New York Journal of Mathematics

New York J. Math. **31** (2025) 1066–1097.

Weighted norm inequalities for maximal operator and extrapolation on variable Lebesgue spaces underlying nonhomogeneous trees

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ABSTRACT. This paper aims to study the weighted strong and weak norm inequalities for maximal operator and the extrapolation theorem on variable Lebesgue spaces in the context of nonhomogeneous trees equipped with flow measures. For this, we introduce a class of continuity condition, which play a key role in this paper. Furthermore, in virtue of extrapolation, the weighted norm inequalities for the sharp maximal operator are established.

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1. Introduction

The theory of harmonic analysis in the context of trees originally emerged from the study of automorphism groups and discrete Laplace operators, as shown in [10, 12]. Since the 1970s, this topic has attracted numerous mathematicians. For research on the properties of certain operators on trees, one can refer to [18, 27, 28]. In 2023, Levi, Santagati, Tabacco and Vallarino [22] investigated infinite nonhomogeneous trees with flow measures satisfying the local doubling condition. They obtained the weak and strong-type estimates for maximal

Received May 16, 2025.
2010 *Mathematics Subject Classification.* 42B25; 42B35; 05C05.
Key words and phrases. variable Lebesgue spaces, weights, maximal operator, extrapolation, trees.
The research was supported by Natural Science Foundation of China (Grant No. 12461021).
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ISSN 1076-9803/2025

operators, developed the Calderón-Zygmund decomposition theory, and introduced Hardy spaces and BMO spaces in this setting. Analogous to the classical Muckenhoupt weight theory initially established by Muckenhoupt in [23] and the discrete Muckenhoupt weights first developed by Saker and Agarwal in [30]. In 2024, Ottazzi, Santagati and Vallarino [26] developed the A_p weight theory on this type of infinite nonhomogeneous trees. Our work focuses on the weighted variable Lebesgue spaces in this setting.

The origins of variable Lebesgue spaces can be traced back to the work of Orlicz [25] in 1931. In addition to their intrinsic research value, these spaces also have significant applications in partial differential equations and variational integrals with non-standard growth conditions, as seen, for example [17, 15]. In the field of physics, variable Lebesgue spaces play a crucial role in the mathematical modeling of electrorheological fluids, as seen in [14]. However, it is widely acknowledged that the modern study of variable Lebesgue spaces began with the work of Kováčik and Rákosník [19] in 1991, who extended certain properties of classical Lebesgue spaces to variable Lebesgue spaces. Later, researchers focused much attention on finding the conditions for the exponent function to obtain the boundedness of the Hardy-Littlewood maximal operator on variable Lebesgue spaces, for detailed information, refer to [29, 11, 2, 3, 7].

In variable Lebesgue spaces, there are two different ways to establish the $A_{p(\cdot)}$ weights: one treats the weight as a measure, and the other as a multiplier. In 2012, Cruz-Uribe, Fiorenza and Neugebauer [6] introduced a class of $A_{p(\cdot)}$ weights using the second approach and established the boundedness of the Hardy-Littlewood maximal operator. This method had previously been used for fractional integral non-diagonal weighted inequalities and two-weight norm inequalities, as in [24, 4]. In 2017, Cruz-Uribe and Wang [8] extended the Rubio de Francia extrapolation theorem based on the $A_{p(\cdot)}$ weights from [6]. In the context of trees, we will also adopt the second approach to define $A_{p(\cdot)}$ weights. Besides, there has been extensive work and exploration to determine the conditions required for the boundedness of the Hardy-Littlewood maximal operator on weighted variable Lebesgue spaces, as seen in[1, 5, 16, 20, 21]. In 2022, Cruz-Uribe and Cummings [9] generalized the boundedness of the Hardy-Littlewood maximal operator on weighted variable Lebesgue spaces to homogeneous spaces.

Inspired by the aforementioned works, the main goal of this paper is to study the necessary and sufficient conditions for the boundedness of maximal operators on weighted variable Lebesgue spaces underlying infinite nonhomogeneous trees equipped with flow measures. In addition, we also extend the Rubio de Francia extrapolation theorem. The organization of the paper is as follows: In Section 2, we review the definitions and existing results related to these nonhomogeneous trees. Moreover, we provide precise definitions of (weighted) variable Lebesgue spaces, globally Hölder continuity condition, and $A_{p(\cdot)}$ weights, along with the necessary lemmas for proving the main results of the paper. In Section 3, the necessary and sufficient conditions for the weighted boundedness of maximal operators are presented. In Section 4, we establish the Rubio de Francia extrapolation theorem, and as an application, a sharp maximal operator weighted norm inequality is derived.

Finally, we illustrate some of the notations in the paper. We always denote by C a positive constant, which is independent of the main parameters, but it may vary from line to line. For a measurable set E, we denote by χ_E the characteristic function of E. $\lceil\cdot\rceil$ represents the ceiling functions, $\mathbb{Z}$ denotes the set of all integers, $\mathbb{C}$ is the set of complex numbers and $\mathbb{N} = \{0, 1, 2, \cdots\}$.

2. Preliminary

2.1. Nonhomogeneous trees equipped with flow measures. Let $T = (\mathcal{V}, \mathcal{E})$ be a tree, which is a connected and acyclic graph, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges. If we endow $\mathcal{V}$ with a natural distance d, then the tree T can be uniquely represented by this metric space, see [12]. For $x,\ y \in \mathcal{V}$, we say $x,\ y$ are neighbours if and only if $d(x, y) = 1$ and denote as $x \sim y$. According to the definition of a tree, there exists a unique path between any two vetices x and y, which is referred to as the geodesic and denoted as $[x, y]$. By fixing a reference vertex called the origin of the tree, denote as o and selecting a semi-infinite geodesic that passes through the origin and extends to the boundary of the tree, the boundary vertex is denoted as ζ. This tree can be viewed as hanging downward from the vetex ζ. These choices naturally induce a partial order ralation $\le$, namely, $x \le y$ if and only if $y \in [x, \zeta)$. Defining the projection of x on the geodesic $[o, \zeta)$ as

$$F(x) = \arg\min_{y\in[o,\zeta)} d(x, y).$$

Further, the level of x is defined as

$$l(x) = d(o, F(x)) - d(F(x), x), \quad \forall x \in \mathcal{V}.$$

After endowing the tree T with aforementioned level struction, for each vertex $x \in \mathcal{V}$, the parent and the sons of x are defined respectively as

$$p(x) = \{y \in \mathcal{V} : l(y) = l(x) + 1,\ d(x, y) = 1\},$$
$$s(x) = \{y \in \mathcal{V} : l(y) = l(x) - 1,\ d(x, y) = 1\}.$$

Next, in order to define the integral, it is necessary to equip the tree T with a measure. We refer to the definition of the flow measure in [22] as follows.

Definition 2.1. A flow measure is a function $\mu : \mathcal{V} \to (0, \infty)$ that satisfies the following conditions

$$\mu(A) = \sum_{x\in A} \mu(x),\ A \subset \mathcal{V} \quad \text{and} \quad \mu(x) = \sum_{y\in s(x)} \mu(y).$$

For a function $f : \mathcal{V} \to \mathbb{C}$ and a subeset $A \subset \mathcal{V}$, the integral of f over A is defined as

$$\int_A f(y)d\mu(y) = \sum_{y\in A} f(y)\mu(y).$$

In the following discussion, we will always base our exposition on the flow measure μ and the definition of the integral. According to the definition of the flow measure, it is straightforward to observe that the measure of vertices increases with the level of the hierarchy, and the rate of growth accelerates progressively. Consequently, this measure clearly does not satisfy the doubling condition for balls defined in the usual sense. To obtain a relationship analogous to the doubling condition, it is necessary to employ two specific types of sets in place of balls.

Definition 2.2. [26] For $h_1, h_2 \in \mathbb{N}$, $h_1 < h_2$, the trapezoid rooted at $x \in \mathcal{V}$ is defined by

$$R(x) := R_{h_1}^{h_2}(x) := \{y \in \mathcal{V} : y \leq x, h_1 \leq d(x,y) < h_2\}.$$

It is worth noting that $\mu(R_{h_1}^{h_2}(x)) = (h_2 - h_1)\mu(x)$. Given a constant $\beta > 12$, the trapezoid R with root x is called admissible if $2 \leq \frac{h_2}{h_1} \leq \beta$ or $R = \{x\}$. Further, the envelope of $R(x)$ is defined as the set

$$\tilde{R}(x) := R_{\lceil \frac{h_1}{\beta} \rceil}^{h_2 \beta}(x).$$

This envelope is also referred to as the Calderón-Zygmund set. We denote by $\mathscr{R}$ the family of all trapezoids and $\tilde{\mathscr{R}}$ the family of all Calderón-Zygmund sets.

Next, several results on the space $(\mathcal{V}, d, \mu)$ from [22] and [26] that are required for this paper are presented.

Lemma 2.3. *Let R be an admissible trapezoid, then $\mu(\tilde{R}) \leq 2\beta\mu(R)$.*

Lemma 2.4. *Let R_1, $R_2 \in \mathscr{R}$ with roots x_1, x_2 respectively, such that $R_1 \cap R_2 \neq \emptyset$ and $\mu(x_1) \geq \mu(x_2)$. Then $R_2 \subset \tilde{R}_1$.*

Lemma 2.5. *Given $\lambda > 0$, $f \in L^1(\mathcal{V})$, set $\Omega = \{x \in \mathcal{V} : M_{\mathscr{R}} f(x) > \lambda\}$ and $\mathscr{S}_0 = \left\{R \in \mathscr{R} : \frac{1}{\mu(R)} \int_R |f(x)| d\mu(x) > \lambda\right\}$, then there exist pairwise disjoint admissible trapezoids $R_j \in \mathscr{S}_0$, $j \in \mathbb{N} \backslash \{0\}$ such that $\Omega \subset \bigcup_j \tilde{R}_j$.*

Lemma 2.6. *There exists a family $\{R_j\}_j \subset \mathscr{R}$ such that $R_j \subset R_{j+1}$ and $\cup_j R_j = \mathcal{V}$.*

Lemma 2.7. *Let R be an admissible trapezoid. Then $\tilde{R}$ is contained in the union of at most 4 admissible trapezoids $Q_1, \ldots, Q_4$ with $\mu(Q_i) \approx_\beta \mu(R)$.*

Lemma 2.8. *Let μ be a flow measure. Assume that $x_0 = o$ and $x_{k+1} = p(x_k)$ for $k \geq 0$. For every $r \in \mathbb{N}$, it holds*

$$\mu(S_r(o)) = \mu(x_{r-1}) + \mu(x_r),$$

where $S_r(o)$ represents the sphere centered at o with radius r and is denoted as $S_r(o) = \{x \in \mathcal{V} : d(x,o) = r\}$.

Lemma 2.9. *Let μ be a flow measure and satisfies locally doubling condition, then there exists a constant $c \geq 2$ such that $\mu(x) \leq c\mu(y)$, $\forall x \in \mathcal{V}, y \in s(x)$.*

2.2. Variable Lebesgue spaces.

Definition 2.10. Define $\mathcal{P}(\mathcal{V})$ as the set of all functions $p(\cdot) : \mathcal{V} \to [1, \infty]$. For given $p(\cdot) \in \mathcal{P}(\mathcal{V})$ and the set $E \in \mathcal{V}$, we define

$$p_-(E) := \inf_{x \in E} p(x), \ p_+(E) := \sup_{x \in E} p(x).$$

For convenience, we write $p_- = p_-(\mathcal{V}), \ p_+ = p_+(\mathcal{V})$.

Next, we define three canonical subsets of $\mathcal{V}$.

Definition 2.11. Let $p(\cdot) \in \mathcal{P}(\mathcal{V})$, define

$$\mathcal{V}_1 := \{x \in \mathcal{V} : p(x) = 1\}, \ \mathcal{V}_\infty := \{x \in \mathcal{V} : p(x) = \infty\}, \ \mathcal{V}_* := \mathcal{V} \backslash (\mathcal{V}_1 \cup \mathcal{V}_\infty).$$

Definition 2.12. Given $p(\cdot) \in \mathcal{P}(\mathcal{V})$ and a function $f : \mathcal{V} \to \mathbb{C}$, the modular functional of f associated with $p(\cdot)$ is defined as

$$\rho_{p(\cdot)}(f) = \int_{\mathcal{V} \backslash \mathcal{V}_\infty} |f(x)|^{p(x)} d\mu(x) + \|f\|_{L^\infty(\mathcal{V}_\infty)}.$$

If f is unbounded on $\mathcal{V}_\infty$, we define $\rho_{p(\cdot)}(f) = \infty$.

