} \langle \overline{e_n}, f_m \rangle_{\partial A}\, f_m. \tag{22}$$

The proof now follows by a direct computation after substituting e_m and f_m (from (1), (2)) in the above relation (22). □

Lemma 3.7. *For $\phi, \psi \in L^\infty(\partial A)$,*

$$[T_{\phi\psi}] \equiv [T_{\phi_C \psi_C}] \oplus [T_{\phi_{C_0} \psi_{C_0}}]$$

if and only if $H^*_{\overline{\phi_C}} H_{\psi_C}$ and $H^*_{\overline{\phi_{C_0}}} H_{\psi_{C_0}}$ are compact.

Proof. Note that, for $\phi, \psi \in L^\infty(\partial A)$ we have $\phi_C,\ \phi_{C_0},\ \psi_C,\ \psi_{C_0} \in L^\infty(\mathbb{T})$. By Theorem 3.5 above

$$[T_\phi] \longrightarrow [T_{\phi_C}] \oplus [T_{\phi_{C_0}}], \quad [T_\psi] \longrightarrow [T_{\psi_C}] \oplus [T_{\psi_{C_0}}] \text{ and}$$
$$[T_\phi][T_\psi] \equiv [T_\phi T_\psi] \equiv [T_{\phi_C} T_{\psi_C}] \oplus [T_{\phi_{C_0}} T_{\psi_{C_0}}]. \tag{23}$$

By Lemma 3.3, the relation (23) further reduces to

$$[T_\phi T_\psi] \equiv [T_{\phi_C \psi_C} - H^*_{\overline{\phi_C}} H_{\psi_C}] \oplus [T_{\phi_{C_0}\psi_{C_0}} - H^*_{\overline{\phi_{C_0}}} H_{\psi_{C_0}}]. \tag{24}$$

Again for $T_{\phi\psi}$ with $\phi, \psi \in L^\infty(\partial A)$, $[T_{\phi\psi}] \equiv [T_{(\phi\psi)_C}] \oplus [T_{(\phi\psi)_{C_0}}]$ (by Theorem 3.5) and hence

$$[T_{\phi\psi}] \equiv [T_{\phi_C\psi_C}] \oplus [T_{\phi_{C_0}\psi_{C_0}}], \tag{25}$$

where the second relation follows from $(\phi\psi)_C = \phi_C\psi_C$, and $(\phi\psi)_{C_0} = \psi_{C_0}\psi_{C_0}$. Now by (24) and (25), $[T_\phi T_\psi] \equiv [T_{\phi\psi}]$ if and only if $H^*_{\overline{\phi_C}} H_{\psi_C}$ and $H^*_{\overline{\phi_{C_0}}} H_{\psi_{C_0}}$ are compact. □

To achieve our goal, we now investigate the compactness of the Hankel operators on $A_{1,R}$. Recall that the boundary ∂A of $A_{1,R}$ consist of the circles C and C_0 with $\mathbb{D}$ and D_0 being corresponding interior, and exterior including the point at ∞ respectively.

Let A and A_0 be the algebra of continuous functions on $\mathbb{D} \cup C$ and $D_0 \cup C_0$ which are holomorphic in $\mathbb{D}$ and D_0 respectively. Also, let Y and Y_0 be the closure of A and A_0 in $L^2(C)$ and $L^2(C_0)$ respectively. Clearly, $Y = H^2(C)$ and $Y_0 = H^2(C_0)$. Since $L^2(C), L^2(C_0) \subseteq L^2(\partial A)$ are closed, the subspaces Y, and Y_0 are also closed in $L^2(\partial A)$. Note that, $L^2(\partial A) = H^2(\partial A) \oplus H^2(\partial A)^\perp$ and for $\phi \in L^\infty(\partial A)$, $H_\phi : H^2(\partial A) \to H^2(\partial A)^\perp$ is compact if and only if the operator $\widetilde{H_\phi} = \begin{pmatrix} 0 & 0 \\ H_\phi & 0 \end{pmatrix}$ on $H^2(\partial A) \oplus H^2(\partial A)^\perp$ is compact. Note by Lemma 3.10 in [1], $H^2(\partial A) = Y + Y_0$ and hence by Lemma 3.9 in [1], $\widetilde{H_\phi}$ is compact if $\widetilde{H_\phi} P_Y$ and $\widetilde{H_\phi} P_{Y_0}$ are compact, where P_Y, P_{Y_0} are orthogonal projections of $L^2(\partial A)$ onto Y and Y_0 respectively. This is equivalent to $H_\phi P_Y$ and $H_\phi P_{Y_0}$ being compact.

