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Hermitian u-invariants under quadratic field extensions

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Abstract. The hermitian u-invariants of a central simple algebra with involution are studied. In this context, a new technique is obtained to give bounds for the behavior of these invariants under a quadratic field extension. This is applied to obtain bounds in terms of the index of the algebra and the u-invariant of the base field.

Contents

1. Introduction

The concept of the u-invariant naturally extends from quadratic forms over fields to hermitian forms over central simple algebras with an involution. This paper is devoted to the problem of bounding the hermitian u-invariants of central simple algebras of exponent 2. Recall that such algebras can be represented as a tensor product of a finite number of quaternion algebras. In the presence of a suitable separable quadratic subfield, one can bound the hermitian u-invariant of a central simple algebra with involution in terms of the hermitian u-invariant of a subalgebra stable under the involution. This relies on a method used in characteristic different from 2 by E. Bayer-Fluckiger and R. Parimala in the construction of an exact sequence of Witt groups of hermitian forms [1, Appendix 2 and §3.1], which plays an essential role in their classification results for hermitian forms over central simple algebras with involution over fields of cohomological dimension 2. It is used later in [10], [11] and [15]

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to obtain upper bounds for the hermitian u-invariants of a central simple algebra of exponent 2 in terms of the u-invariant of the base field. In this article, we refine this method and apply it to study the behavior of the hermitian u-invariants under 2-extensions and in particular multiquadratic extensions. In this way, we obtain bounds on the hermitian u-invariants of a central simple algebra with involution in terms of the degree and the u-invariant of a splitting 2-extension (Corollary 5.2 and Corollary 6.4). For certain algebras with involution of small index, we obtain in Theorem 5.5 and Theorem 6.1 improvements to previously existing bounds. Unfortunately, at present we are not able to produce new examples that would give new insight as to the optimality of these bounds. This seems much related to the analogous problem concerning bounds on the growth of the u-invariant for finite field extensions.

We denote by $\mathbb{N}$ the set of natural numbers including 0 and set $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.

2. Hermitian forms and involutions

Our main references are [7] for the theory of hermitian forms and [8] for the theory of algebras with involution. The notions and basic facts from the theory of central simple algebras that we need are mostly covered by [14, Chap. 8].

Throughout this article let F denote a field. Let A be a central simple F-algebra. The *degree*, *index* and *exponent of* A are denoted by $\mathsf{deg}A$, $\mathsf{ind}\,A$ and $\mathsf{exp}A$, respectively. Given another central simple F-algebra B, we write $A \sim B$ to indicate that A and B are Brauer equivalent. By Wedderburn's Theorem [14, Chap. 8, Cor. 1.6], we have $A \simeq \mathbb{M}_s(D)$ for a central F-division algebra D, unique up to isomorphism, and a unique $s \in \mathbb{N}^+$. Then $\mathsf{ind}\,A = \mathsf{deg}D$ and $\mathsf{deg}A = s \cdot \mathsf{ind}\,A$. We say that A is *split* if $D = F$, or equivalently, if $\mathsf{ind}\,A = 1$. By [14, Chap. 8, Theorem 1.8], every finitely generated A-right module V decomposes into a direct sum of simple A-right modules. Moreover, every simple A-right module is isomorphic to D^s with the action induced by an isomorphism $A \to \mathbb{M}_s(D)$ and the multiplication by matrices from the right. The number of simple components in a decomposition of V as a direct sum of simple A-right modules is called the *rank of* V (over A) and denoted by $\mathsf{rk}_A\, V$; equivalently, it is given by the equality $\mathsf{rk}_A\, V = \frac{\mathsf{dim}_F V}{s \cdot \mathsf{dim}_F D}$. In particular, we have $\mathsf{rk}_A\, A = s$, and if A is a division algebra, then $\mathsf{rk}_A\, V$ is the dimension of V as an A-right vector space. Given a field extension K/F, we obtain from A a central simple K-algebra $A_K = A \otimes_F K$.

An *involution on* a ring A is an anti-automorphism $\sigma : A \to A$ such that $\sigma^2 = \mathsf{id}_A$. If A is an F-algebra, σ is an involution on A, and K/F is a field extension, then $\sigma \otimes \mathsf{id}_K$ is an involution on $A_K = A \otimes_F K$, which we denote by σ_K. If K/F is a quadratic étale extension (that is, either $K \simeq F \times F$ or K/F is a separable quadratic field extension), then the non-trivial F-automorphism of K is an involution on K, which we call the *canonical involution of* K/F and which we denote by $\mathsf{can}_{K/F}$.

Let now A be a central simple F-algebra and σ an involution on A which is F-linear. We fix a field extension K/F such that A_K is split. Hence $A_K \simeq \mathsf{End}_K V$ for a K-vector space V with $\mathsf{dim}_K V = \mathsf{deg} A$, and under this isomorphism of K-algebras, σ_K corresponds to a K-linear involution on $\mathsf{End}_K V$, which is adjoint to a (nonsingular) alternating or symmetric K-bilinear form b on V. Moreover, whether b is alternating depends neither on the choice of the field extension K/F nor on the isomorphism; see [8, Prop. 2.6]. We call the involution σ *symplectic* if the bilinear form b is alternating, and we call it *orthogonal* otherwise.

For a ring A, we set $\mathsf{Z}(A) = \{x \in A \mid xy = yx \text{ for all } y \in A\}$, which is a subring called the *center of* A. By an *F-algebra with involution*, we mean a pair (A, σ) where A is a finite-dimensional F-algebra and σ is an F-linear involution on A such that $F = \{x \in \mathsf{Z}(A) \mid \sigma(x) = x\}$ and A has no non-trivial two-sided ideal I with $\sigma(I) = I$. There are two kinds of situations for this to occur:

(1) A is a central simple F-algebra and σ is an F-linear involution.
(2) $K = \mathsf{Z}(A)$ is a quadratic étale extension of F and $\sigma|_K$ is an automorphism of order 2 of K with F as its fixed field. In this case, either K/F is a separable quadratic field extension and A is a central simple K-algebra, or $K \simeq F \times F$ and $A \simeq B \times B^{\mathsf{op}}$ for a central simple F-algebra B and where B^{op} denotes its opposite algebra (which coincides with B as a set), and σ corresponds to the so-called *switch map* sw_B on $B \times B^{\mathsf{op}}$, which is defined by $\mathsf{sw}_B(b, b') = (b', b)$ for $b, b' \in B$.

If K is either a field or a product of two copies of a field, then an involution σ on a K-algebra A is called *unitary* if $\sigma|_K \neq \mathsf{id}_K$, and in this case (A, σ) is an F-algebra with involution for $F = \{x \in K \mid \sigma(x) = x\}$. We then also call σ a *K/F-unitary involution on A*. Note that, while the fixed field F in this situation is determined by σ, a central simple K-algebra A can have several unitary involutions with different fixed fields, and this situation will also occur crucially in our study.

A unitary involution σ on an F-algebra A with $\mathsf{Z}(A) \simeq F \times F$ is also called *unitary of inner type*. (The term is motivated by a relation to a corresponding notion for linear algebraic groups.) For any F-algebra with involution (A, σ) where σ is not unitary of inner type, we have that $K = \mathsf{Z}(A)$ is a field, K/F is separable with $[K : F] \leqslant 2$ and A is a central simple K-algebra.

We recall the following criteria for the existence of involutions on a central simple algebra.

Theorem 2.1 (Albert)**.** *Let A be a central simple F-algebra. There exists an orthogonal involution on A if and only if $\mathsf{exp} A \leqslant 2$. There exists a symplectic involution on A if and only if $\mathsf{exp} A \leqslant 2$ and $\mathsf{deg} A$ is even.*

Proof. By [8, Theorem 3.1], an F-linear involution on A exists if and only if $\mathsf{exp} A \leqslant 2$, and using [8, Cor. 2.8], the statement follows. □

For a separable quadratic field extension K/F and a central simple K-algebra B, we denote by $\mathsf{cor}_{K/F} B$ the corestriction algebra of B with respect to K/F as defined in [5, §8].

Theorem 2.2 (Albert-Riehm-Scharlau)**.** *Let K/F be a separable quadratic field extension and let B be a central simple K-algebra. There exists a K/F-unitary involution on B if and only if $\mathsf{cor}_{K/F} B$ is split.*

Proof. See [8, Theorem 3.1]. □

It follows in particular from Theorem 2.1 and Theorem 2.2 that, if a central simple algebra A has an involution σ, then every central simple algebra over $\mathsf{Z}(A)$ which is Brauer equivalent to A carries an involution whose restriction to $\mathsf{Z}(A)$ is the same as for σ.

In the sequel, let K be a field, let B be a central simple K-algebra, and let γ be an involution on B. Then $F = \{x \in K \mid \gamma(x) = x\}$ is a subfield of K, and we obtain that (B, γ) is an F-algebra with involution.