Remark 2.13. Let $p(\cdot) \in \mathcal{P}(\mathcal{V})$, according to the aforementioned definition, on one hand, if $p_+(\mathcal{V} \backslash \mathcal{V}_\infty) < \infty$, then for $\lambda \geq 1$,

$$\rho_{p(\cdot)}(\lambda f) \leq \lambda^{p_+(\mathcal{V} \backslash \mathcal{V}_\infty)} \rho_{p(\cdot)}(f).$$

On the other hand, if $p_+ < \infty$, then for $\lambda \geq 1$,

$$\lambda^{p_-} \rho_{p(\cdot)}(f) \leq \rho_{p(\cdot)}(\lambda f) \leq \lambda^{p_+} \rho_{p(\cdot)}(f).$$

For $0 < \lambda < 1$, the converse inequalities hold.

Definition 2.14. Given $p(\cdot) \in \mathcal{P}(\mathcal{V})$ and $f : \mathcal{V} \to \mathbb{C}$, the variable Lebesgue space $L^{p(\cdot)}(\mathcal{V})$ is defined as the set of all functions f such that $\rho_{p(\cdot)}(f/\lambda) < \infty$ for some $\lambda > 0$. Futher, we define the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\mathcal{V})} := \inf \left\{ \lambda > 0 : \rho_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}.$$

If the set on the right-hand side of the equation is the empty set, we write $\|f\|_{L^{p(\cdot)}(\mathcal{V})} = \infty$. We always write $\|f\|_{L^{p(\cdot)}(\mathcal{V})} = \|f\|_{p(\cdot)}$.

Remark 2.15. Obviously, $\rho_{p(\cdot)} \left(\frac{f}{\lambda} \right)$ is continuous and monotonically decreasing with respect to λ. Further, $\rho_{p(\cdot)} \left(\frac{f}{\lambda} \right) \to 0$ as $\lambda \to \infty$. Therefore, $f \in L^{p(\cdot)}(\mathcal{V})$ is equivalent to $\|f\|_{p(\cdot)} < \infty$.

Remark 2.16. Let $(\mathcal{V}, d, \mu)$ be a metric measure space. Suppose $p(\cdot) \in \mathcal{P}(\mathcal{V})$, then $\| \cdot \|_{p(\cdot)}$ is a norm and $L^{p(\cdot)}(\mathcal{V})$ is a Banach space from [7].

Next, we discuss some fundamental results related to exponent functions $p(\cdot)$ and the norm $\| \cdot \|_{p(\cdot)}$. The proofs of these results are almost identical to those in [7], and therefore, we omit the detailed proof here.

Lemma 2.17. *Let $p(\cdot) \in \mathcal{P}(\mathcal{V})$, then $\rho_{p(\cdot)}$ and $\|\cdot\|_{p(\cdot)}$ have the following properties:*

(i) *If $|f(x)| \le |g(x)|$ for all $x \in \mathcal{V}$, then $\rho_{p(\cdot)}(f) \le \rho_{p(\cdot)}(g)$ and $\|f\|_{p(\cdot)} \le \|g\|_{p(\cdot)}$.*

(ii) *For $\lambda > 0$,*

$$\rho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \le \frac{\rho_{p(\cdot)}(f)}{\lambda}.$$

(iii) *If $\|f\|_{p(\cdot)} \le 1$, then $\rho_{p(\cdot)}(f) \le \|f\|_{p(\cdot)}$. If $\|f\|_{p(\cdot)} > 1$, then $\rho_{p(\cdot)}(f) \ge \|f\|_{p(\cdot)}$.*

(iv) $\frac{1}{2}\|f\|_{p(\cdot)} \le \sup_{\substack{g\in L^{p'(\cdot)},\\ \|g\|_{p'(\cdot)}\le 1}} \int_{\mathcal{V}} |f(x)||g(x)|\, d\mu(x) \le 2\|f\|_{p(\cdot)}$.

As a corollary of the third property, the following relationship between norm and modular holds.

Lemma 2.18. *Let $p(\cdot) \in \mathcal{P}(\mathcal{V})$. For any set $E \subset \mathcal{V}$, the following inequalities hold*

$$\min\left\{\|f\chi_E\|_{p(\cdot)}^{p_-(E)}, \|f\chi_E\|_{p(\cdot)}^{p_+(E)}\right\} \le \rho_{p(\cdot)}(f\chi_E) \le \max\left\{\|f\chi_E\|_{p(\cdot)}^{p_-(E)}, \|f\chi_E\|_{p(\cdot)}^{p_+(E)}\right\}.$$

The following lemma is about the generalized Hölder's inequality for the norm of variable Lebesgue spaces.

Lemma 2.19. *Let $p(\cdot)$, $r(\cdot)$, $q(\cdot) \in \mathcal{P}(\mathcal{V})$ satisfy*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Then there exists a constant L such that for all $f \in L^{q(\cdot)}(\mathcal{V})$ and $g \in L^{r(\cdot)}(\mathcal{V})$, $fg \in L^{p(\cdot)}(\mathcal{V})$ with the norm

$$\|fg\|_{p(\cdot)} \le L\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

Definition 2.20. Given a function $r(\cdot) : \mathcal{V} \to [0,\infty]$. $r(\cdot)$ is called globally Hölder continuous, if there exists a constant $B > 0$ and $r_0 \in [1,\infty)$ such that, for all $x \in \mathcal{V}$,

$$|r(x) - r_0| \le \frac{B}{1 + d(x,o)}.$$

We denote $r(\cdot)$ satisfying the aforementioned conditions as $r(\cdot) \in GH(\mathcal{V})$.

Remark 2.21. Let $r(\cdot) \in \mathcal{P}(\mathcal{V})$. If $r_+ < \infty$, then $r(\cdot) \in GH(\mathcal{V})$ is equivalent to $\frac{1}{r(\cdot)} \in GH(\mathcal{V})$, i.e., there exists a constant B' such that, for all $x \in \mathcal{V}$,

$$\left|\frac{1}{r(x)} - \frac{1}{r_0}\right| \le \frac{B'}{1 + d(x,o)}.$$

In fact, for given $x \in \mathcal{V}$,

$$\left|\frac{r(x) - r_0}{(r_+)^2}\right| \le \left|\frac{1}{r(x)} - \frac{1}{r_0}\right| \le \left|\frac{r(x) - r_0}{(r_-)^2}\right|.$$

Lemma 2.22. *Let $r(\cdot) \in GH(\mathcal{V})$, and assume $r_+ < \infty$. Then there exists a constant $C > 0$ such that, for any admissible trapezoid $R \subset \mathcal{V}$ and $x \in R$,*

$$\mu(R)^{r(x)-r_+(R)} \leq C, \quad \mu(R)^{r_-(R)-r(x)} \leq C.$$

Proof. Given an admissible trapezoid R, according to the definition of admissible trapezoids, it is easy to know that there exists at least one $y \in R$ such that $r(y) = r_+(R)$.

For $\mu(R) \geq 1$, the conclusion clearly holds.

For $\mu(R) < 1$, by Definition 2.20,

$$|r(x) - r(y)| \leq |r(x) - r_0| + |r(y) - r_0| \leq \frac{2B}{1 + d(z, o)},$$

where z is x or y such that $d(z, o) = \min\{d(x, o), d(y, o)\}$. Then, by Lemma 2.9

$$\mu(R)^{r(x)-r_+(R)} = \mu(R)^{-|r(x)-r(y)|} \leq \mu(R)^{-\frac{2B}{1+d(z,o)}}$$
$$\leq \mu(z)^{-\frac{2B}{1+d(z,o)}} \leq \left(c^{-d(z,o)}\mu(o)\right)^{-\frac{2B}{1+d(z,o)}} \leq C.$$

The proof of the second inequality is quite similar to that of the first one, hence we have omitted it for brevity. □

Remark 2.23. Using the same method of proof, it follows that the above conclusion also holds for $\tilde{R}$.

Lemma 2.24. *Consider a function $r(\cdot) : \mathcal{V} \to [0, \infty)$ such that $r(\cdot) \in GH(\mathcal{V})$ with $0 < r_-, r_0 < \infty$, and define $K(x) = c^{-Nd(x,o)}$, where c is defined in Lemma 2.9 and $N > 2/r_-$. There exists a constant C such that for any set E and any function F satisfying $0 \leq F(y) \leq 1$ for all $y \in E$, the following holds*

$$\int_E F(y)^{r(y)} d\mu(y) \leq C \int_E F(y)^{r_0} d\mu(y) + \int_E K(y)^{r_-} d\mu(y);$$

$$\int_E F(y)^{r_0} d\mu(y) \leq C \int_E F(y)^{r(y)} d\mu(y) + \int_E K(y)^{r_-} d\mu(y).$$

Proof. We will prove the first inequality; the proof of the second inequality follows a nearly identical approach. Denote E as $E = E_1 \cup E_2$, where $E_1 = \{y \in E : F(y) \leq K(y)\}$ and $E_2 = \{y \in E : K(y) < F(y)\}$. Then

$$\int_{E_1} F(y)^{r(y)} d\mu(y) \leq \int_{E_1} K(y)^{r(y)} d\mu(y) \leq \int_{E_1} K(y)^{r_-} d\mu(y).$$

Furthermore, by the $GH(\mathcal{V})$ condition, we have

$$K(y)^{-|r(y)-r_0|} = c^{N|r(y)-r_0|d(y,o)} \leq c^{NB}.$$

From $F(y) \le 1$,

$$\begin{aligned}\int_{E_2} F(y)^{r(y)} d\mu(y) &\le \int_{E_2} F(y)^{r_0} F(y)^{-|r(y)-r_0|} d\mu(y)\\ &\le \int_{E_2} F(y)^{r_0} K(y)^{-|r(y)-r_0|} d\mu(y)\\ &\le c^{NB} \int_{E_2} F(y)^{r_0} d\mu(y).\end{aligned}$$

□

Remark 2.25. The condition $N > 2/r_-$ ensures that the last term on the right-hand side of both inequalities, $\int_E K(y)^{r_-} d\mu(y) = \sum_{y\in E} K(y)^{r_-}\mu(y)$, is convergent. Indeed, by Lemma 2.8 and Lemma 2.9, it follows that

$$\begin{aligned}&\int_E K(y)^{r_-} d\mu(y)\\ &\le \int_{\mathcal{V}} \frac{d\mu(y)}{c^{2d(y,o)}} \le \mu(o) + \sum_{k=1}^{\infty} \int_{S_k(o)} \frac{d\mu(y)}{c^{2d(y,o)}} \le \mu(o) + \sum_{k=1}^{\infty} \frac{\mu(x_k) + \mu(x_{k-1})}{c^{2k}}\\ &\le \mu(o) + \sum_{k=1}^{\infty} \frac{2\mu(x_k)}{c^{2k}} \le \mu(o) + \sum_{k=1}^{\infty} \frac{2c^k \mu(o)}{c^{2k}} < \infty.\end{aligned}$$

Remark 2.26. The same argument shows that, for any nonnegative function $h(y)$, the following inequalities are valid

$$\int_E F(y)^{r(y)} h(y) d\mu(y) \le C \int_E F(y)^{r_0} h(y) d\mu(y) + \int_E K(y)^{r_-} h(y) d\mu(y);$$

$$\int_E F(y)^{r_0} h(y) d\mu(y) \le C \int_E F(y)^{r(y)} h(y) d\mu(y) + \int_E K(y)^{r_-} h(y) d\mu(y).$$

2.3. $A_{p(\cdot),\mathscr{R}}$ condition and weighted variable Lebesgue spaces. First, let's review the definition of $A_{p,\mathscr{R}}$ and $A_{\infty,\mathscr{R}}$ over $(\mathcal{V}, d, \mu)$.

Definition 2.27. [26] Let $p \in (1, \infty)$ and W be a weight. W is called an $A_{p,\mathscr{R}}$ weight if

$$[W]_{A_{p,\mathscr{R}}} := \sup_{R\in\mathscr{R}} \left(\frac{1}{\mu(R)} \int_R W(y) d\mu(y)\right) \left(\frac{1}{\mu(R)} \int_R W(y)^{-1/(p-1)} d\mu(y)\right)^{p-1} < \infty,$$

W is called an $A_{1,\mathscr{R}}$ weight if

$$[W]_{A_{1,\mathscr{R}}} := \sup_{R\in\mathscr{R}} \left(\frac{1}{\mu(R)} \int_R W(y) d\mu(y)\right) \|W^{-1}\|_{L^\infty(R)} < \infty,$$

and $A_{\infty,\mathscr{R}} := \bigcup_{p>1} A_{p,\mathscr{R}}$.