For $\phi \in L^\infty(\partial A)$, $\phi \circ \pi, \phi \circ \pi_0 \in L^\infty(\mathbb{T})$ and so one can consider the Hankel operators $H^Y_\phi : Y \to Y^\perp (= L^2(C) \ominus Y)$ and $H^{Y_0}_\phi : Y_0 \to Y_0^\perp (= L^2(C_0) \ominus Y_0)$ defined by $H^Y_\phi(f) = P_{Y^\perp}\phi f$, $f \in Y$ and $H^{Y_0}_\phi(g) = P_{Y_0^\perp}\phi g$ $g \in Y_0$; $P_{Y^\perp}, P_{Y_0^\perp}$ being the orthogonal projections of $L^2(C), L^2(C_0)$ onto $Y^\perp$ and $Y_0^\perp$ respectively. Our goal is to relate the components $H_\phi P_Y, H_\phi P_{Y_0}$ with H^Y_ϕ and $H^{Y_0}_\phi$ respectively and find the criteria for compactness of H^Y_ϕ and $H^{Y_0}_\phi$.

Lemma 3.8. *$H^{Y_0}_\phi$ and H^Y_ϕ are compact if and only if ϕ_{C_0} and ϕ_C belong to $H^\infty + C$.*

Proof. We will deal with the components $H_\phi P_{Y_0}$, and $H^{Y_0}_\phi$ only, as $H_\phi P_Y, H^Y_\phi$ can be dealt with exactly in the same way. Corresponding to the homeomorphisms π, π_0, define the maps $\widetilde{\pi}$ and $\widetilde{\pi_0}$ from $L^2(C)$ and $L^2(C_0)$ respectively to

$L^2(\mathbb{T})$ by $\widetilde{\pi}(f) = f\circ\pi$ and $\widetilde{\pi_0}(g) = g\circ\pi_0$. Then $\{\widetilde{e_n}\circ\pi^{-1}\}_{n\in\mathbb{Z}}$ and $\{\widetilde{e_n}\circ\pi_0^{-1}\}_{n\in\mathbb{Z}}$ form orthonormal bases for $L^2(C)$ and $L^2(C_0)$ respectively, where $\{\widetilde{e_n}\}_{n\in\mathbb{Z}}$, with $\widetilde{e_n}(z) = z^n$, $z \in \mathbb{T}$, is the standard orthonormal basis for $L^2(\mathbb{T})$. Note that, the sets $\{\widetilde{e_n}\circ\pi^{-1}\}_{n\in\mathbb{Z}_{\geq 0}}$ and $\{\widetilde{e_n}\circ\pi_0^{-1}\}_{n\in\mathbb{Z}_{\geq 0}}$ form orthonormal bases for Y and Y_0 respectively.

Let us define $U : Y^\perp \longrightarrow H^2(\mathbb{D})^\perp$ and $U_0 : Y_0^\perp \longrightarrow H^2(\mathbb{D})^\perp$ by

$$U\Big(\sum_{n\leq -1}\langle f,\ \widetilde{e_n}\circ\pi^{-1}\rangle\widetilde{e_n}\circ\pi^{-1}\Big) = \sum_{n\leq -1}\langle f\circ\pi, \widetilde{e_n}\rangle\widetilde{e_n}, \quad f \in Y \tag{26}$$

$$U_0\Big(\sum_{n\leq -1}\langle f,\ \widetilde{e_n}\circ\pi_0^{-1}\rangle\widetilde{e_n}\circ\pi_0^{-1}\Big) = \sum_{n\leq -1}\langle f\circ\pi_0, \widetilde{e_n}\rangle\widetilde{e_n}, \quad f \in Y_0. \tag{27}$$

Clearly, U and U_0 are unitary maps. We show that the following diagram is commutative.