Let V be a finitely generated B-right module and $\varepsilon \in \{\pm 1\}$. A bi-additive map $h : V \times V \to B$ is called an *ε-hermitian form over* (B, γ) if it satisfies the following:

- $h(v\alpha, w\beta) = \gamma(\alpha)h(v, w)\beta$ for all $v, w \in V, \alpha, \beta \in B$,
- $h(w, v) = \varepsilon\gamma(h(v, w))$ for all $v, w \in V$.

We also call h simply a *hermitian* (resp. *skew-hermitian*) *form* when $\varepsilon = 1$ (resp. $\varepsilon = -1$). For an ε-hermitian form h over (B, γ), we will denote by $\mathsf{rk}\, h$ the rank of the underlying B-right module V. We refer to [7, Chap. I, §2.2 & §3.4] for the basic concepts of isometry ($\simeq$) and orthogonal sum ($\perp$) for ε-hermitian forms.

Consider an ε-hermitian form h over (B, γ) defined on the B-right module V. For any B-right submodule U of V, the restriction of h to $U \times U$ defines an ε-hermitian form over (B, γ) which we denote by $h|_U$. A B-right submodule U of V is called *totally isotropic* (*with respect to* h) if $h|_U$ is the zero map. The form h is *nonsingular* if for any $v \in V \smallsetminus \{0\}$ there exists $w \in V$ such that $h(v, w) \neq 0$. The form h is *isotropic* if there exists some $v \in V \smallsetminus \{0\}$ such that $h(v, v) = 0$, and *anisotropic* otherwise. The form h is *hyperbolic* if it is nonsingular and $V = U \oplus U'$ for two B-right submodules U and U' which are totally isotropic with respect to h. In particular, any non-trivial hyperbolic ε-hermitian form is isotropic. If $\mathsf{char} K \neq 2$ or $\gamma|_K \neq \mathsf{id}_K$, then any rank-2 nonsingular isotropic ε-hermitian form over (B, γ) is hyperbolic.

For a field extension M/K, we obtain a finitely generated B_M-right module $V_M = V \otimes_K M$ and an ε-hermitian form $h_M : V_M \times V_M \to B_M$ over (B_M, γ_M) given by $h_M(v \otimes \alpha, w \otimes \beta) = h(v, w) \otimes \alpha\beta$ for $v, w \in V, \alpha, \beta \in M$.

Let V be a finitely generated B-right module. Let $h : V \times V \to B$ be a nonsingular hermitian or skew-hermitian form over (B, γ). According to [8, §4.A], h determines an involution ad_h on $\mathsf{End}_B V$ satisfying

$$h(v, f(w)) = h(\mathsf{ad}_h(f)(v), w) \quad \text{for all } v, w \in V \text{ and all } f \in \mathsf{End}_B V.$$

We call ad_h the *adjoint involution of h*. Viewing K naturally embedded into $\mathsf{End}_B V$, we have that $\mathsf{ad}_h|_K = \gamma|_K$. All involutions on $\mathsf{End}_B V$ arise in this way from some nonsingular hermitian form.

Theorem 2.3.

(*a*) *Assume that* $\mathsf{char} K \neq 2$ *and* $\gamma|_K = \mathsf{id}_K$. *Then any* K*-linear involution* σ *on* $\mathsf{End}_B V$ *is the adjoint involution* ad_h *of some nonsingular hermitian or skew-hermitian form* h *over* (B,γ)*, which is unique up to a factor in* $F^\times$*. Moreover, the involutions* σ *and* γ *are both orthogonal or both symplectic if* h *is hermitian, whereas precisely one of them is orthogonal and the other one symplectic if* h *is skew-hermitian.*

(*b*) *Assume that* $\gamma|_K \neq \mathsf{id}_K$. *Then any involution* τ *on* $\mathsf{End}_B V$ *with* $\tau|_K = \gamma|_K$ *is the adjoint involution* ad_h *for some nonsingular hermitian form* h *over* (B,γ)*, which is unique up to a factor in* $F^\times$*.*

Proof. See [8, Theorem 4.2]. □

We denote by $\mathsf{Ad}_B(h)$ the F-algebra with involution $(\mathsf{End}_B V, \mathsf{ad}_h)$.

Let (A,σ) be an F-algebra with involution. We say that σ (or (A,σ)) is *isotropic* if there exists an $a \in A \smallsetminus \{0\}$ such that $\sigma(a)a = 0$, and *anisotropic* otherwise. If there exists an element $e \in A$ such that $e^2 = e$ and $\sigma(e) = 1-e$, then σ (or (A,σ)) is called *hyperbolic*. In particular, any hyperbolic involution is isotropic.

Proposition 2.4. *Let* K *be a field,* D *a central* K*-division algebra and* γ *an involution on* D*. Assume that* $\mathsf{char} K \neq 2$ *or* γ *is unitary. Let* V *be a finite-dimensional* D*-right vector space and* $h : V \times V \to D$ *a nonsingular* ε*-hermitian form over* (D,γ)*. Then* $\mathsf{Ad}_D(h)$ *is isotropic (resp. hyperbolic) if and only if* h *is isotropic (resp. hyperbolic).*

Proof. The statement for hyperbolicity is given in [8, Prop. 6.7]. We prove the statement for isotropy.

We set $(A,\sigma) = (\mathsf{End}_D V, \mathsf{ad}_h)$. Assume that (A,σ) is isotropic. Then there exists some $f \in A \smallsetminus \{0\}$ such that $\sigma(f)\circ f = 0$. Since $f \neq 0$, there exists $v \in V$ with $f(v) \neq 0$. We have $h(f(v), f(v)) = h((\sigma(f)\circ f)(v), v) = h(0,v) = 0$. Thus h is isotropic.

Conversely, assume that h is isotropic. There exists some $v \in V \smallsetminus \{0\}$ such that $h(v,v) = 0$. Since h is nonsingular, we can find a nonzero vector $w \in V$ such that $h(v,w) \neq 0$. Consider $f : V \to V, u \mapsto vh(w,u)$. Then $f \in \mathsf{End}_D V$, and $f \neq 0$ as $f(v) = vh(w,v) \neq 0$. We have that

$$\begin{aligned} h((\sigma(f)\circ f)(u), u') &= h(f(u), f(u')) = h(vh(w,u), vh(w,u')) \\ &= \gamma(h(w,u))h(v,v)h(w,u') = 0 \end{aligned}$$

for any $u, u' \in V$. Using again that h is nonsingular, it follows that $\sigma(f)\circ f = 0$. Therefore (A,σ) is isotropic. □

3. Hermitian u-invariants

Let K be a field and B a central simple K-algebra. Let $\varepsilon \in \{\pm 1\}$ and let γ be an involution on B. Following [12, Chap. 9, Definition 2.4], we set

$$u(B,\gamma,\varepsilon) = \sup\{\mathsf{rk}\, h \mid h \text{ anisotropic } \varepsilon\text{-hermitian form over } (B,\gamma)\} \in \mathbb{N} \cup \{\infty\},$$

and we call $u(B,\gamma,\varepsilon)$ the *ε-hermitian u-invariant* of (B,γ).

For later use, we recall a well-known statement relating the u-invariant and the rank of an ε-hermitian form to the rank of totally isotropic subspaces.

Lemma 3.1. *Let D be a central K-division algebra and γ an involution on D. Let $\varepsilon \in \{\pm 1\}$ and let h be an ε-hermitian form over (D,γ). Then the D-right vector space on which h is defined contains a totally isotropic subspace U with respect to h with $\mathrm{rk}\, U \geqslant \frac{1}{2}(\mathrm{rk}\, h - u(D,\gamma,\varepsilon))$.*

Proof. Let V be the D-right vector space on which h is defined. Let U be a maximal subspace of V that is totally isotropic with respect to h. We set $U' = \{w \in V \mid h(w,u) = 0 \text{ for all } u \in U\}$ and note that this is a D-right subspace of V with $U \subseteq U'$. Viewing D as a D-right vector space, we set $U^* = \mathrm{Hom}_D(U,D)$ and view U^* as a D-right vector space by letting $f.d = \gamma(d)f$ for $f \in U^*$ and $d \in D$. Then $\dim_D U^* = \dim_D U$, and the map $V \to U^*, v \mapsto h(v,*)$ is D-linear and its kernel is U'. We conclude that $\dim_D U' + \dim_D U \geqslant \dim_D V$. Hence $\mathrm{rk}\, U' + \mathrm{rk}\, U \geqslant \mathrm{rk}\, V$. We further have that $U' = U \oplus U''$ for a D-right subspace U'' of U'. Then $\mathrm{rk}\, U'' = \mathrm{rk}\, U' - \mathrm{rk}\, U \geqslant \mathrm{rk}\, V - 2\,\mathrm{rk}\, U$, and it follows from the choice of U that $h|_{U''}$ is anisotropic. Hence $\mathrm{rk}\, U'' \leqslant u(D,\gamma,\varepsilon)$. We conclude that $2\,\mathrm{rk}\, U \geqslant \mathrm{rk}\, V - \mathrm{rk}\, U'' \geqslant \mathrm{rk}\, h - u(D,\gamma,\varepsilon)$. □

The following statement is well-known. In [10, Prop. 2.2], it is shown for the case where B is a division algebra. We include a proof that does not require this hypothesis.