Lemma 2.28. [26] *Let $W \in A_{p,\mathscr{R}}$ for some $p \in [1,\infty)$. Then for each $R \in \mathscr{R}$, there exists a constant $C > 0$ such that*

$$W(\tilde{R}) \leq C[W]^2_{A_{p,\mathscr{R}}} W(R).$$

The following statements are about the properties of $A_{\infty,\mathscr{R}}$ and the proof of these properties are same as [13], so we omit details.

Lemma 2.29. *Let W be a weight. The following statements are equivalent:*

(i) $W \in A_{\infty,\mathscr{R}}$.

(ii) *There exist $0 < \theta \leq 1 \leq C < \infty$ such that for each $R \in \mathscr{R}$ and each set $S \subset R$*

$$\frac{\mu(S)}{\mu(R)} \leq C\left(\frac{W(S)}{W(R)}\right)^{\theta}.$$

(iii) *For any $\alpha \in (0,1)$, there exists $\beta \in (0,1)$ such that for all $R \in \mathscr{R}$ and each set $S \subset R$*

$$\mu(S) \geq \alpha\mu(R) \Longrightarrow W(S) \geq \beta W(R).$$

(iv) *For any $\alpha \in (0,1)$, there exists $\beta \in (0,1)$ such that for all $R \in \mathscr{R}$ and each set $S \subset R$*

$$\mu(S) \leq \alpha\mu(R) \Longrightarrow W(S) \leq \beta W(R).$$

Definition 2.30. Given $p(\cdot) \in \mathcal{P}(\mathcal{V})$ and a weight ω. We say $\omega \in A_{p(\cdot),\mathscr{R}}$ if there exists a postive constant C such that

$$\|\omega\chi_R\|_{p(\cdot)}\|\omega^{-1}\chi_R\|_{p'(\cdot)} \leq C\mu(R),$$

where $p'(\cdot)$ is the conjugate exponent function such that $1/p(\cdot) + 1/p'(\cdot) = 1$.

Remark 2.31. It is evident from the definition of $A_{p(\cdot),\mathscr{R}}$ that if $\omega \in A_{p(\cdot),\mathscr{R}}$, then $\omega^{-1} \in A_{p'(\cdot),\mathscr{R}}$.

Lemma 2.32. *Given $p(\cdot) \in \mathcal{P}(\mathcal{V})$, if $\omega \in A_{p(\cdot),\mathscr{R}}$, then there exists a constant C depending on $p(\cdot)$ and ω such that given any admissible trapezoid $R \in \mathscr{R}$ and $S \subset R$,*

$$\frac{\mu(S)}{\mu(R)} \leq C\frac{\|\omega\chi_S\|_{p(\cdot)}}{\|\omega\chi_R\|_{p(\cdot)}}.$$

Proof. For any given admissible trapezoid $R \in \mathscr{R}$ and set $S \subset R$, by generalized Hölder's inequality and the $A_{p(\cdot),\mathscr{R}}$ condition, we get

$$\begin{aligned}\mu(S) &= \int_{\mathcal{V}} \omega(x)\chi_S(x)\omega(x)^{-1}\chi_R(x)d\mu(x) \leq C\|\omega\chi_S\|_{p(\cdot)}\|\omega^{-1}\chi_R\|_{p'(\cdot)} \\ &\leq C\|\omega\chi_S\|_{p(\cdot)}\|\omega\chi_R\|^{-1}_{p(\cdot)}\mu(R).\end{aligned}$$

□

Lemma 2.33. *Given an exponent function $p(\cdot)$ satisfying the globally Hölder condition $GH(\mathcal{V})$, if $\omega \in A_{p(\cdot),\mathscr{R}}$, then there exists a constant C depending on $p(\cdot)$ and ω such that for all admissible trapezoids $R \in \mathscr{R}$,*

$$\|\omega\chi_R\|_{p(\cdot)}^{p_-(R)-p_+(R)} \le C.$$

Proof. Fix an admissible trapezoid $R \in \mathscr{R}$. Obviously it suffices to assume that $\|\omega\chi_R\|_{p(\cdot)} \le 1$. Set

$$Q_0 = R_{h_1}^{h_2}(x_0) = R_{h_1}^{h_2}(o),\ Q_n = R_{h_1+n}^{h_2+3n}(x_{2n}),\ h_1,\ h_2,\ n \in \mathbb{N}.$$

It is easy to show that Q_n is an admissible trapezoid and $Q_0 \subset Q_1 \subset \cdots Q_n \subset \cdots$. From this construction, there exists $n_0 \in \mathbb{N}$ such that $o \in Q_{n_0}$. Let

$$R_0 = Q_{n_0} = R_{h_1+n_0}^{h_2+3n_0}(x_{2n_0}),\ R_k = Q_{n_0+k},\ k \in \mathbb{N}.$$

Thus $o \in R_0 \subset \cdots \subset R_k$. It is observed that the only way to leave R_0 from any vertex within it is through its bases. Thus we define upper base $\overline{b}$ and lower base $\underline{b}$ of R_0 as follows

$$\overline{b} = \{x \in R_0 : l(x) = l(x_{2n_0}) - h_1 - n_0\},$$
$$\underline{b} = \{x \in R_0 : l(x) = l(x_{2n_0}) - h_2 - 3n_0 + 1\}.$$

Then, for any $r > 0$,

$$\{x \in \mathcal{V} : d(x, R_0) \le r\} = \{x \in \mathcal{V} : x \in R_0 \vee d(x, \overline{b}) \le r \vee d(x, \underline{b}) \le r\}.$$

On one hand, the nodes x that lie above R_0 and satisfy the condition $d(x, \overline{b}) \le r$ are the nodes above R_0 with the level $l(\overline{b}) \le l(x) \le l(\overline{b}) + r$, that is

$$l(x_{2n_0}) - h_1 - n_0 \le l(x) \le l(x_{2n_0}) - h_1 - n_0 + r.$$

It implies that

$$l(x) \le l(x_{2(n_0+r)}) - h_1 - n_0 - 2r + r = l(x_{2(n_0+r)}) - h_1 - n_0 - r.$$

On the other hand, the nodes x that lie below R_0 and satisfy the condition $d(x, \underline{b}) \le r$ are the nodes above R_0 with the level $l(\underline{b}) - r \le l(x) \le l(\overline{b})$, that is

$$l(x_{2n_0}) - h_2 - 3n_0 + 1 - r \le l(x) \le l(x_{2n_0}) - h_2 - 3n_0 + 1.$$

It implies that

$$l(x) \ge l(x_{2(n_0+r)}) - h_2 + 3n_0 - 2r - r + 1 = l(x_{2(n_0+r)}) - h_2 + 3n_0 - 3r + 1.$$

Therefore $\{x \in \mathcal{V} : d(x, R_0) \le r\} \subset R_r$. Further, it is follows that $B_r(o) \subset R_r$, where $B_r(o)$ represents the ball centered at o with radius r and is denoted as $B_r(o) = \{x \in \mathcal{V} : d(x, o) \le r\}$.

We will consider the case $\mu(R) \le \mu(R_0)$. If $\mu(R_0) \le \mu(R)$, only exchange the roles of R and R_0. Since $\mu(R) \le \mu(R_0)$, there exists a constant r_0 independent of R such that

$$\mu(x_R) \le \mu(R) \le \mu(R_0) \le \mu(x_{R_{r_0}}) \le \mu(R_{r_0}).$$

Write $P_k = R_{r_0+k}$, $k \in \mathbb{N}$. If $R \cap P_0 \neq \emptyset$, by Lemma 2.4, we have $R \subset \tilde{P}_0$. Futher, by Lemma 2.7, Hölder's inequality and the $A_{p(\cdot),\mathscr{R}}$ condition, it deduces that

$$\begin{aligned}\mu(R) &= \int_R \omega(x)\omega(x)^{-1} d\mu(x) \leq C\|\omega\chi_R\|_{p(\cdot)}\|\omega^{-1}\chi_R\|_{p'(\cdot)} \\ &\leq C\|\omega\chi_R\|_{p(\cdot)}\|\omega^{-1}\chi_{\tilde{P}_0}\|_{p'(\cdot)} \leq C\|\omega\chi_R\|_{p(\cdot)} \sum_{j=1}^{4} \|\omega^{-1}\chi_{P_0^j}\|_{p'(\cdot)} \\ &\leq C\|\omega\chi_R\|_{p(\cdot)} \sum_{j=1}^{4} \left(\mu(P_0^j)\|\omega\chi_{P_0^j}\|_{p(\cdot)}^{-1}\right),\end{aligned}$$

where P_0^j is the admissble trapezoid such that $\tilde{P}_0 \subset \bigcup_j^4 P_0^j$. Thus, by Lemma 2.22,

$$\begin{aligned}\|\omega\chi_R\|_{p(\cdot)}^{p_-(R)-p_+(R)} &\leq C\mu(R)^{p_-(R)-p_+(R)} \left(1 + \sum_{j=1}^{4} \left(\mu(P_0^j)\|\omega\chi_{P_0^j}\|_{p(\cdot)}^{-1}\right)\right)^{p_+(R)-p_-(R)} \\ &\leq C\left(1 + \sum_{j=1}^{4} \left(\mu(P_0^j)\|\omega\chi_{P_0^j}\|_{p(\cdot)}^{-1}\right)\right)^{p_+-p_-} \leq C.\end{aligned}$$

If $R \cap P_0 = \emptyset$. Let $d(o,R) = r' > r_0$, then when $k = r' - r_0$, $B_{r'}(0) \subset P_k$. It implies that $P_0 \cap P_k \neq \emptyset$ and $R \cap P_k \neq \emptyset$. Thus by Lemma 2.4, $P_0 \subset \tilde{P}_k$ and $R \subset \tilde{P}_k$. Repeating the above calculations and by Lemma 2.7, we can obtain

$$\mu(R) \leq C\|\omega\chi_R\|_{p(\cdot)} \sum_{j=1}^{4} \left(\mu(P_k^j)\|\omega\chi_{P_k^j}\|_{p(\cdot)}^{-1}\right) \leq C\|\omega\chi_R\|_{p(\cdot)}\mu(P_k) \sum_{j=1}^{4} \left(\|\omega\chi_{P_k^j}\|_{p(\cdot)}^{-1}\right),$$

where P_k^j is the admissble trapezoid such that $\tilde{P}_k \subset \bigcup_j^4 P_k^j$. Thus, by Lemma 2.22,

$$\begin{aligned}&\|\omega\chi_R\|_{p(\cdot)}^{p_-(R)-p_+(R)} \\ &\leq C\mu(R)^{p_-(R)-p_+(R)}\mu(P_k)^{p_+(R)-p_-(R)} \left(1 + \sum_{j=1}^{4} \left(\|\omega\chi_{P_k^j}\|_{p(\cdot)}^{-1}\right)\right)^{p_+(R)-p_-(R)}.\end{aligned}$$

We claim $\mu(P_k)^{p_+(R)-p_-(R)} \leq C$, then by Lemma 2.22, it follows that

$$\|\omega\chi_R\|_{p(\cdot)}^{p_-(R)-p_+(R)} \leq C.$$

Indeed, by the definition of admissible trapezoids, there exist $x_1, x_2 \in R$ such that $p_-(R) = p(x_1)$, $p_+(R) = p(x_2)$. By $GH(\mathcal{V})$ condition,

$$p_+(R) - p_-(R) \leq |p(x_2) - p_0| + |p(x_1) - p_0| \leq \frac{2B}{1 + d(z,o)},$$

where z is x_1 or x_2 such that $d(z,o) = \min\{d(x_1,o), d(x_2,o)\}$. Therefore,

$$\mu(P_k)^{p_+(R)-p_-(R)} \le \mu(P_k)^{\frac{2B}{1+d(z,o)}} \le C.$$

□

Lemma 2.34. *Given an exponent function $p(\cdot) \in \mathcal{P}(\mathcal{V})$ satisfying the globally Hölder condition $GH(\mathcal{V})$, if $\omega(x) \in A_{p(\cdot),\mathscr{R}}$, then $W(\cdot) = \omega(\cdot)^{p(\cdot)} \in A_{\infty,\mathscr{R}}$.*

Proof. For any given admissible trapezoid $R \in \mathscr{R}$ and $S \subset R$, by Lemma 2.29, it suffices to prove that there exists a constant $C > 0$ such that

$$\frac{\mu(S)}{\mu(R)} \le C\left(\frac{W(S)}{W(R)}\right)^{1/p_+}.$$

To compelete the proof, we consider three cases.