$$\begin{array}{ccc} Y_0 & \xrightarrow{\widetilde{\pi_0}} & H^2(\mathbb{D}) \\ {\scriptstyle H_\phi^{Y_0}}\big\downarrow & & \big\downarrow {\scriptstyle H_{\phi_{C_0}}(=H_{\phi\circ\pi_0})} \\ Y_0^\perp & \xrightarrow{U_0} & H^2(\mathbb{D})^\perp \end{array}$$

Indeed, for $f \in Y_0$, $\widetilde{\pi_0}(f) = f\circ\pi_0 = \tilde{f}$ (say), and hence

$$H_\phi^{Y_0}\widetilde{\pi_0}^{-1}(\tilde{f}) = H_\phi^{Y_0}(f) = P_{Y_0{}^\perp}(\phi f). \tag{28}$$

Since $\{\widetilde{e_n}\circ\pi_0^{-1}\}_{n\leq -1}$ is an orthonormal basis of $Y_0^\perp$,

$$P_{Y_0{}^\perp}(\phi f) = \sum_{n\leq -1}\langle \phi f, \widetilde{e_n}\circ\pi_0^{-1}\rangle\widetilde{e_n}\circ\pi_0^{-1}. \tag{29}$$

By equations (27), (28) and (29),

$$U_0 H_\phi^{Y_0}\widetilde{\pi_0}^{-1}(\tilde{f}) = \sum_{n\leq -1}\langle(\phi f)\circ\pi_0, \widetilde{e_n}\rangle\widetilde{e_n}. \tag{30}$$

Since $\phi f\circ\pi_0 = (\phi\circ\pi_0)(f\circ\pi_0)$ and $\phi\circ\pi_0 = \phi_{C_0}$, (30) becomes

$$U_0 H_\phi^{Y_0}\widetilde{\pi_0}^{-1}(\tilde{f}) = \sum_{n\leq -1}\langle \phi_{C_0}\tilde{f}, \widetilde{e_n}\rangle\widetilde{e_n} = H_{\phi_{C_0}}(\tilde{f}). \tag{31}$$

Since U_0 and $\widetilde{\pi_0}^{-1}$ are invertible, by (31) $H_\phi^{Y_0}$ is compact if and only if $H_{\phi_{C_0}}$ is compact, which is, by the result of Hartman (see Theorem 3.4) equivalent to $\phi_{C_0} \in H^\infty + C$.

□

We now establish the following relation between $H_\phi P_{Y_0}$, $H_\phi P_Y$ and $H_\phi^{Y_0}$, H_ϕ^Y.

Lemma 3.9. *$H_\phi P_{Y_0}$ is compact if and only if $H_\phi^{Y_0}$ is compact. $H_\phi P_Y$ is compact if and only if H_ϕ^Y is compact.*

Proof. We will prove the first statement. The second one can be proved similarly. Note that, $\phi \in L^\infty(\partial A)$ implies $\phi \in L^\infty(C_0)$ and recall

$$L^2(C_0) = Y_0 \oplus Y_0^\perp = H^2(C_0) \oplus H^2(C_0)^\perp \subseteq H^2(\partial A) \oplus H^2(\partial A)^\perp.$$

Then for $f \in Y_0$ and $\phi \in L^\infty(\partial A)$, $\phi f \in L^2(C_0)$ and one can write

$$\phi f = y_1 \oplus y_2 \quad \text{for some } y_1 \in Y_0, \quad y_2 \in Y_0^\perp. \tag{32}$$

Again, as $Y_0^\perp \subseteq H^2(\partial A) \oplus H^2(\partial A)^\perp$, there exist $x_1 \in H^2(\partial A)$ and $x_2 \in H^2(\partial A)^\perp$ such that

$$H_\phi^{Y_0} f = y_2 = x_1 \oplus x_2. \tag{33}$$

On the other hand, for the same f in Y_0, it follows by equation (32)

$$H_\phi P_{Y_0}(f) = H_\phi(f) = P_{H^2(\partial A)^\perp}(\phi f) = P_{H^2(\partial A)^\perp}(y_1 \oplus y_2). \tag{34}$$