Proposition 3.2. *Let γ and γ' involutions on B.*

(1) *Let $\varepsilon \in \{\pm 1\}$. If γ and γ' are either both orthogonal or both symplectic, then $u(B,\gamma,\varepsilon) = u(B,\gamma',\varepsilon)$. If γ is orthogonal and γ' is symplectic, then $u(B,\gamma,\varepsilon) = u(B,\gamma',-\varepsilon)$.*
(2) *If γ and γ' are both unitary and $\gamma|_K = \gamma'|_K$, then $u(B,\gamma,1) = u(B,\gamma',-1)$.*

Proof. In both parts, we have $\gamma|_K = \gamma'|_K$. Hence $\gamma\circ\gamma'$ is a K-automorphism of B. We obtain by the Skolem-Noether Theorem that $\gamma\circ\gamma' = \mathrm{Int}(b)$ for some $b \in B^\times$. Then $\gamma = \mathrm{Int}(b)\circ\gamma'$.

(1) If γ and γ' are either both orthogonal or both symplectic, then we set $\varepsilon' = 1$, otherwise we set $\varepsilon' = -1$. Then $\gamma'(b) = \varepsilon' b$, by [8, Prop. 2.7]. Let V be a finitely generated B-right module and $h : V \times V \to B$ an ε-hermitian form over (B,γ). One easily verifies that $b^{-1}h : V \times V \to B$ is an $\varepsilon\varepsilon'$-hermitian form over (B,γ'). Clearly, h is isotropic if and only if $b^{-1}h$ is isotropic, and $\mathrm{rk}\, h = \mathrm{rk}\, b^{-1}h$. Similarly, if $h' : V \times V \to B$ is an $\varepsilon\varepsilon'$-hermitian form over (B,γ'), then $bh : V \times V \to B$ is an ε-hermitian form over (B,γ). This shows that $u(B,\gamma',\varepsilon\varepsilon') = u(B,\gamma,\varepsilon)$.

(2) Set $\lambda = b\gamma'(b)^{-1}$. Then $\gamma'(\lambda)\lambda = 1$, whereby $-\lambda^{-1}\gamma'(-\lambda^{-1}) = 1$. We further obtain that $\mathrm{Int}(b)\circ\gamma'\circ\mathrm{Int}(b) = \mathrm{Int}(\lambda)\circ\gamma'$, hence $\mathrm{id}_B = \gamma\circ\gamma = \mathrm{Int}(\lambda)$, whereby $\lambda \in B^\times \cap \mathrm{Z}(B) = K^\times$. By Hilbert's Theorem 90 applied to the quadratic extension K/F where $F = \{x \in K \mid \gamma'(x) = x\}$, there exists $\lambda' \in K^\times$ such that $\lambda'^{-1}\gamma'(\lambda') = -\lambda^{-1}$. Set $b' = (\lambda')^{-1}b$. Then $\mathrm{Int}(b') = \mathrm{Int}(b)$ and thus $\gamma =$

$\mathsf{Int}(b')\circ\gamma'$. Consider a finitely generated B-right module V. Given a hermitian form $h : V \times V \to B$ over (B,γ), we obtain that $b'^{-1}h$ is a skew-hermitian form over (B,γ'). Clearly h is isotropic if and only if $b'^{-1}h$ is isotropic, and $\mathsf{rk}\, h = \mathsf{rk}\, b'^{-1}h$. Similarly, if $h' : V \times V \to B$ is a skew-hermitian form over (B,γ'), then $b'h$ is a hermitian form over (B,γ). This shows that $u(B,\gamma',-1) = u(B,\gamma,1)$. □

By means of Proposition 3.2, we can reduce the number of different hermitian u-invariants to be considered for a central simple F-algebra.

Assume for now that $\mathsf{char} F \neq 2$. Given a central simple F-algebra B with $\mathsf{exp} B \leqslant 2$, we fix an arbitrary orthogonal involution γ on B, which exists by Theorem 2.1, and we set

$$u^+(B) = u(B,\gamma,1) \quad \text{and} \quad u^-(B) = u(B,\gamma,-1),$$

observing that, by Proposition 3.2 (1), the definition does not depend on the particular choice of γ. For a given central simple F-algebra B with $\mathsf{exp} B > 2$, we set $u^+(B) = u^-(B) = 0$. We call $u^+(B)$ the *orthogonal u-invariant of B* and $u^-(B)$ the *symplectic u-invariant of B*. These notations go back to [10, Remark 2.3]. Note that $u^-(B) = 0$ if B is split.

Let us point out the relation between the orthogonal u-invariant and the classical u-invariant of a field. Recall that the field F is *nonreal* if -1 is a sum of squares in F and *real* otherwise. If F is nonreal, one defines

$$u(F) = \mathsf{sup}\{\mathsf{dim}(q) \mid q \text{ anisotropic quadratic form over } F\}.$$

This is called the *u-invariant of F*. We refer to [12, Chap. 8] for a treatment of the u-invariant, including a discussion of how to extend this notion to cover real fields, which however is not relevant for this article. We note that when $\mathsf{char}(F) = 2$, the definition of $u(F)$ here corresponds to what is denoted by $\hat{u}(F)$ in [12, Chap. 8, Sect. 4].

Example 3.3. Assume that $\mathsf{char} F \neq 2$. We consider $u^+(B)$ for the central simple F-algebra $B = F$. By the 1-1-correspondence between symmetric bilinear forms and quadratic forms over F, we obtain that $u^+(F) = u(F)$.

We now define the hermitian u-invariant for unitary involutions in a similar way. Here we make no assumption on the characteristic of F. Consider a separable quadratic field extension K/F and a central simple K-algebra B. If there exists a K/F-unitary involution γ on B, (which by Theorem 2.2 holds if and only if the corestriction algebra $\mathsf{cor}_{K/F}\, B$ is split), then we set

$$u(B/F) = u(B,\gamma,1),$$

observing that this is independent of the specific choice of the K/F-unitary involution γ, in view of Proposition 3.2 (2). If B does not admit any K/F-unitary involution, then we set $u(B/F) = 0$. We call $u(B/F)$ the *F-unitary u-invariant of B*.

Note that the hermitian u-invariants depend only on the Brauer class of the algebra. This was pointed out in [15, Lemma 2.1].

Proposition 3.4. *Let K/F be a separable field extension with $[K : F] \leqslant 2$. Let B and B' be central simple K-algebras such that $B \sim B'$. Then we have $u^+(B) = u^+(B')$, $u^-(B) = u^-(B')$ and $u(B/F) = u(B'/F)$.*

Proof. This follows by [7, Theorem 9.3.5] together with Theorem 2.3. □

Let $\mathsf{Br}(F)$ denote the Brauer group of F and let $\mathsf{Br}_2(F)$ denote its 2-torsion subgroup. For $\alpha \in \mathsf{Br}(F)$, we take the central F-division algebra D such that $\alpha = [D]$ and define

$$u^+(\alpha) = u^+(D) \qquad \text{and} \qquad u^-(\alpha) = u^-(D)\,.$$

Similarly, given a separable quadratic field extension K/F and $\alpha \in \mathsf{Br}(K)$, we take the central K-division algebra D with $\alpha = [D]$ and define

$$u(\alpha/F) = u(D/F)\,.$$

Proposition 3.5. *Let (A, σ) be an F-algebra with involution. For (a) and (b), assume that $\mathsf{char} F \neq 2$.*

(a) *If σ is orthogonal and $\mathsf{deg} A > \mathsf{ind}\, A \cdot u^+(A)$, then σ is isotropic.*
(b) *If σ is symplectic and $\mathsf{deg} A > \mathsf{ind}\, A \cdot u^-(A)$, then σ is isotropic.*
(c) *If σ is unitary and $\mathsf{deg} A > \mathsf{ind}\, A \cdot u(A/F)$, then σ is isotropic.*

Proof. Suppose that σ is anisotropic. Then $K = \mathsf{Z}(A)$ is a field. Let D be the central K-division algebra Brauer equivalent to A. Then $\mathsf{deg} D = \mathsf{ind}\, A$. We fix an involution γ on D with $\gamma|_K = \sigma|_K$ and such that γ is orthogonal if and only if σ is orthogonal. By Theorem 2.3, $(A, \sigma) \simeq \mathsf{Ad}_D(h)$ for some nonsingular hermitian form over (D, γ). Then $\mathsf{deg} A = \mathsf{ind}\, A \cdot \mathsf{rk}\, h$.