Case I. If $\|\omega\chi_R\|_{p(\cdot)} \le 1$, then $\|\omega\chi_S\|_{p(\cdot)} \le 1$, by Lemma 2.18, it follows that

$$\|\omega\chi_S\|_{p(\cdot)} \le W(S)^{\frac{1}{p_+(S)}} \le W(S)^{\frac{1}{p_+(R)}} \quad \text{and} \quad \|\omega\chi_R\|_{p(\cdot)} \ge W(R)^{\frac{1}{p_+(R)}},$$

which together with Lemmas 2.32 and 2.33 implies that

$$\begin{aligned}\frac{\mu(S)}{\mu(R)} &\le C\frac{\|\omega\chi_S\|_{p(\cdot)}}{\|\omega\chi_R\|_{p(\cdot)}^{\frac{p_-(R)}{p_+(R)}}\|\omega\chi_R\|_{p(\cdot)}^{1-\frac{p_-(R)}{p_+(R)}}} \le C\left(\frac{W(S)}{W(R)}\right)^{\frac{1}{p_+(R)}}\|\omega\chi_R\|_{p(\cdot)}^{\frac{p_-(R)}{p_+(R)}-1}\\ &\le C\left(\frac{W(S)}{W(R)}\right)^{\frac{1}{p_+(R)}} \le C\left(\frac{W(S)}{W(R)}\right)^{\frac{1}{p_+}}.\end{aligned}$$

Case II. If $\|\omega\chi_S\|_{p(\cdot)} \le 1 \le \|\omega\chi_R\|_{p(\cdot)}$, then again by Lemma 2.18 we get that

$$\|\omega\chi_S\|_{p(\cdot)} \le W(S)^{\frac{1}{p_+(S)}} \le W(S)^{\frac{1}{p_+(R)}} \quad \text{and} \quad \|\omega\chi_R\|_{p(\cdot)} \ge W(R)^{\frac{1}{p_+(R)}}.$$

By this and Lemma 2.32 yield that

$$\frac{\mu(S)}{\mu(R)} \le C\frac{\|\omega\chi_S\|_{p(\cdot)}}{\|\omega\chi_R\|_{p(\cdot)}} \le C\left(\frac{W(S)}{W(R)}\right)^{\frac{1}{p_+(R)}}.$$

Case III. It remains to consider the case that $1 \le \|\omega\chi_S\|_{p(\cdot)} \le \|\omega\chi_R\|_{p(\cdot)}$. Let $\lambda := \|\omega\chi_R\|_{p(\cdot)}$, using Lemma 2.24 to obtain that

$$\int_R \lambda^{-p_0}\omega(x)^{p(x)}d\mu(x) \le C\left(\int_R\left(\frac{\omega(x)}{\lambda}\right)^{p(x)}d\mu(x)\right) + \int_R \frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}}d\mu(x).$$

By Lemma 2.18, the first integral on the right-hand side equals 1. Now we estimate the second integral on the right-hand side. According the construction from Lemma 2.33,

$$\begin{aligned}&\int_{\mathcal{V}}\frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}}d\mu(x)\\&\leq\int_{R_0}\frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}}d\mu(x)+\sum_{k=1}^{\infty}\int_{R_k\setminus R_{k-1}}\frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}}d\mu(x)\\&\leq c^{-Np_-}W(R_0)+C\sum_{k=1}^{\infty}\frac{1}{c^{Np_-(k-1)}}W(R_k)\\&\leq c^{-Np_-}W(R_0)+C\sum_{k=1}^{\infty}\frac{1}{c^{Np_-(k-1)}}\max\left\{\|\omega\chi_{R_k}\|_{p(\cdot)}^{p_-},\|\omega\chi_{R_k}\|_{p(\cdot)}^{p_+}\right\}.\end{aligned}$$

By Lemma 2.32 and Lemma 2.9,

$$\|\omega\chi_{R_k}\|_{p(\cdot)}\leq C\frac{\mu(R_k)}{\mu(R_0)}\|\omega\chi_{R_0}\|_{p(\cdot)}\leq C\frac{h_2-h_1+2n_0+2k}{h_2-h_1+2n_0}c^{2k}.$$

Combining these two estimates we have

$$\int_{\mathcal{V}}\frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}}d\mu(x)\leq c^{-Np_-}W(R_0)+C\sum_{k=1}^{\infty}\left(\frac{h_2-h_1+2n_0+2k}{h_2-h_1+2n_0}\right)^{p_+}\frac{(c^{2k})^{p_+}}{c^{Np_-(k-1)}}.$$

By choosing N sufficiently large, we can make the right-hand side less than 1. From above argument, it follows that

$$W(R)^{\frac{1}{p_0}}\leq(C+1)^{\frac{1}{p_0}}\|\omega\chi_R\|_{p(\cdot)}.$$

We now repeat the above argument, replacing R with S and exchanging the roles of p_0 and $p(\cdot)$. Let $\lambda:=\|\omega\chi_S\|_{p(\cdot)}$. We also use Lemma 2.24 to obtain

$$1=\int_S\left(\frac{\omega(x)}{\lambda}\right)^{p(x)}d\mu(x)\leq C\int_S\lambda^{-p_0}\omega(x)^{p(x)}d\mu(x)+\int_S\frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}}d\mu(x).$$

The same argument as before implies that the second term on the right-hand of the above inequality is less than $1/2$. We rearrange terms and get

$$\|\omega\chi_S\|_{p(\cdot)}\leq CW(S)^{1/p_0}.$$

By Lemma 2.32, we conclude that

$$\frac{\mu(S)}{\mu(R)}\leq C\frac{\|\omega\chi_S\|_{p(\cdot)}}{\|\omega\chi_R\|_{p(\cdot)}}\leq C\left(\frac{W(S)}{W(R)}\right)^{1/p_0}\leq C\left(\frac{W(S)}{W(R)}\right)^{1/p_+}.$$

□

Corollary 2.35. *Given an exponent function $p(\cdot)\in\mathcal{P}(\mathcal{V})$ satisfying $GH(\mathcal{V})$ condition, if $\omega\in A_{p(\cdot),\mathscr{R}}$, R is an admissible trapezoid and $S\subset R$ satisfying $\|\omega\chi_S\|_{p(\cdot)}\geq 1$, then*

$$\frac{\mu(S)}{\mu(R)}\leq C\left(\frac{W(S)}{W(R)}\right)^{1/p_0}.$$

Corollary 2.36. *Given an exponent function $p(\cdot) \in \mathcal{P}(\mathcal{V})$ satisfying $GH(\mathcal{V})$ condition, if $\omega \in A_{p(\cdot),\mathscr{R}}$, R is an admissible trapezoid satisfying $\|\omega\chi_R\|_{p(\cdot)} \geq 1$, then*

$$\frac{1}{C}\|\omega\chi_R\|_{p(\cdot)} \leq W(R)^{\frac{1}{p_0}} \leq C\|\omega\chi_R\|_{p(\cdot)}.$$

Remark 2.37. According to Lemma 2.28, we also obtain $W(\tilde{R})^{\frac{1}{p_0}} \leq C\|\omega\chi_{\tilde{R}}\|_{p(\cdot)}$.

We now define weighted variable Lebesgue spaces as follows.

Definition 2.38. Let $p(\cdot) \in \mathcal{P}(\mathcal{V})$ and ω be a weight. The weighted variable Lebesgue space $L^{p(\cdot)}_{\omega}(\mathcal{V})$ on $(\mathcal{V}, d, \mu)$ is defined to be the set of all functions on $(\mathcal{V}, d, \mu)$ such that

$$\|f\|_{L^{p(\cdot)}_{\omega}(\mathcal{V})} := \|f\omega\|_{L^{p(\cdot)}(\mathcal{V})} < \infty.$$

Remark 2.39. According to the definition of $\|\cdot\|_{p(\cdot)}$, it is easily show that $\|\cdot\|_{L^{p(\cdot)}_{\omega}}$ is a norm.

3. The weighted norm inequalities for maximal operator

In this section, we show that the boundedness of maximal operator associated with admissible trapezoids on variable exponent Lebesgue spaces.

Definition 3.1. [22] The maximal operator $M_{\mathscr{R}}$ associated with admissible trapezoid $R \in \mathscr{R}$ is defined as

$$M_{\mathscr{R}}f(x) = \sup_{R\in\mathscr{R}, R\ni x} \frac{1}{\mu(R)} \int_R |f(y)|d\mu(y),$$

where the supremum is taken over all $R \in \mathscr{R}$ such that $x \in R$. Let σ be a weight. Define the maximal function $M_{\sigma,\mathscr{R}}$ associate with the weight σ as follows

$$M_{\sigma,\mathscr{R}}f(x) := \sup_{R\in\mathscr{R}, R\ni x} \frac{1}{\sigma(R)} \int_R |f(y)|\sigma(y)d\mu(y).$$

Lemma 3.2. [26] *Given a weight $\sigma \in A_\infty$, for $1 < p < \infty$ and $f \in L^p_\sigma(\mathcal{V})$,*

$$\int_{\mathcal{V}} M_{\sigma,\mathscr{R}}f(x)^p\sigma(x)d\mu(x) \leq C\int_{\mathcal{V}} |f(x)|^p\sigma(x)d\mu(x).$$

Remark 3.3. If $\mathscr{R}$ is replaced with $\tilde{\mathscr{R}}$, by extending $\tilde{R}$ in a manner similar to R in [22, Theorem 3.3], the aforementioned strong-type inequality for $\tilde{\mathscr{R}}$ can still hold.

Now, we state the boundedness of maximal operator associated with admissible trapezoids on variable Lebesgue spaces as our main results in this section.

Theorem 3.4. *Let ω be a weight. For $p(\cdot) : \mathcal{V} \to [1, \infty)$, assume that $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in GH(\mathcal{V})$, then*

$$\|(M_{\mathscr{R}}f)\omega\|_{p(\cdot)} \leq C\|f\omega\|_{p(\cdot)}$$

if and only if $\omega \in A_{p(\cdot),\mathscr{R}}$. *If* $p_- \geq 1$, *then the strong-type inequality can be replaced by the following weak-type inequality*

$$\|t\chi_{\{x\in\mathcal{V}:M_{\mathscr{R}}>t\}}\omega\|_{p(\cdot)} \leq C\|f\omega\|_{p(\cdot)},\ t>0.$$

3.1. Proof of necessarty of Theorem 3.1. Since the strong-type inequality implies the weak-type inequality, it suffices to prove that the latter ensures the $A_{p(\cdot),\mathscr{R}}$ condition. According to the definition of $A_{p(\cdot),\mathscr{R}}$, it suffices to prove there exists a constant $C>0$ such that for all $R\in\mathscr{R}$,

$$\|\omega\chi_R\|_{p(\cdot)}\|\omega^{-1}\chi_R\|_{p'(\cdot)} \leq C\mu(R).$$

Fix an admissible trapezoid $R\in\mathscr{R}$. Without loss of generality, we may assume that $\|\omega^{-1}\chi_R\|_{p'(\cdot)}=1$ by the homogeneity of $\|\cdot\|_{p'(\cdot)}$. Thus, we only need to prove that

$$\|\omega\chi_R\|_{p(\cdot)} \leq C\mu(R).$$

Define two sets as follows,

$$\Omega'_0=\{x\in R:\ p'(x)<\infty\} \quad \text{and} \quad \Omega'_\infty=\{x\in R:\ p'(x)=\infty\}.$$

From Definition 2.14, for $0<\lambda<1$, it follows that

$$1\leq \rho_{p'(\cdot)}\left(\frac{\omega^{-1}\chi_R}{\lambda}\right)=\int_{\Omega'_0}\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}d\mu(x)+\lambda^{-1}\|\omega^{-1}\chi_{\Omega'_\infty}\|_\infty.$$

On one hand, if $\lambda^{-1}\|\omega^{-1}\chi_{\Omega'_\infty}\|_\infty > \frac{1}{2}$, let b be a constant such that $b > \|\omega^{-1}\chi_{\Omega'_\infty}\|_\infty^{-1} = \inf_{x\in\Omega'_\infty}\omega(x)$. Then there exists a set $E\subset\Omega'_\infty$ and $\mu(E)>0$ such that $\omega(x)\leq b$ for $x\in E$. Set $f=\chi_E$. From $p'(x)=1$ on Ω'_∞,

$$\|f\omega\|_{p(\cdot)}=\|\omega\chi_E\|_{p(\cdot)}=\omega(E).$$

Since $M_{\mathscr{R}}f(x)\geq\frac{\mu(E)}{\mu(R)}$ for all $x\in R$, by the weak-type inequality, for $t<\frac{\mu(E)}{\mu(R)}$,

$$t\|\omega\chi_R\|_{p(\cdot)}\leq t\|\omega\chi_{\{x\in\mathcal{V}:M_{\mathscr{R}}>t\}}\|_{p(\cdot)}\leq C\|f\omega\|_{p(\cdot)}=C\omega(E).$$