Since $y_1 \in Y_0 \subseteq H^2(\partial A)$, equations (33) and (34) together imply

$$H_\phi P_{Y_0}(f) = x_2, \tag{35}$$

and finally by (33) and (35)

$$P_{H^2(\partial A)^\perp} H_\phi^{Y_0} = H_\phi P_{Y_0}. \tag{36}$$

Note that, $H_\phi^{Y_0}$ can be decomposed as

$$H_\phi^{Y_0} = P_{H^2(\partial A)} H_\phi^{Y_0} \oplus P_{H^2(\partial A)^\perp} H_\phi^{Y_0}. \tag{37}$$

We now show that the component $P_{H^2(\partial A)} H_\phi^{Y_0}$ can be written as the product $T P_{H^2(\partial A)^\perp} H_\phi^{Y_0}$, for some suitable $T : H^2(\partial A)^\perp \to H^2(\partial A)$. Then the conclusion of the theorem will follow by (36) and (37).

Since $y_2 \in Y_0^\perp = L^2(C_0) \ominus H^2(C_0)$, we have $y_2 \circ \pi_0 \in H^2(\mathbb{D})^\perp$ (see the diagram above) and hence $\overline{y_2 \circ \pi_0} \in H^2(\mathbb{D})$. Since π_0 is a homeomorphism, $\overline{y_2} \in Y_0 \subseteq H^2(\partial A)$ and one can write

$$\overline{y_2} = \sum_{n\in\mathbb{Z}} \langle \overline{y_2}, e_n \rangle_{L^2(\partial A)} e_n, \tag{38}$$

where $\{e_n\}_{n\in\mathbb{Z}}$ is as in (1). Then by (33) and (38)

$$y_2 = \sum_{n\in\mathbb{Z}} \langle e_n, \overline{y_2} \rangle_{L^2(\partial A)} \overline{e_n} = H_\phi^{Y_0} f. \tag{39}$$

Clearly by Lemma 3.6 , the equation (39) further reduces to

$$y_2 = \sum_{n\in\mathbb{Z}} \langle e_n, \overline{y_2} \rangle_{L^2(\partial A)} \Big[\frac{2R^n}{1+R^{2n}} e_{-n} + \frac{1-R^{2n}}{1+R^{2n}} f_{-n} \Big]. \tag{40}$$

Again by (39) and (40),

$$P_{H^2(\partial A)} H_\phi^{Y_0} f = P_{H^2(\partial A)} y_2 = \sum_{n\in\mathbb{Z}} \langle e_n, \overline{y_2} \rangle_{L^2(\partial A)} \Big(\frac{2R^n}{1+R^{2n}} \Big) e_{-n}, \quad \text{and} \tag{41}$$

$$P_{H^2(\partial A)^\perp} H_\phi^{Y_0} f = P_{H^2(\partial A)^\perp} y_2 = \sum_{n\in\mathbb{Z}} \langle e_n, \overline{y_2}\rangle_{L^2(\partial A)} \Big(\frac{1-R^{2n}}{1+R^{2n}}\Big) f_{-n}. \tag{42}$$

Let us now define $T : H^2(\partial A)^\perp \longrightarrow H^2(\partial A)$ by

$$f_{-n} \longrightarrow \frac{2R^n}{1-R^{2n}} e_{-n}, \quad \text{for all } n \in \mathbb{Z}. \tag{43}$$

Then we have $T = T_2T_1$, where $T_1 : H^2(\partial A)^\perp \to H^2(\partial A)^\perp$ and $T_2 : H^2(\partial A)^\perp \to H^2(\partial A)$ are defined by

$$T_1(f_n) = \frac{2R^n}{1-R^{2n}} f_n \quad \text{for all } n \in \mathbb{Z},$$

and $T_2(f_n) = e_n \; \forall n \in \mathbb{Z}$. Then by (41), (42), and (43)

$$P_{H^2(\partial A)} H_\phi^{Y_0} f = P_{H^2(\partial A)} y_2 = TP_{H^2(\partial A)^\perp} y_2 = TP_{H^2(\partial A)^\perp} H_\phi^{Y_0} f. \tag{44}$$

Since $f \in Y_0$ is arbitrary, we finally have

$$P_{H^2(\partial A)} H_\phi^{Y_0} = TP_{H^2(\partial A)^\perp} H_\phi^{Y_0}. \tag{45}$$

□

Note that T_1 and T in the above proof are compact. As we mentioned earlier, one can proceed similarly to conclude that $H_\phi P_Y$ is compact if and only if H_ϕ^Y is compact. Now, we are in a position to complete the proof of Theorem 1.4.