Since σ is anisotropic, by Proposition 2.4, h is anisotropic. If σ is orthogonal, then so is γ, and it follows that $\mathsf{rk}\, h \leqslant u^+(D) = u^+(A)$. If σ is symplectic, then so is γ, and we obtain that $\mathsf{rk}\, h \leqslant u^-(D) = u^-(A)$. Finally, if σ is unitary, then as $\gamma|_K = \sigma|_K$, we conclude that $\mathsf{rk}\, h \leqslant u(D/F) = u(A/F)$. □

Within the study of the u-invariant for quadratic forms, a central topic is its behavior under field extensions.

Proposition 3.6. *Assume that F is nonreal and let L/F be a finite 2-extension. For $n \in \mathbb{N}$ with $[L : F] = 2^n$, we have $u(L) \leqslant (\frac{3}{2})^n \cdot u(F)$.*

Proof. See [12, Chap. 9, Cor. 2.2] for the case where $n = 1$. From this case, the statement follows by induction on n. □

We recall the following fact about the unitary u-invariant in the split case.

Proposition 3.7. *Assume that F is nonreal. Let K/F be a separable quadratic field extension. The following hold:*

(a) *$u(K/F) \leqslant \frac{1}{2} u(F)$.*
(b) *If $u(K/F) > 2$, then there exists an anisotropic quadratic 3-fold Pfister form over F which becomes hyperbolic over K.*

Proof. We recall from [14, Chap. 10, §1] that, for an arbitrary nonsingular hermitian form over $(K, \mathsf{can}_{K/F})$, the rule $x \mapsto h(x,x)$ defines a quadratic form q_h over F with $\dim \mathsf{q}_h = 2\,\mathsf{rk}\,h$ which becomes hyperbolic over K, and that h is isotropic if and only if q_h is isotropic.

(a) As $u(K/F) = u(K, \mathsf{can}_{K/F}, 1)$, this follows directly.

(b) Assume that $u(K/F) > 2$. We may then choose an anisotropic hermitian form h over $(K/F, \mathsf{can}_{K/F})$ with $\mathsf{rk}\,h = 3$. Then q_h is an anisotropic 6-dimensional quadratic form over F which becomes hyperbolic over K. It follows that q_h is similar to a subform of some quadratic 3-fold Pfister form ρ over F. Since q_h is anisotropic, ρ is not hyperbolic. Since ρ is a Pfister form, it follows that ρ is anisotropic and becomes hyperbolic over K. □

Using systems of quadratic forms, Mahmoudi obtained in [10] the following upper bounds on the hermitian u-invariants in terms of the u-invariant of the base field, which we restate here in our setup.

Proposition 3.8 (Mahmoudi)**.** *Assume that* $\mathsf{char} F \neq 2$ *and F is nonreal.*

(a) *Let $\alpha \in \mathsf{Br}(F)$ and $n = \mathsf{ind}\,\alpha$. Then*

$$u^+(\alpha) \leqslant \tfrac{(n+1)(n^2+n+2)}{8n} \cdot u(F) \quad \textit{and} \quad u^-(\alpha) \leqslant \tfrac{(n-1)(n^2-n+2)}{8n} \cdot u(F).$$

(b) *Let K/F be a quadratic field extension, $\alpha \in \mathsf{Br}(K)$ and $n = \mathsf{ind}\,\alpha$. Then*

$$u(\alpha/F) \leqslant \tfrac{n^2+1}{4} \cdot u(F).$$

Proof. See [10, Prop. 3.6]. □

The hermitian u-invariants were further studied in [11] and [15].

Theorem 3.9 (Parihar-Suresh)**.** *Assume that F is nonreal with* $\mathsf{char} F \neq 2$. *Let K/F be a quadratic field extension and $\alpha \in \mathsf{Br}(F)$. Then $u^+(\alpha_K) \leqslant \frac{3}{2}u^+(\alpha)$, $u^-(\alpha_K) \leqslant \frac{3}{2}u^-(\alpha)$ and $u(\alpha_K/F) \leqslant \frac{1}{2}u^+(\alpha) + u^-(\alpha)$.*

Proof. See [11, Theorems 4.2 & 4.3]. □

Using Theorem 3.9, the following bounds were obtained in [15].

Theorem 3.10 (Wu)**.** *Assume that F is nonreal and* $\mathsf{char} F \neq 2$. *Let $\alpha \in \mathsf{Br}(F)$. Assume that $n \in \mathbb{N}$ is such that* $\mathsf{ind}\,\alpha = 2^n$ *and α is given by a tensor product of n F-quaternion algebras. Then the following hold:*

(a) $u^+(\alpha) \leqslant \frac{1}{5}(4 + (\frac{3}{2})^{2n})u(F)$.

(b) $u^-(\alpha) \leqslant \frac{1}{5}(-1 + (\frac{3}{2})^{2n})u(F)$.

(c) $u(\alpha_K/F) \leqslant \frac{1}{5}(1 + (\frac{3}{2})^{2n+1})u(F)$ *for any quadratic field extension K/F.*

Proof. This is [15, Theorem 1.3]. The proof is by induction on n. For $n = 1$, (a), (b), and (c) are established in [10, Cor. 3.4], [14, Chap. 10, Theorem 1.7], and [11, Cor. 4.4], respectively. □

Remark 3.11. The hypothesis in Theorem 3.10 means that the Brauer class α is represented by a division algebra which is a tensor product of n quaternion algebras. The hypothesis stated in [15, Theorem 1.3] is weaker, but the author notified us that it needs to be strengthened, due to the results used in the proof.

4. Isotropy via quadratic reduction

Given a central simple algebra with involution and a separable quadratic subextension stable under the involution, one obtains a simple subalgebra carrying a pair of involutions. Accordingly, we can obtain from a hermitian form over this algebra with involution a pair of hermitian forms over the subalgebra with respect to two different involutions such that the isotropy of this pair (i.e., the existence of a common nonzero isotropic vector) is equivalent to the isotropy of the original hermitian form. By these means, one can compare the hermitian u-invariant of a central simple algebra with involution with the hermitian u-invariants of a subalgebra. This method to study the hermitian u-invariants stems from [10, Prop. 3.1]. The purpose of this section is to extend this method by relaxing the hypotheses under which it can be applied. This will be achieved with Theorem 4.6. Our treatment will cover the case of characteristic 2 for the unitary case. As described above, the method relies on the presence of a quadratic subextension. In Proposition 4.2 we discuss another technique to deal with the complementary situation where we cannot find any quadratic field extension of the center contained in the algebra. Together, these methods will allow us in the subsequent sections to obtain bounds on the hermitian u-invariants of a central simple algebra with involution in terms of the degree and the u-invariant of a splitting 2-extension; see Corollary 5.2 and Corollary 6.4.

In this section, let K be a field.

Proposition 4.1. *Let D be a central K-division algebra and γ an involution on D. Assume that $\mathsf{char} K \neq 2$ if $\gamma|_K = \mathsf{id}_K$. Let $\varepsilon \in \{\pm 1\}$ and h be a nonsingular ε-hermitian form over (D, γ). Let M/K be a separable quadratic field extension such that D_M is a division algebra. Then*

$$h \simeq h' \perp h_1 \perp \ldots \perp h_n$$

for some $n \in \mathbb{N}$ and ε-hermitian forms $h', h_1, \ldots, h_n$ over (D, γ) such that h'_M is anisotropic and, for $1 \leqslant i \leqslant n$, we have that $\mathsf{rk}\, h_i = 2$ and $(h_i)_M$ is hyperbolic.

Proof. See [2, Prop. 4.3] and its proof. (In [2], the general assumption is that $\gamma|_K = \mathsf{id}_K$ and $\mathsf{char} K \neq 2$, but the proof [2, Prop. 4.3] is valid in general.) □

The following proposition extends [10, Prop. 3.1]. (Here the assumptions on α are less restrictive, and we cover characteristic 2 for the unitary case.) This extension is complementary to those obtained in [11, §4], while based on similar arguments.

Proposition 4.2. *Let K/F be a separable field extension with $[K : F] \leqslant 2$. Let M/K be a quadratic field extension such that M/F is separable and M contains* *a quadratic extension of F distinct from K. Let $\alpha \in \mathrm{Br}(K)$ be such that* $\mathrm{ind}\,\alpha_M = \mathrm{ind}\,\alpha$. *Then the following hold:*

(*a*) *If* $\mathrm{char} F \neq 2$ *and* $K = F$*, then*

$$u^+(\alpha) \leqslant u^+(\alpha_M) + 2u(\alpha_M/F) \quad \textit{and} \quad u^-(\alpha) \leqslant u^-(\alpha_M) + 2u(\alpha_M/F).$$

(*b*) *If* $[K : F] = 2$*, then M/F has a quadratic subextension L/F such that $M = KL$ and*

$$u(\alpha/F) \leqslant 3 \cdot u(\alpha_M/L).$$

Proof. If $K = F$, then we assume that $\mathrm{char} F \neq 2$. Let D be the central K-division algebra such that $\alpha = [D]$. If $K = F$ and $u^+(\alpha) = 0$, respectively if $[K : F] = 2$ and $u(\alpha/F) = 0$, then the claim of (a), respectively of (b) holds trivially. We may assume that this is not the case. We may thus fix an F-linear involution γ on D with $F = \{x \in K \mid \gamma(x) = x\}$ which is orthogonal or unitary (depending on $[K : F]$).