Taking the supremum over all such t, we obtain that

$$\frac{\mu(E)}{\mu(R)}\|\omega\chi_R\|_{p(\cdot)}\leq C\omega(E).$$

Further,

$$\mu(R)^{-1}\|\omega\chi_R\|_{p(\cdot)}\leq\frac{C\omega(E)}{\mu(E)}=\frac{C\int_E\omega(x)d\mu(x)}{\mu(E)}\leq Cb.$$

Now, taking the infimum over all such b, it follows that

$$\mu(R)^{-1}\|\omega\chi_R\|_{p(\cdot)}\leq C\|\omega^{-1}\chi_{\Omega'_\infty}\|_\infty^{-1}\leq\frac{2C}{\lambda}\leq 4C.$$

On the other hand, assume that

$$\int_{\Omega'_0}\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}d\mu(x)>\frac{1}{2}.$$

Since $1 < (p_+)' \le p'(x) \le (p_-)' \le \infty$, to avoid this complication, define

$$F_N = \{x \in \Omega_0' : p'(x) < N\}, \quad N > 1.$$

Then there exists sufficiently large N such that

$$\frac{1}{2} < \int_{F_N} \left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu(x) < \lambda^{-(p_-)'} \int_{F_N} \omega(x)^{-p'(x)} d\mu(x) \le \lambda^{-(p_-)'} < \infty.$$

Since the integral of $\left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)}$ over F_N equals the sum of the value of it for each $x \in F_N$, we can take $E \subset F_N$ such that

$$\frac{1}{2} < \int_E \left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu(x) < 1.$$

Define $f(x) := \frac{\omega(x)^{-p'(x)}}{\lambda^{p'(x)-1}} \chi_E$, it is easy to check that

$$\frac{1}{2} < \rho_{p(\cdot)}(f\omega) = \int_E \left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)} d\mu(x) < 1.$$

Therefore, $\|f\omega\|_{p(\cdot)} \le 1$. Furthermore, for all $x \in R$,

$$M_{\mathscr{R}} f(x) \ge \frac{1}{\mu(R)} \int_R f(x) d\mu(x) = \frac{\lambda}{\mu(R)} \int_E \left(\frac{\omega(x)^{-1}}{\lambda}\right)^{p'(x)} \ge \frac{\lambda}{2\mu(R)}.$$

Taking $t < \frac{\lambda}{2\mu(R)}$, by the weak-type inequality,

$$t\|\omega\chi_R\|_{p(\cdot)} \le t\|\omega\chi_{\{x\in\mathcal{V}: M_{\mathscr{R}} f(x) > t\}}\|_{p(\cdot)} \le C\|f\omega\|_{p(\cdot)} \le C,$$

if we take the supremum over all such t, the $A_{p(\cdot),\mathscr{R}}$ condition is obtained as desired.

3.2. Proof of sufficiency of Theorem 3.1. For any given $f \in L^{p(\cdot)}_\omega(\mathcal{V})$, since $M_R f = M_R(|f|)$, we may assume that f is non-negative. We can also assume that $\|f\omega\|_{p(\cdot)} = 1$ by the homogeneity of $\|\cdot\|_{p(\cdot)}$. Define $\sigma(x) := \omega(x)^{-p'(x)}$, decompose $f = f_1 + f_2$, where $f_1 := f\chi_{\{x\in\mathcal{V}: f\sigma^{-1} > 1\}}$ and $f_2 := f\chi_{\{x\in\mathcal{V}: f\sigma^{-1} \le 1\}}$. Then $M_{\mathscr{R}} f(x) \le M_{\mathscr{R}} f_1(x) + M_{\mathscr{R}} f_2(x)$ for all $x \in \mathcal{V}$, and by Lemma 2.17,

$$\int_{\mathcal{V}} |f_i(x)|^{p(x)} \omega(x)^{p(x)} d\mu(x) \le \|f_i\omega\|_{p(\cdot)} \le \|f\omega\|_{p(\cdot)} \le 1. \tag{1}$$

By this and Lemma 2.18, it suffices to prove that there exists a positive constant C such that

$$I_i := \int_{\mathcal{V}} M_{\mathscr{R}} f_i(x)^{p(x)} \omega(x)^{p(x)} d\mu(x) < C, \quad i = 1, 2. \tag{2}$$

We first prove (2) for f_1. Let $A > 1$. For each $k \in \mathbb{Z}$, let

$$\Omega_k = \{x \in \mathcal{V} : M_{\mathscr{R}} f_1(x) > A^k\}.$$

Then $\mathcal{V} = \bigcup_{k\in\mathbb{Z}} \Omega_k \setminus \Omega_{k+1}$. Further, by Lemma 2.5, there exists a set of pairwise disjoint admissible trapezoids $\left\{R_j^k\right\}_j$ such that

$$\Omega_k \subset \bigcup_j \tilde{R}_j^k \quad \text{and} \quad \frac{1}{\mu\left(R_j^k\right)} \int_{R_j^k} f_1(y)d\mu(y) > A^{k-1}.$$

From the above argument, it is easily seen that

$$\frac{1}{\mu\left(\tilde{R}_j^k\right)} \int_{\tilde{R}_j^k} f_1(y)d\mu(y) > \frac{\mu\left(R_j^k\right)}{\mu\left(\tilde{R}_j^k\right)} \int_{R_j^k} f_1(y)d\mu(y) > \frac{A^{k-1}}{2\beta}.$$

For each k, define $\left\{S_j^k\right\}$ as follows

$$S_1^k = (\Omega_k \backslash \Omega_{k+1}) \cap \tilde{R}_1^k,\ S_j^k = \left[(\Omega_k \setminus \Omega_{k+1}) \cap \tilde{R}_j^k\right] \Big\backslash \left(\bigcup_{l=1}^{j-1} S_l^k\right),\ j \geq 2.$$

Therefore, the set $\left\{S_j^k\right\}$ is a pairwise disjoint family for all $j \in \mathbb{N}$ and $k \in \mathbb{Z}$, and $\Omega_k \backslash \Omega_{k+1} = \bigcup_j S_j^k$. Further,

$$\begin{aligned}
I_1 &= \sum_{k=1}^{\infty} \int_{\Omega_k \backslash \Omega_{k+1}} M_{\mathscr{R}} f_1(x)^{p(x)} \omega(x)^{p(x)} d\mu(x) \\
&\leq \sum_{k=1}^{\infty} \int_{\Omega_k \backslash \Omega_{k+1}} \left(A^{k+1}\right)^{p(x)} \omega(x)^{p(x)} d\mu(x) \\
&\leq C \sum_{k,j} \int_{S_j^k} \left(\frac{1}{\mu(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} f_1(y)d\mu(y) \right)^{p(x)} \omega(x)^{p(x)} d\mu(x) \\
&= C \sum_{k,j} \int_{S_j^k} \left(\int_{\tilde{R}_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)d\mu(y) \right)^{p(x)} \mu(\tilde{R}_j^k)^{-p(x)} \omega(x)^{p(x)} d\mu(x).
\end{aligned}$$

Note that $f_1\sigma^{-1} \geq 1$ or $f_1\sigma^{-1} = 0$ and $p(y) \geq 1$ for all $y \in \mathcal{V}$. By (1), we have

$$\begin{aligned}
\int_{\tilde{R}_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)d\mu(y) &\leq \int_{\tilde{R}_j^k} \left(f_1(y)\sigma(y)^{-1}\right)^{p(y)} \sigma(y)d\mu(y) \\
&= \int_{\tilde{R}_j^k} (f_1(y)\omega(y))^{p(y)} d\mu(y) \leq 1.
\end{aligned} \tag{3}$$

Define $p_{jk} = p_-(\tilde{R}_j^k)$. Since $p(x) \geq p_{jk} \geq p_-$ for $x \in S_j^k \subset \tilde{R}_j^k$, by (3) and Hölder's inequality, it deduces that

$$\begin{aligned}
&\sum_{k,j} \int_{S_j^k} \left(\int_{\tilde{R}_j^k} f_1(y)\sigma(y)^{-1}\sigma(y)d\mu(y) \right)^{p(x)} \mu(\tilde{R}_j^k)^{-p(x)}\omega(x)^{p(x)}d\mu(x) \\
&\leq \sum_{k,j} \left(\int_{\tilde{R}_j^k} \left(f_1(y)\sigma(y)^{-1}\right)^{\frac{p(y)}{p_{jk}}} \sigma(y)d\mu(y) \right)^{p_{jk}} \left(\int_{S_j^k} \mu(\tilde{R}_j^k)^{-p(x)}\omega(x)^{p(x)}d\mu(x) \right) \\
&\leq \sum_{k,j} \left(\frac{1}{\sigma(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} \left(f_1(y)\sigma(y)^{-1}\right)^{\frac{p(y)}{p_{jk}}} \sigma(y)d\mu(y) \right)^{p_{jk}} \left(\int_{S_j^k} \sigma(\tilde{R}_j^k)^{p_{jk}} \left(\frac{\mu(\tilde{R}_j^k)}{\omega(x)} \right)^{p(x)} d\mu(x) \right) \\
&\leq \sum_{k,j} \left(\frac{1}{\sigma(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} \left(f_1(y)\sigma(y)^{-1}\right)^{\frac{p(y)}{p_-}} \sigma(y)d\mu(y) \right)^{p_-} \left(\int_{S_j^k} \sigma(\tilde{R}_j^k)^{p_{jk}} \left(\frac{\mu(\tilde{R}_j^k)}{\omega(x)} \right)^{p(x)} d\mu(x) \right).
\end{aligned}$$

Since $\omega \in A_{p(\cdot),\mathscr{R}}$ implies that $\omega^{-1} \in A_{p'(\cdot),\mathscr{R}}$, by Lemma 2.34, we have $\sigma \in A_\infty$. Further, by Lemma 2.28 and Lemma 2.7,

$$\begin{aligned}
&\int_{S_j^k} \sigma(\tilde{R}_j^k)^{p_{jk}} \mu(\tilde{R}_j^k)^{-p(x)}\omega(x)^{p(x)}d\mu(x) \\
&\leq C \int_{S_j^k} \sigma(R_j^k)^{p_{jk}} \mu(R_j^k)^{-p(x)}\omega(x)^{p(x)}d\mu(x) \\
&\leq C \int_{\tilde{R}_j^k} \sigma(R_j^k)^{p_{jk}} \mu(R_j^k)^{-p(x)}\omega(x)^{p(x)}d\mu(x) \\
&\leq C \sum_{l=1}^{4} \int_{Q_{j,l}^k} \sigma(R_j^k)^{p_{jk}} \mu(Q_{j,l}^k)^{-p(x)}\omega(x)d\mu(x) \\
&= C \sum_{l=1}^{4} \left(\frac{\sigma(R_j^k)}{\sigma(Q_{j,l}^k)} \right)^{p_{jk}} \int_{Q_{j,l}^k} \sigma(Q_{j,l}^k)^{p_{jk}} \mu(Q_{j,l}^k)^{-p(x)}\omega(x)^{p(x)}d\mu(x).
\end{aligned}$$

We claim that

$$\int_{Q_{j,l}^k} \sigma(Q_{j,l}^k)^{p_{jk}} \mu(Q_{j,l}^k)^{-p(x)}\omega(x)^{p(x)}d\mu(x) \leq C\sigma(Q_{j,l}^k).$$

Then, by Lemma 3.2,

$$
\begin{aligned}
I_1 &\le C\sum_{k,j}\left(\frac{1}{\sigma(\tilde{R}_j^k)}\int_{\tilde{R}_j^k}\left(f_1(y)\sigma(y)^{-1}\right)^{\frac{p(y)}{p_-}}\sigma(y)d\mu(y)\right)^{p_-}\sum_{l=1}^{4}\sigma(Q_{j,l}^k)\\
&\le C\sum_{k,j}\sum_{l=1}^{4}\int_{Q_{j,l}^k}\left(M_{\sigma,\tilde{\mathscr{R}}}(f_1\sigma^{-1})(x)^{\frac{p(\cdot)}{p_-}}\right)^{p_-}\sigma(x)d\mu(x)\\
&\le C\int_{\mathcal{V}}\left(M_{\sigma,\tilde{\mathscr{R}}}(f_1\sigma^{-1})(x)^{\frac{p(\cdot)}{p_-}}\right)^{p_-}\sigma(x)d\mu(x)\\
&\le C\int_{\mathcal{V}}\left(f_1(x)\sigma(x)^{-1}\right)^{p(x)}\sigma(x)d\mu(x)\le C.
\end{aligned}
$$

To finalize the estimation for I_1, we need to prove, for all $j\in\mathbb{N}$ and $k\in\mathbb{Z}$,

$$
\int_{Q_{j,l}^k}\sigma(Q_{j,l}^k)^{p_{jk}}\mu(Q_{j,l}^k)^{-p(x)}\omega(x)^{p(x)}d\mu(x)\le C\sigma(Q_{j,l}^k). \tag{4}
$$

From the definition of $A_{p(\cdot),\mathscr{R}}$ and $\omega\in A_{p(\cdot),\mathscr{R}}$, there exists a constant $C>0$ such that

$$
\left\|\frac{\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}\omega\chi_{Q_{j,l}^k}}{\mu(Q_{j,l}^k)}\right\|_{p(\cdot)}\le C.
$$

By this and Lemma 2.18, it follows that

$$
\int_{Q_{j,l}^k}\left(\frac{\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}\omega(x)}{\mu(Q_{j,l}^k)}\right)^{p(x)}d\mu(x)\le C.
$$

Further, we change (4) to

$$
\left(\frac{\sigma(Q_{j,l}^k)}{\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}}\right)^{p_-(Q_{j,l}^k)}\int_{Q_{j,l}^k}\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{p_-(Q_{j,l}^k)-p(x)}\left(\frac{\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(x)}\omega(x)}{\mu(Q_{j,l}^k)}\right)^{p(x)}d\mu(x)
$$
$$
\le C\sigma(Q_{j,l}^k).
$$

Now, it suffices to prove that there exists a positive constant C such that

$$
\left(\frac{\sigma(Q_{j,l}^k)}{\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(x)}}\right)^{p_-(Q_{j,l}^k)}\le C\sigma(Q_{j,l}^k) \tag{5}
$$

and

$$
\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{p_-(Q_{j,l}^k)-p(x)}\le C. \tag{6}
$$

The aforementioned inequalities will be proved by dividing it into two cases.