Proof. We know that H_ϕ is compact if and only if $H_\phi P_Y$ and $H_\phi P_{Y_0}$ are compact. This, by Lemma 3.9, is equivalent to the compactness of $H_\phi^{Y_0}$ and H_ϕ^Y. By Lemma 3.8, this is equivalent to ϕ_{C_0} and ϕ_C belonging to $H^\infty + C$. □

The proof of Theorem 1.5 is now easy.

Proof. Assume that ϕ and ψ are as in the statement of Theorem 1.5. If $T_\phi T_\psi = 0$, it follows from Lemma 3.7 that $T_{\phi\psi}$ is compact. Since the only compact Toeplitz operators on the Hardy space over any domain are the zero operators ([1], Corollary 2.12), we have $\phi\psi = 0$ on ∂A and further by Lemma 3.2, either $\phi \equiv 0$ or $\psi \equiv 0$. □

4. Toeplitz operators on the Bergman space of the annulus

Recall that, the Bergman space $B^2(A_{1,R})$ is the space of all square integrable holomorphic functions on $A_{1,R}$ i.e.,

$$B^2(A_{1,R}) = \{f : A_{1,R} \to \mathbb{C}, \text{ holomorphic and } \int_{A_{1,R}} |f(z)|^2 \, dA(z) < \infty\},$$

where $dA(z) = dxdy$ is the area measure. For $f, g \in B^2(A_{1,R})$, the norm and inner product of the space are given by

$$\|f\|^2_{B^2(A_{1,R})} = \frac{1}{2\pi} \int_{A_{1,R}} |f(z)|^2 dA(z),$$

$$\langle f, g\rangle_{B^2(A_{1,R})} = \frac{1}{2\pi}\int_{A_{1,R}} f(g)\overline{g(z)}dA(z).$$

It is well-known that $B^2(A_{1,R})$ is a closed subspace of $L^2(A_{1,R}, dA)$ and an orthonormal basis of the space is given by the following lemma.

Lemma 4.1. *The set* $\{\sqrt{\frac{2(n+1)}{1-R^{2(n+1)}}}z^n\}_{n\in\mathbb{Z}\setminus\{-1\}}\bigcup\{\frac{z^{-1}}{(\log\frac{1}{R})^{1/2}}\}$ *is an orthonormal basis of* $B^2(A_{1,R})$.

Proof. Follows by an easy and straightforward computation. □

We now introduce the Mellin transform, radial, and quasi-homogeneous functions that will be useful in our context.

Definition 4.2. *The Mellin transform* $\widehat{f}$ *of of a function* $f \in L^1([R,1], rdr)$ *is defined by*

$$\widehat{f}(z) = \int_R^1 f(r)r^{z-1}dr.$$

In fact, the Mellin transform is defined for suitable functions defined on $(0,\infty)$. In the above, the function is considered to be zero on $(0,R)\cup(1,\infty)$. For a function $\psi \in L^1([0,1], rdr)$ $\big($considered to be zero on $(1,\infty)\big)$, the Mellin transform $\widehat{\psi}$ is well-defined on $\{z \in \mathbb{C} : \ \mathrm{Re}\, z \geq 2\}$ and analytic on $\{z \in \mathbb{C} : \mathrm{Re}\, z > 2\}$ (see [8],[11],[12]). Also, it is shown in ([8]) that, a function can be determined by the values of a certain number of its Mellin coefficients. We state it as the following lemma:

Lemma 4.3. *Let* $f \in L^1([0,1], rdr)$*. If there exist* $n_0, p \in \mathbb{Z}$ *such that*

$$\widehat{f}(n_0 + pk) = 0 \quad \text{for all } k \in \mathbb{N},$$

then $f = 0$.