Note that D_M is a division algebra because $\mathrm{ind}\,\alpha_M = \mathrm{ind}\,\alpha$, and $\gamma_M = \gamma \otimes \mathrm{id}_M$ is an F-linear involution on D_M.

Let $\varepsilon \in \{\pm 1\}$ and consider an anisotropic ε-hermitian form h over (D, γ), defined on some finite-dimensional D-right vector space V. By Proposition 4.1, $h \simeq h' \perp h''$ for some ε-hermitian forms h' and h'' such that h'_M is anisotropic and h''_M is hyperbolic. In particular, $\mathrm{rk}\, h' = \mathrm{rk}\, h'_M \leqslant u(D_M, \gamma_M, \varepsilon)$.

Set $(A, \sigma) = \mathrm{Ad}_D(h'')$. Then (A, σ) is an F-algebra with involution and $\deg A = \mathrm{rk}\, h'' \cdot \deg D$. Since h is anisotropic, so is h'', and hence σ is anisotropic. As h''_M is hyperbolic and $\mathrm{Ad}_{D_M}(h''_M) \simeq (A_M, \sigma_M)$, we obtain that σ_M is hyperbolic. It follows by [4, Theorem 1.15 and Theorem 1.16] that we can identify M with an F-subalgebra of A with $\sigma(M) = M$ and $\sigma|_M \neq \mathrm{id}_M$.

Let $C = \{a \in A \mid ab = ba \text{ for all } b \in M\}$, the centralizer of M in A. Then C is a central simple M-algebra and $\sigma|_C$ is an anisotropic unitary involution on C. By Theorem 2.2, since $[C] = [D_M] = \alpha_M$ in $\mathrm{Br}(M)$, we can choose an involution $\tilde{\gamma}$ on D_M with $\tilde{\gamma}|_M = \sigma|_M$. It follows by Theorem 2.3 that $(C, \sigma|_C) \simeq \mathrm{Ad}_{D_M}(\tilde{h})$ for some anisotropic hermitian form $\tilde{h}$ over $(D_M, \tilde{\gamma})$. Hence $\mathrm{rk}\, \tilde{h} \leqslant u(D_M, \tilde{\gamma}, 1)$. Since $\deg D = \mathrm{ind}\,\alpha = \mathrm{ind}\,\alpha_M = \deg D_M$ and $\deg D \cdot \mathrm{rk}\, h'' = \deg A = 2 \deg C = 2 \deg D_M \cdot \mathrm{rk}\, \tilde{h}$, we now conclude that $\mathrm{rk}\, h'' = 2\, \mathrm{rk}\, \tilde{h} \leqslant 2u(D_M, \tilde{\gamma}, 1)$.

It follows that $\mathrm{rk}\, h = \mathrm{rk}\, h' + \mathrm{rk}\, h'' \leqslant u(D_M, \gamma_M, \varepsilon) + 2u(D_M, \tilde{\gamma}, 1)$.

This argument shows that

$$u(D, \gamma, \varepsilon) \leqslant u(D_M, \gamma_M, \varepsilon) + 2u(D_M, \tilde{\gamma}, 1)\,.$$

If $\mathrm{char} F \neq 2$ and $K = F$, then by taking the two possible choices of $\varepsilon \in \{\pm 1\}$, we obtain the two inequalities in (a).

Assume now that $[K : F] = 2$. We take $\varepsilon = 1$. Set $\gamma_0 = \gamma_M$ and $\gamma_1 = \tilde{\gamma}$, and for $i = 0, 1$ let $L_i = \{x \in M \mid \gamma_i(x) = x\}$. Then the above inequality yields that $u(\alpha/F) = u(D, \gamma, 1) \leqslant u(D_M, \gamma_0, 1) + 2u(D_M, \gamma_1, 1) = u(\alpha_M/L_0) + 2u(\alpha_M/L_1)$. Hence letting $k \in \{0, 1\}$ be such that $u(\alpha_M/L_k) \geqslant u(\alpha_M/L_{1-k})$, we conclude that $u(\alpha/F) \leqslant 3u(\alpha_M/L_k)$, which establishes ($b$). □

Proposition 4.3. *Let D be a central simple K-algebra and γ an involution on D. Let $F = \{x \in K \mid \gamma(x) = x\}$ and L/F be a separable quadratic field extension contained in D and linearly disjoint from K/F. Then there exists $j \in D^\times$ such that $\mathsf{Int}(j)|_L = \mathsf{can}_{L/F}$, and for any such j, there exists an involution γ' on D which is either symplectic or unitary and such that $\gamma'|_K = \gamma|_K$, $\gamma'|_L = \mathsf{can}_{L/F}$ and $\gamma'(j) = -j$.*

Proof. By the Skolem-Noether Theorem, there exists an element $j \in D^\times$ such that $\mathsf{Int}(j)|_{KL} = \mathsf{can}_{KL/K}$. Since $[KL : K] = [L : F] = 2$, we obtain that $\mathsf{Int}(j)|_L = \mathsf{can}_{L/F} \neq \mathsf{id}_L$. Note that $L(j^2)$ is a subfield of D. As $\mathsf{Int}(j)|_L \neq \mathsf{id}_L$ we have $j \notin L(j^2)$ and $L \not\subseteq F(j^2)$. Hence $[L(j^2) : F(j^2)] = [L : F] = 2$. Set $H = L(j^2) \oplus jL(j^2)$ and $Q = KL(j^2) \oplus jKL(j^2)$. Then H is an $F(j^2)$-quaternion algebra and Q is a $K(j^2)$-quaternion algebra contained in D. Note that H and $K(j^2)$ are $F(j^2)$-subalgebras of Q and that j^2 commutes with KL and hence with Q. Multiplication in Q induces an isomorphism of $F(j^2)$-algebras $H \otimes_{F(j^2)} K(j^2) \to Q$. Let can_H denote the canonical involution on H and let τ be the involution on Q corresponding with $\mathsf{can}_H \otimes \mathsf{can}_{K(j^2)/F(j^2)}$ under this isomorphism. Then τ is symplectic or unitary, $\tau(j) = -j$ and $\tau|_K = \mathsf{can}_{K/F} = \gamma|_K$, where $\mathsf{can}_{K/F} = \mathsf{id}_F$ when $K = F$. Hence, by [8, Theorem 4.14], there exists an involution γ' on D which is symplectic or unitary and such that $\gamma'|_Q = \tau$. In particular, $\gamma'|_K = \gamma|_K$, $\gamma'|_L = \mathsf{can}_{L/F}$ and $\gamma'(j) = -j$. □

Proposition 4.4. *Let D be a central K-division algebra and γ an involution on D. Let $F = \{x \in K \mid \gamma(x) = x\}$. Let L/F be a separable quadratic field extension contained in D, linearly disjoint from K/F and with $\gamma|_L = \mathsf{can}_{L/F}$. Let $j \in D^\times$ be such that $\gamma(j) = -j$ and $\mathsf{Int}(j)|_L = \mathsf{can}_{L/F}$. Set $\tilde{D} = C_D(L)$, $\gamma_0 = \gamma|_{\tilde{D}}$ and $\gamma_1 = (\mathsf{Int}(j^{-1}) \circ \gamma)|_{\tilde{D}}$. Then $\tilde{D}$ is an L-division algebra with $\mathsf{Z}(\tilde{D}) = LK$ and such that $D = \tilde{D} \oplus j\tilde{D}$. Furthermore, the following hold:*

(*a*) *The maps γ_0 and γ_1 are involutions on $\tilde{D}$, and γ_0 is unitary. If γ is unitary, then so is γ_1. If $\mathsf{char} F \neq 2$, then γ is orthogonal if and only if γ_1 is symplectic, and vice-versa.*

(*b*) *Let $\pi_0, \pi_1 : D \to \tilde{D}$ be the F-linear maps such that $\mathsf{id}_D = \pi_0 + j\pi_1$. Let $\varepsilon \in \{\pm 1\}$ and let $h : V \times V \to D$ be a nonsingular ε-hermitian form over (D, γ) defined on a finite-dimensional D-right vector space V. Then $\pi_0 \circ h : V \times V \to \tilde{D}$ is a nonsingular ε-hermitian form over $(\tilde{D}, \gamma_0)$ and $\pi_1 \circ h : V \times V \to \tilde{D}$ is a nonsingular $(-\varepsilon)$-hermitian form over $(\tilde{D}, \gamma_1)$.*

Proof. It is easy to see that $\gamma(\tilde{D}) = \tilde{D}$, $\mathsf{Z}(\tilde{D}) = KL$, $D = \tilde{D} \oplus j\tilde{D}$ and that γ_0 and γ_1 are involutions on $\tilde{D}$.