For $\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)} \geq 1$. It is clear that (6) is valid. By Lemma 2.18, it deduces that

$$\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)} \geq \sigma(Q_{j,l}^k)^{\frac{1}{(p)'_+(Q_{j,l}^k)}}.$$

Hence,

$$\left(\frac{\sigma(Q_{j,l}^k)}{\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(x)}}\right)^{p_-(Q_{j,l}^k)} \leq C\left(\sigma(Q_{j,l}^k)^{1-\frac{1}{(p')_+(Q_{j,l}^k)}}\right)^{p_-(Q_{j,l}^k)} = C\sigma(Q_{j,l}^k),$$

where we have used the fact that $1 - \frac{1}{(p')_+(Q_{j,l}^k)} = \frac{1}{p_-(Q_{j,l}^k)}$.

For $\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)} < 1$. By Lemma 2.18, it deduces that

$$\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{(p')_+(Q_{j,l}^k)} \leq \sigma(Q_{j,l}^k) \leq \|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{(p')_-(Q_{j,l}^k)}.$$

From $\omega^{-1} \in A_{p'(\cdot),\mathscr{R}}$ and Lemma 2.33, we conclude that

$$\begin{aligned}
&\left(\frac{\sigma(Q_{j,l}^k)}{\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}}\right)^{p_-(Q_{j,l}^k)} \\
&\leq \left(\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{(p')_-(Q_{j,l}^k)-1}\right)^{p_-(Q_{j,l}^k)} \\
&= \|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{\left((p')_+(Q_{j,l}^k)-1\right)p_-(Q_{j,l}^k)} \|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{\left((p')_-(Q_{j,l}^k)-(p')_+(Q_{j,l}^k)\right)p_-(Q_{j,l}^k)} \\
&\leq C\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{\left((p')_+(Q_{j,l}^k)-1\right)p_-(Q_{j,l}^k)} \\
&\leq C\left(\sigma(Q_{j,l}^k)^{1-\frac{1}{(p')_+(Q_{j,l}^k)}}\right)^{p_-(Q_{j,l}^k)} = C\sigma(Q_{j,l}^k).
\end{aligned}$$

(5) is obtained. It remains to prove (6). By the definition of $p'(\cdot)$,

$$\begin{aligned}
&p(x) - p_-(Q_{j,l}^k) \\
&= \frac{p'(x)}{p'(x)-1} - \frac{(p')_+(Q_{j,l}^k)}{(p')_+(Q_{j,l}^k)-1} \\
&= \frac{(p')_+(Q_{j,l}^k) - p'(x)}{[p'(x)-1]\,[(p')_+(Q_{j,l}^k)-1]} \leq \frac{(p')_+(Q_{j,l}^k)-(p')_-(Q_{j,l}^k)}{[(p')_- - 1]^2}.
\end{aligned}$$

Therefore, from Lemma 2.33, it deduces that

$$\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{p_-(Q_{j,l}^k)-p(x)} \leq C\|\omega^{-1}\chi_{Q_{j,l}^k}\|_{p'(\cdot)}^{(p')_-(Q_{j,l}^k)-(p')_+(Q_{j,l}^k)} \leq C.$$

Now we prove (2) for I_2. Decompose the integral of Mf_2 as we did above for Mf_1 to get, with the same notation as before,

$$I_2 = \sum_{k=1}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} (Mf_2(x))^{p(x)} \omega(x)^{p(x)} d\mu(x)$$
$$\leq \sum_{k=1}^{\infty} \int_{\Omega_k \setminus \Omega_{k+1}} (A^{k+1})^{p(x)} \omega(x)^{p(x)} d\mu(x)$$
$$\leq C \sum_{k,j} \int_{S_j^k} \left(\frac{1}{\mu(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} f_2(y) d\mu(y) \right)^{p(x)} \omega(x)^{p(x)} d\mu(x).$$

By Lemma 2.6, fix an admissible trapezoid $R \in \{R_j\}_j$ such that $\sigma(R) < \infty$. Set

$$\mathcal{G}_1 := \{(k,j) : \tilde{R}_j^k \subset R\},$$
$$\mathcal{G}_2 := \{(k,j) : \tilde{R}_j^k \not\subset R\}.$$

Therefore, it follows that

$$I_2 \leq C \left(\sum_{(k,j)\in\mathcal{G}_1} + \sum_{(k,j)\in\mathcal{G}_2} \right) \int_{S_j^k} \left(\frac{1}{\mu(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} f_2(y) d\mu(y) \right)^{p(x)} \omega(x)^{p(x)} d\mu(x)$$
$$=: I_{21} + I_{22}.$$

For I_{21}. From $0 \leq f_2 \sigma^{-1} \leq 1$ and (4), it deduces that

$$I_{21} \leq \sum_{(k,j)\in\mathcal{G}_1} \int_{S_j^k} \left(\frac{1}{\mu(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} \sigma(y) d\mu(y) \right)^{p(x)} \omega(x)^{p(x)} d\mu(x)$$
$$\leq \sum_{(k,j)\in\mathcal{G}_1} \int_{S_j^k} \sigma(\tilde{R}_j^k)^{p(x)-p_{jk}} \sigma(\tilde{R}_j^k)^{p_{jk}} \mu(\tilde{R}_j^k)^{-p(x)} \omega(x)^{p(x)} d\mu(x)$$
$$\leq \sum_{(k,j)\in\mathcal{G}_1} (1+\sigma(\tilde{R}_j^k))^{p_+-p_-} \int_{S_j^k} \sigma(\tilde{R}_j^k)^{p_{jk}} \mu(\tilde{R}_j^k)^{-p(x)} \omega(x)^{p(x)} d\mu(x)$$
$$\leq C(1+\sigma(R))^{p_+-p_-} \sum_{(k,j)\in\mathcal{G}_1} \sum_{l=1}^{4} \sigma(Q_{j,l}^k)$$
$$\leq C(1+\sigma(R))^{p_+-p_-} \sigma(R).$$

For I_{22}. Since $\tilde{R}_j^k$ is a set composed of discrete points and μ is a flow measure, there exists an appropriate constant $B > 1$ such that

$$\sup_{x\in\tilde{R}_j^k} \mu(x) \leq B \inf_{x\in\tilde{R}_j^k} \mu(x). \tag{7}$$

To estimate I_{22} we need to divide $\mathcal{G}_2$ into two subsets:

$$\mathcal{G}_{21} = \{(k,j) \in \mathcal{G}_2 : \sigma(\tilde{R}_j^k) \leq 1\}, \quad \mathcal{G}_{22} = \{(k,j) \in \mathcal{G}_2 : \sigma(\tilde{R}_j^k) > 1\}.$$

Then

$$I_{22} \le C\left(\sum_{(k,j)\in\mathcal{G}_{21}} + \sum_{(k,j)\in\mathcal{G}_{22}}\right)\int_{S_j^k}\left(\frac{1}{\mu(\tilde{R}_j^k)}\int_{\tilde{R}_j^k} f_2(y)d\mu(y)\right)^{p(x)}\omega(x)^{p(x)}d\mu(x)$$
$$=: I_{22}^1 + I_{22}^2.$$

We first estimate I_{22}^1. Since $f\sigma^{-1}\le 1$, by Lemma 2.24, it deduces that

$$I_{22}^1 \le C\sum_{(k,j)\in\mathcal{G}_{21}}\int_{S_j^k}\left(\frac{1}{\sigma(\tilde{R}_j^k)}\int_{\tilde{R}_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)d\mu(y)\right)^{p_0}\left(\frac{\sigma(\tilde{R}_j^k)}{\mu(\tilde{R}_j^k)}\right)^{p(x)}\omega(x)^{p(x)}d\mu(x)$$
$$+ C\sum_{(k,j)\in\mathcal{G}_{21}}\int_{S_j^k}\frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}}\left(\frac{\sigma(\tilde{R}_j^k)}{\mu(\tilde{R}_j^k)}\right)^{p(x)}d\mu(x) =: I_{22}^{11} + I_{22}^{12}.$$

For $(j,k)\in\mathcal{G}_{21}$, by $\sigma(\tilde{R}_j^k)\le 1$ and (4), we obtain

$$I_{22}^{11} \le C\sum_{(k,j)\in\mathcal{G}_{21}}\left(\frac{1}{\sigma(\tilde{R}_j^k)}\int_{\tilde{R}_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)d\mu(y)\right)^{p_0}\int_{S_j^k}\sigma(\tilde{R}_j^k)^{p_{jk}}\left(\frac{\omega(x)}{\mu(\tilde{R}_j^k)}\right)^{p(x)}d\mu(x)$$
$$\le C\sum_{(k,j)\in\mathcal{G}_{21}}\left(\frac{1}{\sigma(\tilde{R}_j^k)}\int_{\tilde{R}_j^k} f_2(y)\sigma(y)^{-1}\sigma(y)d\mu(y)\right)^{p_0}\sum_{k=1}^{4}\sigma(Q_{j,l}^k).$$

From the proof of Lemma 2.34, we can choose N sufficiently large such that

$$\int_{\mathcal{V}}\frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}}d\mu(x)\le 1,$$

using a same argument as this, replacing $\omega(\cdot)^{p(\cdot)}$ by σ, we can also obtain

$$\int_{\mathcal{V}}\frac{\sigma(x)}{c^{Np_-d(x,o)}}d\mu(x)\le 1.$$

Therefore, from Lemma 3.2 and (1)

$$I_{22}^{11} \le C\sum_{(k,j)\in\mathcal{G}_{21}}\sum_{k=1}^{4}\int_{Q_{j,l}^k}\left(M_{\sigma,\mathscr{R}}(f_2\sigma^{-1})(x)\right)^{p_0}\sigma(x)d\mu(x)$$
$$\le C\int_{\mathcal{V}}\left(M_{\sigma,\mathscr{R}}(f_2\sigma^{-1})(x)\right)^{p_0}\sigma(x)d\mu(x)$$
$$\le C\int_{\mathcal{V}}(f_2(x)\sigma(x)^{-1})^{p_0}\sigma(x)d\mu(x)$$
$$\le C\int_{\mathcal{V}}(f_2(x)\sigma(x)^{-1})^{p(x)}\sigma(x)d\mu(x) + \int_{\mathcal{V}}\frac{\sigma(x)}{c^{Np_-d(x,o)}}d\mu(x)$$
$$\le C\int_{\mathcal{V}} f_2(x)^{p(x)}\omega(x)^{p(x)}d\mu(x) + \int_{\mathcal{V}}\frac{\sigma(x)}{c^{Np_-d(x,o)}}d\mu(x)\le C.$$