Definition 4.4. *A function* $f \in L^1(A_{1,R}, dA)$ *is called radial if*

$$f(z) = f_1(|z|) \quad R \leq |z| \leq 1,$$

for a function f_1 *on* $[R,1]$*. A function* f *defined on* $A_{1,R}$ *is said to be quasi-homogeneous of degree* $p \in \mathbb{Z}$ *if we can write it as* $e^{ip\theta}\phi$*, where* ϕ *is a radial function.*

Clearly, a radial function is a quasi-homogeneous function of degree zero. Note that, any function $f \in L^2(A_{1,R})$ has the polar decomposition

$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} f_k(r)e^{ik\theta},$$

where f_k are radial in $L^2([R,1], rdr)$. Below, we define the Toeplitz operator on $B^2(A_{1,R})$.

A Toeplitz operator T_f is called a quasi-homogeneous, if its symbol f quasi-homogeneous. We show that a quasi-homogeneous Toeplitz operator on $B^2(A_{1,R})$ is either a weighted shift or a diagonal operator.

Let $f(re^{i\theta}) = f_1(r)e^{ip\theta}$, where f_1 is radial and $p \in \mathbb{Z}$. Set

$$t_n = \begin{cases} \frac{1}{(\log \frac{1}{R})^{1/2}} \text{ if } n = -1 \\ \sqrt{\frac{2(n+1)}{1-R^{2(n+1)}}} \text{ if } n \neq -1. \end{cases} \tag{46}$$

Then, we have the following lemma:

Lemma 4.5. *For $n \in \mathbb{Z}$,*

$$T_f(z^n) = t_{p+n}^2 \widehat{f}_1(p+2n+2) z^{p+n}.$$

Proof. Note by Lemma 4.1, and equation (46), an orthonormal basis of $B^2(A_{1,R})$ is given by $\{t_n z^n\}_{n\in\mathbb{Z}}$. Then for $f = e^{ip\theta} f_1(r)$ with $p \in \mathbb{Z}$, it follows that $\forall n \in \mathbb{Z}$,

$$T_f(z^n) = P_{B^2(A_{1,R})}(fz^n) = \sum_{m\in\mathbb{Z}} \langle fz^n, t_m z^m \rangle_{A_{1,R}} t_m z^m, \quad \text{where} \tag{47}$$

$$\begin{aligned} \langle fz^n, t_m z^m \rangle_{A_{1,R}} &= \frac{1}{2\pi} \int_{A_{1,R}} t_m f(z) z^n \overline{z}^m dA(z) \\ &= \frac{1}{2\pi} t_m \int_R^1 f_1(r) r^{m+n+1} dr \int_0^{2\pi} e^{i(p+n-m)} d\theta \\ &= t_m \widehat{f}_1(n+m+2) \frac{1}{2\pi} \int_0^{2\pi} e^{i(p+n-m)\theta} d\theta, \quad \text{and hence} \end{aligned}$$

$$\langle fz^n, t_m z^m \rangle_{A_{1,R}} = \begin{cases} t_{p+n} \widehat{f}_1(p+2n+2), & \text{if } m = p+n \\ 0, & \text{if } m \neq p+n. \end{cases}$$

Therefore, it follows by (47)

$$T_f(z^n) = t_{p+n}^2 \widehat{f}_1(p+2n+2) z^{p+n}.$$

□

Remark 4.6. T_f is a forward (backward) shift if $p > 0 \quad (p < 0)$, and a diagonal operator if $p = 0$, i.e., f is radial.

We are now ready to prove Theorem 1.6

Proof. By Lemma 4.5, for all $n \in \mathbb{Z}$

$$T_g(z^n) = t_{N+n}^2 \widehat{g_N}(N+2n+2) z^{n+N} + \sum_{k=-\infty}^{N-1} t_{k+n}^2 \widehat{g_k}(k+2n+2) z^{k+n}. \tag{48}$$

Since $\widehat{g_N}(2n_0 + N + 2) \neq 0$,

$$z^{n_0+N} \in \overline{\text{span}}\{T_g(z^{n_0}), z^{n_0+N-1}, z^{n_0+N-2}, ...\}. \tag{49}$$

Similarly for $n = (n_0 + 1)$, $\widehat{g_N}(2n_0 + N + 4) \neq 0$, and equation (48) implies

$$z^{n_0+N+1} \in \overline{\text{span}}\{T_g(z^{n_0+1}), z^{n_0+N}, z^{n_0+N-1}, z^{n_0+N-2}, ...\}. \tag{50}$$