(*a*) Clearly $(\gamma_0)|_L \neq \mathsf{id}_L$, so γ_0 is unitary. Note that $(\gamma_1)|_L = \mathsf{id}_L$ and $(\gamma_1)|_K = \gamma|_K$. Therefore γ_1 is unitary if and only if γ is unitary. Assume now that this is not the case, that is $\gamma|_K = \mathsf{id}_K$ and $F = K$. Since $\gamma(j) = -j$, assuming that $\mathsf{char} F \neq 2$, it follows by [8, Prop. 2.7] that if γ is orthogonal, then γ_1 is symplectic, and vice-versa.

(*b*) Set $h_0 = \pi_0 \circ h$ and $h_1 = \pi_1 \circ h$. Clearly the maps h_0 and h_1 are bi-additive, and for any $v, w \in V$ we have $h(v, w) = h_0(v, w) + jh_1(v, w)$. Let $v, w \in V$ and

$x, y \in \tilde{D}$. We have

$$\begin{aligned} h(vx, wy) = \gamma(x)h(v,w)y &= \gamma(x)h_0(v,w)y + \gamma(x)jh_1(v,w)y \\ &= \gamma_0(x)h_0(v,w)y + j\gamma_1(x)h_1(v,w)y, \end{aligned}$$

whereby $h_0(vx, wy) = \gamma_0(x)h_0(v,w)y$ and $h_1(vx, wy) = \gamma_1(x)h_1(v,w)y$. Furthermore,

$$\begin{aligned} h(w,v) = \varepsilon\gamma(h(v,w)) &= \varepsilon\gamma(h_0(v,w)) - \varepsilon\gamma(h_1(v,w))j \\ &= \varepsilon\gamma_0(h_0(v,w)) - \varepsilon j\gamma_1(h_1(v,w)). \end{aligned}$$

whereby $h_0(w,v) = \varepsilon\gamma_0(h_0(v,w))$ and $h_1(w,v) = -\varepsilon\gamma_1(h_1(v,w))$. Hence, h_0 is an ε-hermitian form over $(\tilde{D}, \gamma_0)$ and h_1 is a $(-\varepsilon)$-hermitian form over $(\tilde{D}, \gamma_1)$.

Note that for $v, w \in V$, since $\gamma(j) = -j$, we have

$$\begin{aligned} h(vj, w) &= h_0(vj, w) + jh_1(vj, w) \\ &= -jh_0(v,w) - j^2h_1(v,w), \end{aligned}$$

and as $j^2 \in \tilde{D}^\times$, we get that $h_0(vj, w) = -j^2h_1(v,w)$ and $h_1(vj,w) = -h_0(v,w)$. As h is nonsingular, this implies in particular that h_0 and h_1 are nonsingular. □

We obtain another generalization of [10, Prop. 3.1].

Proposition 4.5. *Let K/F be a separable field extension with $[K : F] \leqslant 2$. Let M/K be a quadratic field extension such that M/F is separable and M contains a quadratic extension of F distinct from K. Let $\alpha \in \mathsf{Br}(K)$ be such that $\mathsf{ind}\,\alpha_M = \frac{1}{2}\mathsf{ind}\,\alpha$. Then the following hold:*

(*a*) *If $\mathsf{char}F \neq 2$ and $K = F$, then*

$$u^+(\alpha) \leqslant \tfrac{1}{2}u^+(\alpha_M) + u(\alpha_M/F) \quad \textit{and} \quad u^-(\alpha) \leqslant u^-(\alpha_M) + \tfrac{1}{2}u(\alpha_M/F).$$

(*b*) *If $[K : F] = 2$, then M/F has a quadratic subextension L/F such that $M = KL$ and*

$$u(\alpha/F) \leqslant \tfrac{3}{2} \cdot u(\alpha_M/L).$$

Proof. If $K = F$, then we assume that $\mathsf{char}F \neq 2$. Let D be the central K-division algebra such that $\alpha = [D]$. If $K = F$ and $u^+(\alpha) = 0$, respectively if $[K : F] = 2$ and $u(\alpha/F) = 0$, then the claim of (a), respectively of (b) holds trivially. We may assume that this is not the case. We may thus fix an F-linear involution γ on D with $F = \{x \in K \mid \gamma(x) = x\}$ which is orthogonal or unitary (depending on $[K : F]$).

Since $\mathsf{ind}\,\alpha_M = \frac{1}{2}\mathsf{ind}\,\alpha$, we can embed M into D and hence view M as an F-subalgebra of D. Using Proposition 4.3, we choose an element $j \in D^\times$ with $\mathsf{Int}(j)|_M = \mathsf{can}_{M/K}$ and an F-linear involution γ' on D which is symplectic or unitary and such that $\gamma'|_K = \gamma|_K$, $\gamma'(M) = M$ and $\gamma'(j) = -j$.

Set $\tilde{D} = \{x \in D \mid xy = yx \text{ for all } y \in M\}$, the centralizer of M in D, and $\gamma_0 = \gamma'|_{\tilde{D}}$, $\gamma_1 = (\mathsf{Int}(j^{-1})\circ\gamma')|_{\tilde{D}}$. Then $\tilde{D}$ is a central simple M-algebra, and γ_0

and γ_1 are F-linear involutions on $\tilde{D}$, whose properties are, according to Proposition 4.4, determined by those of γ' in the following way: γ_0 is unitary in any case, and if γ' is symplectic, then γ_1 is orthogonal, whereas if γ' is unitary, then so is γ_1.

Let $\varepsilon \in \{\pm 1\}$ and consider an ε-hermitian form h over (D, γ'), defined on some finite-dimensional D-right vector space V. Seeing V as a $\tilde{D}$-right vector space, Proposition 4.4 yields an ε-hermitian form h_0 over $(\tilde{D}, \gamma_0)$ and a $(-\varepsilon)$-hermitian form h_1 over $(\tilde{D}, \gamma_1)$ such that $\mathsf{rk}\, h_0 = \mathsf{rk}\, h_1 = 2 \cdot \mathsf{rk}\, h$.

Suppose now that $\mathsf{rk}\, h > \frac{1}{2} u(\tilde{D}, \gamma_0, \varepsilon) + u(\tilde{D}, \gamma_1, -\varepsilon)$. As $\mathsf{rk}\, h_0 = 2\,\mathsf{rk}\, h$, it follows by Lemma 3.1 that V contains a $\tilde{D}$-subspace W such that $h_0(v, w) = 0$ for all $v, w \in W$ and $\mathsf{rk}_{\tilde{D}}\, W > u(\tilde{D}, \gamma_1, -\varepsilon)$. This implies that $h_1(v, v) = 0$ for some $v \in W \smallsetminus \{0\}$. Then $h(v, v) = h_0(v, v) + j h_1(v, v) = 0$, so h is isotropic. This argument proves that

$$u(D, \gamma', \varepsilon) \leqslant \frac{1}{2} u(\tilde{D}, \gamma_0, \varepsilon) + u(\tilde{D}, \gamma_1, -\varepsilon)\,.$$

Analogously, switching the roles of h_0 and h_1 in the argument, we obtain that

$$u(D, \gamma', \varepsilon) \leqslant u(\tilde{D}, \gamma_0, \varepsilon) + \frac{1}{2} u(\tilde{D}, \gamma_1, -\varepsilon)\,.$$

We are now ready to prove the two parts of the statement. If $K = F$, then using Proposition 3.2, the second inequality for $\varepsilon = -1$ gives the first inequality in (a), while the first inequality for $\varepsilon = 1$ yields the second inequality in (a).

Assume now that $[K : F] = 2$. For $i = 0, 1$, we set $L_i = \{x \in M \mid \gamma_i(x) = x\}$. Then K/F, L_0/F and L_1/F are the three different quadratic subextensions in M/F. Note that $u(\tilde{D}, \gamma_i, \varepsilon) = u(\tilde{D}, \gamma_i, 1)$ for $i = 0, 1$. Take $k \in \{0, 1\}$ such that $u(\tilde{D}, \gamma_k, 1) \geqslant u(\tilde{D}, \gamma_{1-k}, 1)$. We conclude from either of the two inequalities above that $u(\alpha/F) = u(D, \gamma, 1) \leqslant \frac{3}{2} u(\tilde{D}, \gamma_k, 1) = \frac{3}{2} u(\alpha_M/L_k)$, proving (b). □

We merge the main results of this section into one theorem.