Furthermore, from the fact that $\sigma(\tilde{R}_j^k) \le 1$ and inequalities (4) and (7), it deduces that

$$\begin{aligned} I_{22}^{12} &\le C \sum_{(k,j)\in\mathcal{G}_{31}} \int_{S_j^k} \sigma(\tilde{R}_j^k)^{p_{jk}} \mu(\tilde{R}_j^k)^{-p(x)} \frac{\omega(x)}{c^{Np_-d(x,o)}} d\mu(x) \\ &\le C \sum_{(k,j)\in\mathcal{G}_{31}} \sup_{x\in S_j^k} c^{-Np_-d(x,o)} \int_{S_j^k} \sigma(\tilde{R}_j^k)^{p_{jk}} \mu(\tilde{R}_j^k)^{-p(x)} \omega(x)^{p(x)} d\mu(x) \\ &\le C \sum_{(k,j)\in\mathcal{G}_{31}} \sup_{x\in S_j^k} c^{-Np_-d(x,o)} \sum_{l=1}^{4} \sigma(Q_{j,l}^k) \\ &\le C \sum_{(k,j)\in\mathcal{G}_{31}} \sum_{l=1}^{4} \int_{Q_{j,l}^k} \frac{\sigma(x)}{c^{Np_-d(x,o)}} d\mu(x) \le C. \end{aligned}$$

Next we estimate the term I_{22}^2. By the Hölder's inequality

$$\int_{\tilde{R}_j^k} f_2(y)\, d\mu(y) \le C\|f_2\omega\|_{p(\cdot)} \|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p(\cdot)} \le C\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p(\cdot)}.$$

Thus, by Lemma 2.24, it follows that

$$\begin{aligned} &I_{22}^2 \\ &\le C \sum_{(k,j)\in\mathcal{G}_{22}} \int_{S_j^k} \left(\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}^{-1} \int_{\tilde{R}_j^k} f_2(y) d\mu(y) \right)^{p(x)} \left(\frac{\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}}{\mu(\tilde{R}_j^k)} \right)^{p(x)} \omega(x)^{p(x)} d\mu(x) \\ &\le C \sum_{(k,j)\in\mathcal{G}_{22}} \int_{S_j^k} \left(\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}^{-1} \int_{\tilde{R}_j^k} f_2(y) d\mu(y) \right)^{p_0} \left(\frac{\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}}{\mu(\tilde{R}_j^k)} \right)^{p(x)} \omega(x)^{p(x)} d\mu(x) \\ &\quad + C \sum_{(k,j)\in\mathcal{G}_{22}} \int_{S_j^k} \left(\frac{\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}}{\mu(\tilde{R}_j^k)} \right)^{p(x)} \frac{\omega(x)^{p(x)}}{c^{Np_-d(x,o)}} d\mu(x) \\ &=: I_{22}^{21} + I_{22}^{22}. \end{aligned}$$

From the fact of $\omega^{-1} \in A_{p'(\cdot),\mathscr{R}}$ and Corollary 2.36, it yields that

$$\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}^{-p_0} \sigma(\tilde{R}_j^k)^{p_0} \le C\sigma(\tilde{R}_j^k)^{-\frac{p_0}{p'_\infty}+p_0} \le C\sigma(\tilde{R}_j^k).$$

By this and (4), we further get

$$
\begin{aligned}
I_{22}^{21} &= C \sum_{(k,j)\in\mathcal{G}_{32}} \int_{S_j^k} \left(\frac{1}{\sigma(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} f_2(y)d\mu(y) \right)^{p_0} \|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}^{p(x)-p_0} \frac{\sigma(\tilde{R}_j^k)^{p_0}}{\mu(\tilde{R}_j^k)^{p(x)}} \omega(x)^{p(x)} d\mu(x) \\
&\le C \sum_{(k,j)\in\mathcal{G}_{32}} \left(\frac{1}{\sigma(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} f_2(y)d\mu(y) \right)^{p_\infty} \sigma(\tilde{R}_j^k) \int_{S_j^k} \left(\frac{\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}}{\mu(\tilde{R}_j^k)} \right)^{p(x)} \omega(x)^{p(x)} d\mu(x) \\
&\le C \sum_{(k,j)\in\mathcal{G}_{32}} \int_{S_j^k} \left(\frac{1}{\sigma(\tilde{R}_j^k)} \int_{\tilde{R}_j^k} f_2(y)d\mu(y) \right)^{p_\infty} \sum_{l=1}^{4} \sigma(Q_{j,l}^k).
\end{aligned}
$$

The final term, as in the estimate for I_{22}^{11}, is bounded by a constant. Since $\sigma(R_j^k) \ge C\sigma(\tilde{R}_j^k) \ge C$. Then by (7),

$$
\begin{aligned}
I_{22}^{22} &\le C \sum_{(k,j)\in\mathcal{G}_{32}} \sup_{x\in R_j^k} c^{-Np_- d(x,o)} \int_{\tilde{R}_j^k} \left(\frac{\|\omega^{-1}\chi_{\tilde{R}_j^k}\|_{p'(\cdot)}}{\mu(\tilde{R}_j^k)} \right)^{p(x)} \omega(x)^{p(x)} d\mu(x) \\
&\le C \sum_{(k,j)\in\mathcal{G}_{32}} \sup_{x\in R_j^k} c^{-Np_- d(x,o)} \sigma(\tilde{R}_j^k) \\
&\le C \int_{\mathcal{V}} \frac{\sigma(x)}{c^{Np_- d(x,o)}} d\mu(x) \le C.
\end{aligned}
$$

This completes the estimate for f_2 and so the proof of the sufficiency of the $A_{p(\cdot),\mathscr{R}}$ condition for the strong-type inequality. The proof of the weak-type inequality is quite similar to that of the strong-type inequality, with the difference being that p_- can be equal to 1. We can readily modify the proof of the strong-type inequality to demonstrate it.

Let $p_- = 1$. Define f_1 and f_2 as before, then for all t,

$$
\{x \in \mathcal{V} : M_{\mathscr{R}} f(x) > t\} \subset \{x \in \mathscr{R} : M_{\mathscr{R}} f_1(x) > t/2\} \cup \{x \in \mathcal{V} : M_{\mathscr{R}} f_2(x) > t/2\}.
$$

Therefore, it will suffice to prove that f_1 and f_2 each satisfy the weak-type inequality.

We first consider f_1. Fix $t > 0$, by Lemma 2.5, there exists a set of pairwise disjoint admissible trapezoids $\left\{R_j^k\right\}_j$ such that

$$
\{x \in \mathcal{V} : M_{\mathscr{R}} f_1(x) > t\} \subset \bigcup_j \tilde{R}_j,
$$

furthermore

$$
\begin{aligned}
&\int_{\mathcal{V}} t^{p(x)} \chi_{\{x\in\mathcal{V}: M_{\mathscr{R}} f_1(x) > t\}}(x) \omega(x)^{p(x)} d\mu(x) \\
&\le C \sum_j \int_{\tilde{R}_j} \left(\frac{1}{\mu(R_j)} \int_{R_j} f_1(y)d\mu(y) \right)^{p(x)} \omega(x)^{p(x)} d\mu(x).
\end{aligned}
$$

As is the case with the proof of I_1, we can obtain the following inequality.

$$\begin{aligned}&\sum_j \int_{\tilde{R}_j} \left(\frac{1}{\mu(R_j)} \int_{R_j} f_1(y) d\mu(y)\right)^{p(x)} \omega(x)^{p(x)} d\mu(x) \\ &\leq C \sum_j \left(\frac{1}{\sigma(R_j)} \int_{R_j} (f_1(y)\sigma(y)^{-1})^{p(y)} \sigma(y) d\mu(y)\right) \sigma(R_j) \\ &\leq C \int_{\mathcal{V}} \left(M_{\sigma,\mathscr{R}}(f_1\sigma^{-1})(x)^{p(\cdot)}\right) \sigma(x) d\mu(x) \\ &\leq C \int_{\mathcal{V}} \left(f_1(x)\sigma^{-1}(x)\right)^{p(x)} \sigma(x) d\mu(x) \\ &\leq C \int_{\mathcal{V}} (f_1(x)\omega(x))^{p(x)} d\mu(x) \leq C.\end{aligned}$$

The estimates for f_2, though more detailed, can be adapted in exactly the same manner to complete the proof of the weak-type inequality. It completes the proof.

4. Extrapolation and application

The purpose of this section is to obtain the extrapolation theorem for pairs of functions (f, g) in some family $\mathscr{F}$. We first present the extrapolation theorem for $A_{1,\mathscr{R}}$, and then, based on the extrapolation theorem for $A_{1,\mathscr{R}}$, derive the extrapolation theorem for $A_{\infty,\mathscr{R}}$. We begin the discussion with the key technique for proving the extrapolation theorem, namely the Rubio de Francia iterative algorithm.

Theorem 4.1. *Given $r(\cdot) \in \mathscr{P}(\mathcal{V})$, suppose that ω is a weight such that $M_{\mathscr{R}}$ is bounded on $L^{r(\cdot)}_{\omega}(\mathcal{V})$. For a positive function $h(x)$ satisfying $M_{\mathscr{R}}h(x) < \infty$, define*

$$\Pi h(x) := \sum_{k=0}^{\infty} \frac{M^k_{\mathscr{R}} h(x)}{2^k \|M_{\mathscr{R}}\|^k_{L^{r(\cdot)}_{\omega}(\mathcal{V})}},$$

where for $k > 0$, $M^k_{\mathscr{R}} = M_{\mathscr{R}} \circ \cdots \circ M_{\mathscr{R}}$ denotes the iteration of k-th maximal operator $M_{\mathscr{R}}$ and $M^0_{\mathscr{R}} h(x) = h(x)$. And then the following conclusion holds:

(i) $h(x) \leq \Pi h(x)$

(ii) $\|\Pi h(x)\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})} \leq 2\|h\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})}$.

(iii) $\Pi h \in A_{1,\mathscr{R}}$ *and* $[\Pi h]_{A_{1,\mathscr{R}}} \leq 2\|M_{\mathscr{R}}\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})}$.

Proof. (i) This is evident because the term when $k = 0$ is h.

(ii) By direct computation, we obtain

$$\|\Pi h\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})} \leq \sum_{k=0}^{\infty} \frac{\|M^k_{\mathscr{R}} h\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})}}{2^k \|M_{\mathscr{R}}\|^k_{L^{r(\cdot)}_{\omega}(\mathcal{V})}} \leq \sum_{k=1}^{\infty} 2^{-k} \|h\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})} = 2\|h\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})},$$

where we have utilized the boundedness of the maximal operator $M_{\mathscr{R}}$ on the weighted variable exponent Lebesgue space $L^{r(\cdot)}_{\omega}(\mathcal{V})$.

(iii) It suffices to derive the following inequality,

$$M_{\mathscr{R}}(\Pi h)(x) = M_{\mathscr{R}}\left(\sum_{k=0}^{\infty} \frac{M^k_{\mathscr{R}} h(x)}{2^k \|M_{\mathscr{R}}\|^k_{L^{r(\cdot)}_{\omega}(\mathcal{V})}}\right) \leq \sum_{k=0}^{\infty} \frac{M^{k+1}_{\mathscr{R}} h(x)}{2^k \|M_{\mathscr{R}}\|^k_{L^{r(\cdot)}_{\omega}(\mathcal{V})}} \leq 2\|M_{\mathscr{R}}\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})} \Pi h(x).$$

□

Remark 4.2. More generally, for fixed constants $\alpha > 0$ and $\beta \in \mathbb{R}$, and another weight ν, define the operator

$$Hh(x) := \left[\Pi\left(h^{\alpha}\nu^{\beta}\right)(x)\right]^{\frac{1}{\alpha}} \nu^{-\frac{\beta}{\alpha}}.$$

Then the following conclusions hold:

(i) $h(x) \leq Hh(x)$.

(ii) If $\eta(x) := \nu^{\beta/\alpha}\omega^{1/\alpha}$, then $\|Hh\|_{L^{\alpha r(\cdot)}_{\eta}(\mathcal{V})} \leq 2\|h\|_{L^{\alpha r(\cdot)}_{\eta}(\mathcal{V})}$.

(iii) $\nu^{\beta}(Hh)^{\alpha} \in A_{1,\mathscr{R}}$ and $[\nu^{\beta}(Hh)^{\alpha}]_{A_{1,\mathscr{R}}} \leq 2\|M_{\mathscr{R}}\|_{L^{r(\cdot)}_{\omega}(\mathcal{V})}$.

Remark 4.3. Theorem 4.1 also holds when $r(\cdot)$ is a positive constant.

Next, we present extrapolation theorem for $A_{1,\mathscr{R}}$.