Hence, (49) and (50) together yield

$$z^{n_0+N+1} \in \overline{\text{span}}\{T_g(z^{n_0+1}), T_g(z^{n_0}), z^{n_0+N-1}, z^{n_0+N-2}, \ldots\}. \tag{51}$$

Proceeding exactly in the same way, it follows by induction that, for all $l \geq 0$

$$z^{n_0+N+l} \in \overline{\text{span}}\{T_g(z^{n_0+l}), \ldots, T_g(z^{n_0}), z^{n_0+N-1}, z^{n_0+N-2}, \ldots\}. \tag{52}$$

Let $T_f T_g = 0$. Then the relation 52 reduces to

$$T_f(z^{n_0+N+l}) \in \overline{\text{span}}\{T_f(z^{n_0+N-1}), T_f(z^{n_0+N-2}), \ldots\} \text{ for all } l \geq 0. \tag{53}$$

We now consider T_f. for all $n \in \mathbb{Z}$, Lemma 4.5 implies

$$T_f(z^n) = \sum_{k=-\infty}^{M} t_{k+n}^2 \widehat{f_k}(k+2n+2) z^{k+n}, \tag{54}$$

and hence for all $n \in \mathbb{Z}$,

$$T_f(z^n) \in \overline{\text{span}}\{z^{M+n}, z^{M+n-1}, \ldots \ldots\}. \tag{55}$$

Now for $l \geq 0$, equations 53 and 55 together imply

$$T_f(z^{n_0+N+l}) \in \overline{\text{span}}\{z^{M+(n_0+N-1)}, z^{M+(n_0+N-2)}, z^{M+(n_0+N-3)}, \ldots\}. \tag{56}$$

For $M \in \mathbb{Z}$, there exists $l_M \in \mathbb{N}$ such that

$$M + 2(n_0 + N + l_M + 1) \in \mathbb{N}$$

and

$$M + n_0 + N + l_M > M + n_0 + N - 1.$$

Then by 54,

$$T_f(z^{n_0+N+l_M}) = \sum_{k=-\infty}^{M} t_{k+(n_0+N+l_M)}^2 \widehat{f_k}\big(k+2(n_0+N+l_M)+2\big) z^{k+(n_0+N+l_M)}. \tag{57}$$

Now for any $l \geq 0$, we have

$$M + n_0 + N + l_M + l > M + n_0 + N - 1 \tag{58}$$

and again by 54, for all $l \geq 0$,

$$T_f(z^{n_0+N+l_M+l}) = \sum_{k=-\infty}^{M} t_{k+(n_0+N+l_M+l)}^2 \widehat{f_k}(k+2(n_0+N+l_M+l)+2) z^{k+(n_0+N+l_M+l)}. \tag{59}$$

Hence, it follows by (56), (58), and (59)

$$\widehat{f_M}(M + 2(n_0 + N + l_M + l) + 2) = 0, \quad \forall l \geq 0. \tag{60}$$

Now for the radial function $f_M (\in (L^1[R,1], rdr))$, $M + 2(n_0 + N + l_M + 1), 2 \in \mathbb{N}$ such that

$$\widehat{f_M}(M + 2(n_0 + N + l_M + 1) + 2l) = 0 \quad \forall l \geq 0 \quad (\text{ by (60)}).$$

Hence by Lemma 4.3, $f_M = 0$. Similarly, for any $k \in (-\infty, M)$ and for any radial function $f_k(r)$, there exists $l_k \in \mathbb{N}$ such that

$$k + 2(n_0 + N + l_k + 1) \in \mathbb{N}$$

and

$$k + n_0 + N + l_k > M + n_0 + N - 1,$$

and again, by a similar argument as above, we have

$$\widehat{f_k}(k + 2(n_0 + N + l_k + 1) + 2l) = 0 \text{ for all } l \geq 0,$$

and hence by Lemma 4.3, $f_k = 0$. Since $k \in (-\infty, M)$ is an arbitrary integer, it follows that $f_k = 0$ for all $k \in (-\infty, M) \cap \mathbb{Z}$ and hence $f = 0$. □

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