Theorem 4.6. *Let K/F be a separable field extension with $[K : F] \leqslant 2$. Let M/K be a quadratic field extension such that M/F is separable and M contains a quadratic extension of F distinct from K. Let $\alpha \in \mathsf{Br}(K)$.*

(a) *If $\mathsf{char} F \neq 2$ and $K = F$, then*

$$u^+(\alpha) \quad \leqslant \quad \frac{\mathsf{ind}\,\alpha_M}{\mathsf{ind}\,\alpha} \left(u^+(\alpha_M) + 2u(\alpha_M/F) \right)\,.$$

(b) *If $[K : F] = 2$, then M/F has a quadratic subextension L/F such that $M = KL$ and*

$$u(\alpha/F) \quad \leqslant \quad 3 \cdot \frac{\mathsf{ind}\,\alpha_M}{\mathsf{ind}\,\alpha} \cdot u(\alpha_M/L).$$

Proof. This follows from Proposition 4.2 if $\mathsf{ind}\,\alpha_{KM} = \mathsf{ind}\,\alpha$ and from Proposition 4.5 if $\mathsf{ind}\,\alpha_{KM} = \frac{1}{2}\,\mathsf{ind}\,\alpha$. □

In Section 5 and Section 6, we shall apply Theorem 4.6 to obtain bounds on the hermitian u-invariants.

5. Bounds on the unitary u-invariant

In this section, we study the behavior of the unitary u-invariant under 2-extensions.

Theorem 5.1. *Let K/F be a separable quadratic field extension and $\alpha \in \mathsf{Br}(K)$. Let $n \in \mathbb{N}$ and let L/F be a 2-extension linearly disjoint from K/F such that $[L : F] = 2^n$. Then there exists a 2-extension L'/F linearly disjoint from K/F with $KL' = KL$ and such that*

$$\operatorname{ind}\alpha \cdot u(\alpha/F) \leqslant 3^n \cdot \operatorname{ind}\alpha_{KL} \cdot u(\alpha_{KL}/L').$$

Proof. We prove the statement by induction on n. If $n = 0$, then $L = F$ and the inequality holds trivially by taking $L' = F$. Assume now that $n \geqslant 1$. Since L/F is a 2-extension, there exist a family of intermediate fields $(L_i)_{i=0}^n$ with $L_0 = F$, $L_n = L$ and such that L_i/L_{i-1} is a separable quadratic field extension for $1 \leqslant i \leqslant n$. Set $K' = L_1K$.

By Theorem 4.6, there exists a separable quadratic field extension L_1'/F contained in K'/F such that $L_1'K = K'$ and

$$u(\alpha/F) \leqslant \frac{\operatorname{ind}\alpha_{K'}}{\operatorname{ind}\alpha} \cdot 3 \cdot u(\alpha_{K'}/L_1').$$

Since $[L_1' : F] = [K : F] = 2$ and $[K' : F] = 4$, it follows that L_1'/F is linearly disjoint from K/F. Note that there exists a 2-extension L'/L_1' contained in LK, linearly disjoint from K'/L_1' and such that $L'K = LK$. Moreover, for any such extension L'/L_1' we have that $[L' : L_1'] = [LK : L_1K] = [L : L_1] = 2^{n-1}$. The induction hypothesis yields that there exists such an extension L'/L_1' with

$$u(\alpha_{K'}/L_1') \leqslant \frac{\operatorname{ind}\alpha_{KL}}{\operatorname{ind}\alpha_{K'}} \cdot 3^{n-1} \cdot u(\alpha_{KL}/L').$$

Combining the inequalities yields that $u(\alpha/F) \leqslant \frac{\operatorname{ind}\alpha_{KL}}{\operatorname{ind}\alpha} \cdot 3^n \cdot u(\alpha_{KL}/L')$. □

Corollary 5.2. *Let K/F be a separable quadratic field extension and $\alpha \in \mathsf{Br}(K)$. Let $n \in \mathbb{N}$ and let L/F be a 2-extension linearly disjoint from K/F such that $[L : F] = 2^n$ and $\alpha_{KL} = 0$. Then, there exists a 2-extension L'/F linearly disjoint from K/F such that $KL' = KL$ and*

$$\operatorname{ind}\alpha \cdot u(\alpha/F) \leqslant 3^n \cdot u(KL/L').$$

Proof. By Theorem 5.1, there exists a 2-extension L'/F linearly disjoint from K/F with $KL' = KL$ and $\operatorname{ind}\alpha \cdot u(\alpha/F) \leqslant \operatorname{ind}\alpha_{KL} \cdot 3^n \cdot u(\alpha_{KL}/L')$. Since $\alpha_{KL} = 0$, we have $\operatorname{ind}\alpha_{KL} = 1$ and $u(\alpha_{KL}/L') = u(KL/L')$. □

We denote by $\mathrm{I}_{\mathrm{q}}^3 F$ the subgroup of the Witt group of F generated by the Witt equivalence classes of quadratic 3-fold Pfister forms over F. In particular, $\mathrm{I}_{\mathrm{q}}^3 F = 0$ if and only if every quadratic 3-fold Pfister form over F is hyperbolic.

Corollary 5.3. *Assume that* $\mathrm{I}_{\mathrm{q}}^3 F = 0$. *Let* K/F *be a separable quadratic field extension and* $\alpha \in \mathrm{Br}(K)$. *Let* $n \in \mathbb{N}$. *Assume that there exists a 2-extension* L/F *linearly disjoint from* K/F *with* $[L : F] = 2^n$ *and such that* $\alpha_L = 0$. *Then*

$$\operatorname{ind}\alpha \cdot u(\alpha/F) \leqslant 2 \cdot 3^n.$$

Proof. By Corollary 5.2, there exists a 2-extension L'/F linearly disjoint from K/F with $KL' = KL$ such that $\operatorname{ind}\alpha \cdot u(\alpha/F) \leqslant 3^n \cdot u(KL/L')$. As $\mathrm{I}_{\mathrm{q}}^3 F = 0$ and L'/F is a 2-extension, it follows by a repeated use of [6, Theorem 34.22] that $\mathrm{I}_{\mathrm{q}}^3 L' = 0$. Now Proposition 3.7 yields that $u(KL/L') \leqslant 2$. □

Remark 5.4. Note that the bound in Corollary 5.3 does not involve $u(F)$. When $\operatorname{ind}\alpha = 2$ and $\mathrm{I}_{\mathrm{q}}^3 F = 0$, we obtain that $u(\alpha/F) \leqslant 3$. Starting by constructing an example of a central simple algebra of degree 12 with an anisotropic quadratic pair, one can produce an example showing that this bound is optimal. This will be included in a forthcoming article.

Theorem 5.5. *Assume that* F *is nonreal with* $\mathrm{char} F \neq 2$. *Let* K/F *be a quadratic field extension. Let* $\alpha \in \mathrm{Br}(K)$ *be such that* $\operatorname{ind}\alpha \leqslant 4$. *Then* $u(\alpha/F) \leqslant \frac{63}{32} u(F)$.

Proof. We may assume that $\operatorname{ind}\alpha = 4$. It follows by [3, Theorem 7.4] that there exists a separable quadratic field extension L/F linearly disjoint from K/F such that $\operatorname{ind}\alpha_{KL} = 2$. Moreover, in view of Theorem 5.1, we may choose L/F in such way that $u(\alpha/F) \leqslant \frac{3}{2} u(\alpha_{KL}/L)$. Now, $u(\alpha_{KL}/L) \leqslant \frac{7}{8} u(L)$, by Theorem 3.10. Moreover, $u(L) \leqslant \frac{3}{2} u(F)$, by Proposition 3.6. Therefore $u(\alpha/F) \leqslant \frac{63}{32} u(F)$. □

Remark 5.6. Let K/F and α be as in Theorem 5.5. If $\alpha = \beta_K$ for some $\beta \in \mathrm{Br}_2(F)$ with $\operatorname{ind}\beta \leqslant 4$, then Theorem 3.10 yields that $u(\alpha/F) \leqslant \frac{55}{32} u(F)$, which is better than the bound in Theorem 5.5. If we only assume that $\exp\alpha = 2$ and $\operatorname{ind}\alpha = 4$, then we can derive from Theorem 3.10 that $u(\alpha/F) \leqslant \frac{463}{128} u(F)$, by using that $\alpha = (\gamma_1 + \gamma_2 + \gamma_3)_K$ for certain $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{Br}(F)$ with $\operatorname{ind}\gamma_i \leqslant 2$ for $1 \leqslant i \leqslant 3$. Theorem 5.5 yields a sharper bound in this case. Furthermore, if $\exp\alpha = \operatorname{ind}\alpha = 4$, then Theorem 3.10 does not apply, while Theorem 5.5 does.

6. Bounds on the orthogonal u-invariant

In this section, we assume that F is nonreal with $\mathrm{char} F \neq 2$. We study the behavior of the orthogonal u-invariant under multiquadratic field extensions.