Theorem 4.4. *Suppose that for some $q > 0$ and each $W \in A_{1,\mathscr{R}}$,*

$$\int_{\mathcal{V}} f(x)^q W(x) d\mu(x) \leq C \int_{\mathcal{V}} g(x)^q W(x) d\mu(x),\ (f,g) \in \mathscr{F}. \tag{8}$$

Given $p(\cdot) \in \mathscr{P}(\mathcal{V})$, if $p(\cdot) \in GH(\mathcal{V})$, $q < p_- \leq p_+ < \infty$ and $\omega^q \in A_{p(\cdot)/q,\mathscr{R}}$, then

$$\|f\|_{L^{p(\cdot)}_{\omega}(\mathcal{V})} \leq C\|g\|_{L^{p(\cdot)}_{\omega}(\mathcal{V})},\ (f,g) \in \mathscr{F}.$$

Proof. Without loss of generality, we may assume $0 < \|f\|_{L^{p(\cdot)}_{\omega}(\mathcal{V})}, \|g\|_{L^{p(\cdot)}_{\omega}(\mathcal{V})}$, as otherwise there is nothing to prove. By Lemma 2.17 and scaling, there exist $h_1 \in L^{(p(\cdot)/q)'}$ with $\|h_1\|_{(p(\cdot)/q)'} = 1$ such that

$$\|f\omega\|^q_{p(\cdot)} \leq C \int_{\mathcal{V}} f(x)^q \omega(x)^q d\mu(x) \leq C \int_{\mathcal{V}} f(x)^q \omega(x)^q H(h_1(x))(x) d\mu(x).$$

To apply equation (8), we need to ensure that the term on the right-hand side is bounded and that $\omega(x)^q H \in A_{1,\mathscr{R}}$. Indeed, it suffices to set $\alpha = 1$ and $\beta = q$ appropriately. First, when $\alpha = 1$ and $\beta = q$, it follows directly from the Remark 4.2 that $\omega(x)^q H \in A_{1,\mathscr{R}}$ is evidently satisfied. Furthermore, from $\omega^q \in$

$A_{p(\cdot)/q,\mathscr{R}}$, it follows that $\omega^{-q} \in A_{(p(\cdot)/q)',\mathscr{R}}$. By Lemma 2.19 and Theorem 4.1, it follows that

$$\int_{\mathcal{V}} f(x)^q \omega(x)^q H(h_1)(x) d\mu(x) \leq C\|f\omega\|_{p(\cdot)}^q \|Hh_1\|_{(p(\cdot)/p)'} \leq C\|f\omega\|_{p(\cdot)}\|h_1\|_{(p(\cdot)/p)'} < \infty.$$

Thus, by (8) and Lemma 2.19, the following inequality holds

$$\int_{\mathcal{V}} f(x)^q [\omega(x)^q Hh_1(x)] d\mu(x) \leq C \int_{\mathcal{V}} g(x)^q [\omega(x)^q Hh_1(x)] d\mu(x) \leq C\|g\omega\|_{p(\cdot)}^q \|Hh_1\|_{(p(\cdot)/q)'} \leq C\|g\omega\|_{p(\cdot)}.$$

The proof of Theorem 4.4 □

Next, we will present an interesting relationship between $A_{1,\mathscr{R}}$ and $A_{\infty,\mathscr{R}}$. Through this relationship, we can directly derive the extrapolation theorem for $A_{\infty,\mathscr{R}}$.

Lemma 4.5. *Suppose that for some $q > 0$ and every $W \in A_{\infty,\mathscr{R}}$, the following inequality holds*

$$\int_{\mathcal{V}} f(x)^q W(x) d\mu(x) \leq C \int_{\mathcal{V}} g(x)^q W(x) d\mu(x),\ (f, g) \in \mathscr{F}.$$

Then for all $0 < s < q$, $W \in A_{1,\mathscr{R}}$ and $(f, g) \in \mathscr{F}$, it deduces that

$$\int_{\mathcal{V}} f(x)^s W(x) d\mu(x) \leq C \int_{\mathcal{V}} g(x)^s W(x) d\mu(x).$$

Proof. Given $0 < s < q$ and $W \in A_{1,\mathscr{R}}$. Without loss of generality, we may assume that $0 < \|f\|_{L^s_W(\mathcal{V})}, \|g\|_{L^s_W(\mathcal{V})} < \infty$, as otherwise there is nothing to prove. Let $p_0 = q/s > 1$, then $W \in A_{1,\mathscr{R}} \subset A_{p_0',\mathscr{R}}$ and $M_{\mathscr{R}}$ is bounded on $L^{p_0'}_W(\mathcal{V})$. For $f, g \in L^s_W(\mathcal{V})$, define

$$H(x) = \Pi\left(\left(\frac{f}{\|f\|_{L^s_W(\mathcal{V})}}\right)^{\frac{s}{p_0'}} + \left(\frac{g}{\|g\|_{L^s_W(\mathcal{V})}}\right)^{\frac{s}{p_0'}}\right)(x).$$

By Theorem 4.1, it follows that

$$\left(\frac{f(x)}{\|f\|_{L^s_W(\mathcal{V})}}\right)^{\frac{s}{p_0'}} \leq H(x),\ \left(\frac{g(x)}{\|g\|_{L^s_W(\mathcal{V})}}\right)^{\frac{s}{p_0'}} \leq H(x), \tag{9}$$

and $H \in A_{1,\mathscr{R}}$. By Hölder's inequality,

$$\begin{aligned}&\int_{\mathcal{V}} f(x)^s W(x)d\mu(x)\\ &\leq \left(\int_{\mathcal{V}} f(x)^q H(x)^{-p_0} W(x)d\mu(x)\right)^{\frac{1}{p_0}} \left(\int_{\mathcal{V}} H(x)^{p_0'} W(x)d\mu(x)\right)^{\frac{1}{p_0'}}\\ &=: J_1 J_2.\end{aligned}$$

First, we estimate J_1. From $W, H \in A_{1,\mathscr{R}}$, $1 + p_0 > 1$ and the decomposition property of $A_{p,\mathscr{R}}$, it follows that $WH^{-p_0} = WH^{1-(1+p_0)} \subset A_{1+p_0,\mathscr{R}} \subset A_{\infty,\mathscr{R}}$. According to (9), it deduces that

$$\begin{aligned}&\int_{\mathcal{V}} f(x)^q H(x)^{-p_0} W(x)d\mu(x)\\ &\leq \|f\|_{L^s_W(\mathcal{V})}^{\frac{sp_0}{p_0'}} \int_{\mathcal{V}} f(x)^{q-\frac{sp_0}{p_0'}W(x)} d\mu(x) = \|f\|_{L^s_W(\mathcal{V})}^{sp_0} < \infty.\end{aligned}$$

Further, from (8), we have

$$J_1 \leq C\left(\int_{\mathcal{V}} g(x)^q H(x)^{-p_0} W(x)d\mu(x)\right)^{\frac{1}{p_0}} \leq C\int_{\mathcal{V}} g(x)^s W(x)d\mu(x).$$

For J_2, from Theorem 4.4, by direct computation, we obtain $J_2 \leq 4$. It completes the proof of Lemma 4.5 □

Theorem 4.6. *Suppose that for some $q > 0$ and each $W \in A_{\infty,\mathscr{R}}$,*

$$\int_{\mathcal{V}} f(x)^q W(x)d\mu(x) \leq C\int_{\mathcal{V}} g(x)^q W(x)d\mu(x),\ (f, g) \in \mathscr{F}. \tag{10}$$

Given $p(\cdot) \in GH(\mathcal{V})$, if there exists $s \in (0, p_-)$ such that $\omega^s \in A_{p(\cdot)/s,\mathscr{R}}$, then

$$\|f\|_{L^{p(\cdot)}_\omega(\mathcal{V})} \leq C\|g\|_{L^{p(\cdot)}_\omega(\mathcal{V})},\ (f, g) \in \mathscr{F}.$$

Proof. According to Lemma 4.5, by replacing q with s, (10) holds for all $W \in A_{1,\mathscr{R}}$. Combining this with Theorem 4.4, we obtain the desired conclusion. □

Below, as an application of the extrapolation theorem, we will consider the weighted norm inequalities for a class of sharp maximal operators. First, recall that the definition of the sharp maximal operator is given by

$$M^\sharp f(x) = \sup_{R \ni x} \frac{1}{\mu(R)} \int_R |f(x) - f_R| d\mu(x),$$

where the supremum is taken over all $R \in \mathscr{R}$ such that $x \in R$, and f_R denotes the integral average of f, that is $f_R = \frac{1}{\mu(R)} \int_R f(x)d\mu(x)$.

In [22], the authors have established the following "good-λ" inequality.

Lemma 4.7. *For all $\gamma > 0$, $\lambda > 0$ and $f : \mathcal{V} \to \mathbb{C}$, the following inequality holds*

$$\mu(\{x \in \mathcal{V} : M_{\mathcal{D}}f(x) > 2\lambda, M^{\sharp}f(x) < \gamma\lambda\}) \leq C\gamma\mu(\{x \in \mathcal{V} : M_{\mathcal{D}}f(x) > \lambda\}),$$

where $M_{\mathcal{D}}$ denotes the dyadic maximal function, for details, see [22].

Based on the property (iii) of $A_{\infty,\mathscr{R}}$, we can further derive the following result.

Lemma 4.8. *Given $W \in A_{\infty,\mathscr{R}}$, for all $\lambda > 0$ and $\lambda > 0$, the following inequality holds*

$$W(\{x \in \mathcal{V} : M_{\mathcal{D}}f(x) > 2\lambda, M^{\sharp}f(x) \leq \gamma\lambda\}) \leq C\gamma W(\{x \in \mathcal{V} : M_{\mathcal{D}}f(x) > \lambda\}).$$

Our objective is to apply the extrapolation theorem (Theorem 4.6) to demonstrate the comparability of $M^{\sharp}$ and f under the $L_W^{p(\cdot)}(\mathcal{V})$ norm. To this end, we first present the following conclusion.

Lemma 4.9. *Assume $0 < q_0 < \infty$, $W \in A_{\infty,\mathscr{R}}$. Let $q_0 \leq q < \infty$, then there exists a constant $C > 0$ such that for all $M_{\mathcal{D}}f \in L_W^{q_0}(\mathcal{V})$,*

$$\int_{\mathcal{V}} M_{\mathcal{D}}f(x)^q W(x)d\mu(x) \leq C\int_{\mathcal{V}} M^{\sharp}f(x)^q W(x)d\mu(x). \tag{11}$$

Proof. Without loss of generality, we assume $\int_{\mathcal{V}}(M^{\#}f)^q W(x)d\mu(x) < \infty$, as otherwise there would be nothing to prove. For $N > 0$, let

$$I_N = \int_0^N q\lambda^{q-1}W(\{x \in \mathcal{V} : M_{\mathcal{D}}f(x) > \lambda\})d\lambda.$$

From

$$I_N \leq \frac{q}{q_0}N^{q-q_0}\int_0^N q_0\lambda^{q_0-1}W(\{x \in \mathcal{V} : M_{\mathcal{D}}f(x) > \lambda\})d\lambda < \infty,$$

and $M_{\mathcal{D}}f \in L_W^{q_0}(\mathcal{V})$, it follows that I_N is finite. Furthermore,

$$\begin{aligned}
I_N &= 2^q\int_0^{N/2} q\lambda^{q-1}W(\{x \in \mathcal{V} : M_{\mathcal{D}}f(x) > 2\lambda\})d\lambda \\
&\leq 2^q\int_0^{N/2} q\lambda^{q-1}W(\{x \in \mathcal{V} : M_{\mathcal{D}}f(x) > 2\lambda, M^{\sharp}f(x) \leq \gamma\lambda\})d\lambda \\
&\quad + 2^q\int_0^{N/2} q\lambda^{q-1}W(\{x \in \mathcal{V} : M^{\sharp}f(x) > \gamma\lambda\})d\lambda \\
&\leq C\gamma I_N + \frac{2^q}{\gamma^q}\int_0^{\gamma N/2} q\lambda^{q-1}W(\{x \in \mathcal{V} : M^{\sharp}f(x) > \lambda\})d\lambda.
\end{aligned}$$

Fix γ such that $2^q\gamma = 1/2$, then

$$I_N \leq \frac{2^{q+1}}{\gamma^q}\int_0^{\gamma N/2} q\lambda^{q-1}W(\{x \in \mathcal{V} : M^{\sharp}f(x) > \lambda\})d\lambda. \tag{12}$$

Taking the limit as $N \to \infty$ in inequality (12) to get (11). $\square$

By applying Theorem 4.6, we immediately obtain the following result.

Theorem 4.10. *Given $p(\cdot) \in GH(\mathcal{V})$ and a weight ω. If there exists a constant $s \in (0, p_-)$ such that $\omega^s \in A_{p(\cdot)/s,\mathscr{R}}$, then*

$$\|f\|_{L^{p(\cdot)}_{\omega}(\mathcal{V})} \le \|M^{\sharp} f\|_{L^{p(\cdot)}_{\omega}(\mathcal{V})}.$$

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This paper is available via http://nyjm.albany.edu/j/2025/31-39.html.