Theorem 6.1. *Let* $\alpha \in \mathrm{Br}_2(F)$ *with* $\operatorname{ind}\alpha = 8$. *Then*

$$u^+(\alpha) \leqslant \lfloor \tfrac{87}{64} u(F) \rfloor + \lfloor \tfrac{63}{32} u(F) \rfloor \leqslant \tfrac{213}{64} u(F).$$

Proof. By [13], there exists a separable quadratic field extension K/F such that $\operatorname{ind}\alpha_K = 4$. By Theorem 4.6, we have $u^+(\alpha) \leqslant \frac{1}{2} u^+(\alpha_K) + u(\alpha_K/F)$, and by Proposition 3.6, we have $u(K) \leqslant \frac{3}{2} u(F)$. We obtain by Theorem 3.10

that $u^+(\alpha_K) \leqslant \frac{29}{16}u(K) \leqslant \frac{87}{32}u(F)$. Furthermore, by Theorem 5.5, we have that $u(\alpha_K/F) \leqslant \frac{63}{32}u(F)$. This yields the desired inequality. □

Remark 6.2. Let $\alpha \in \mathrm{Br}_2(F)$ with $\mathrm{ind}\,\alpha = 8$, as in Theorem 6.1. From Proposition 3.8, we would obtain the bound $u^+(\alpha) \leqslant \frac{333}{32}u(F)$. This general bound is now considerably improved by Theorem 6.1.

Note that Theorem 3.10 does not apply here without further assumption. If we assume that $\alpha = \gamma_1 + \gamma_2 + \gamma_3$ for certain $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{Br}(F)$ with $\mathrm{ind}\,\gamma_1 = 2$ for $1 \leqslant i \leqslant 3$, then Theorem 3.10 yields that $u^+(\alpha) \leqslant \frac{197}{64}u(F)$, which is slightly better than the bound obtained in Theorem 6.1.

Proposition 6.3. *Let $\alpha \in \mathrm{Br}(F)$, $n \in \mathbb{N}^+$ and let M/F be a multiquadratic field extension with $[M : F] = 2^n$. There exists a subextension L/F of M/F with $[M : L] = 2$ such that*

$$\mathrm{ind}\,\alpha \cdot u^+(\alpha) \leqslant \mathrm{ind}\,\alpha_M \cdot \big(u^+(\alpha_M) + (3^n - 1) \cdot u(\alpha_M/L)\big).$$

Proof. We prove the statement by induction on n. If $n = 1$, then $[M : F] = 2$, and we conclude by Theorem 4.6 that the claimed inequality holds with $L = F$. Assume now that $n > 1$. We fix a quadratic subextension K/F and a multiquadratic subextension M'/F of M/F linearly disjoint from K/F such that $M = M'K$. Then $\alpha_K \in \mathrm{Br}(K)$ and M/K is a multiquadratic field extension with $[M : K] = 2^{n-1}$. Hence, by the induction hypothesis, there exists a 2-extension L_1/K contained in M/K with $[M : L_1] = 2$ and such that

$$\mathrm{ind}\,\alpha_K \cdot u^+(\alpha_K) \leqslant \mathrm{ind}\,\alpha_M \cdot \big(u^+(\alpha_M) + (3^{n-1} - 1) \cdot u(\alpha_M/L_1)\big).$$

Since M'/F is a 2-extension linearly disjoint from K/F with $[M' : F] = 2^{n-1}$, it follows by Theorem 5.1 that there exists a subextension L_2/F of M/F linearly disjoint from K/F such that $L_2K = M'K = M$ and

$$\mathrm{ind}\,\alpha_K \cdot u(\alpha_K/F) \leqslant 3^{n-1} \cdot \mathrm{ind}\,\alpha_M \cdot u(\alpha_M/L_2).$$

By Theorem 4.6 we have

$$\mathrm{ind}\,\alpha \cdot u^+(\alpha) \leqslant \mathrm{ind}\,\alpha_K \cdot \big(u^+(\alpha_K) + 2 \cdot u(\alpha_K/F)\big)\,.$$

If $u(\alpha_M/L_1) \geqslant u(\alpha_M/L_2)$ then we set $L = L_1$, and otherwise we set $L = L_2$. Then $u(\alpha_M/L_i) \leqslant u(\alpha_M/L)$ for $i = 1, 2$, and we conclude that

$$\mathrm{ind}\,\alpha \cdot u^+(\alpha) \leqslant \mathrm{ind}\,\alpha_M \cdot \big(u^+(\alpha_M) + (3^n - 1) \cdot u(\alpha_M/L)\big).$$ □

Corollary 6.4. *Let $\alpha \in \mathrm{Br}(F)$ and $n \in \mathbb{N}$. Let M/F be a multiquadratic field extension such that $[M : F] = 2^n$ and $\alpha_M = 0$. If $u \in \mathbb{N}$ is such that $u(L) \leqslant u$ for every subextension L/F of M/F, then*

$$\mathrm{ind}\,\alpha \cdot u^+(\alpha) \leqslant \tfrac{3^n+1}{2} \cdot u.$$

In any case, we have

$$\mathrm{ind}\,\alpha \cdot u^+(\alpha) \leqslant \frac{(3^n+2)3^{n-1}}{2^n} \cdot u(F)\,.$$

Proof. If $n = 0$, then $\alpha = 0$, whereby $u^+(\alpha) = u^+(F) = u(F)$, so that both parts of the statement hold trivially. Assume now that $n \geqslant 1$. By Proposition 6.3, there exists a 2-extension L/F contained in M/F with $[M : L] = 2$ and such that

$$\operatorname{ind}\alpha \cdot u^+(\alpha) \leqslant \operatorname{ind}\alpha_M \cdot \big(u^+(\alpha_M) + (3^n - 1)\cdot u(\alpha_M/L)\big).$$

Since $\alpha_M = 0$, we have $\operatorname{ind}\alpha_M = 1$, and hence $u^+(\alpha_M) = u(M)$ and further $u(\alpha_M/L) = u(M/L) \leqslant \frac{1}{2}u(L)$, in view of Proposition 3.7. This yields the first part. As M/F and L/F are 2-extensions with $[M : F] = 2^n$ and $[L : F] = 2^{n-1}$, we obtain by Proposition 3.6 that $u(M) \leqslant (\frac{3}{2})^n \cdot u(F)$ and $u(L) \leqslant (\frac{3}{2})^{n-1} \cdot u(F)$. This yields the second part. □

Most bounds that we presented in this article have strictly weaker hypotheses than previously known bounds. The trade-off is that the bounds that we obtain are also a bit weaker, by comparison.

Remark 6.5. Let $n \in \mathbb{N}$. Consider the condition on F that, for any $r \in \mathbb{N}$, every system of r quadratic forms over F in more than $r \cdot 2^n$ variables has a non-trivial zero over F. With this condition, the proof of [10, Prop. 3.6] yields that $u^+(\alpha) \leqslant (1 + \frac{1}{\operatorname{ind}\alpha}) \cdot 2^{n-1}$ for any $\alpha \in \mathrm{Br}_2(F)$.

Note that the condition on systems of quadratic forms also implies that $u(F') \leqslant 2^n$ for every finite field extension F'/F. However, the bound which we get from [10, Prop. 3.6] is far better than what one would obtain by applying Corollary 6.4 with $u = 2^n$.

However, there are fields F for which it is known that $u(F') \leqslant 2^n$ holds for every finite field extension F'/F, while there is no evidence that the stronger condition on systems of quadratic forms over F is satisfied.

A very interesting such case is that of a rational function field

$$F = \mathbb{Q}_p(t_1, \dots, t_{n-2})$$

in $n-2$ variables, where $n \geqslant 3$, over the field of p-adic numbers $\mathbb{Q}_p$ for a prime number p. Here, it is shown in [9, Prop. 2.4, Cor. 2.7] that, for any $r \in \mathbb{N}$, any systems of r quadratic forms over F in more than $r \cdot 2^n$ variables has a solution in some finite extension of odd degree of F, and this is further used in [9, Theorem 3.4] to show that $u(F') \leqslant 2^n$ for every finite extension F'/F.

Since it is not known whether $u^+(\alpha_L) = u^+(\alpha)$ for any $\alpha \in \mathrm{Br}_2(F)$ and a finite extension of odd degree L/F, the bound from Corollary 6.4 is still the best we might have so far. For $n = 4$, that is, $F = \mathbb{Q}_p(t_1, t_2)$, we obtain for example that $\operatorname{ind}\alpha \cdot u^+(\alpha) \leqslant (3^7 + 1) \cdot 8 = 17504$ for any $\alpha \in \mathrm{Br}_2(F)$. From Theorem 3.10, one can get that $u^+(\alpha) \leqslant 946$, which is better when $\operatorname{ind}\alpha \leqslant 16$. It is unknown whether there exists $\alpha \in \mathrm{Br}_2(F)$ with $\operatorname{ind}\alpha > 16$ over this field F.

In [15, Theorem 1.2], precise values for the u-invariants were determined for algebras with involution over function fields of curves over a p-adic field, where p is an odd prime number.

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