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Thurston construction mapping classes with minimal dilatation

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ABSTRACT. Given a pair of filling curves α, β on a surface of genus g with n punctures, we explicitly compute the mapping classes realizing the minimal dilatation over all the pseudo-Anosov maps given by the Thurston construction on α, β. We do so by solving for the minimal spectral radius in a congruence subgroup of $SL_2(\mathbb{Z})$. We apply this result to realized lower bounds on intersection number between α and β to give the minimal dilatation over any Thurston construction pA map on $\Sigma_{g,n}$ given by a filling pair $\alpha \cup \beta$.

CONTENTS

1. Statement of results

Let $\Sigma_{g,n}$ denote the orientable surface of genus g with n punctures and let $\text{Mod}(\Sigma_{g,n})$ denote the associated mapping class group. If $[f] \in \text{Mod}(\Sigma_{g,n})$ is an isotopy class of pseudo-Anosov (pA) homeomorphisms of $\Sigma_{g,n}$, then there is an associated "stretch factor" $\lambda > 1$ which quantifies the scaling of its stable and unstable foliations ([FM12], Section 13.2.3). This "stretch factor" or *dilatation* λ gives multiple perspectives of f.

Among other things, λ is the growth rate of the unstable foliation of f under iteration and $\log(\lambda)$ is the topological entropy of f ([FM12], Theorem 13.2). In addition, there is a bijective correspondence between the set of dilatations in $\text{Mod}(\Sigma_{g,n})$ and the length spectrum of closed geodesics in the moduli space of $\Sigma_{g,n}$. Finally, $\log(\lambda)$ gives the Teichmüller translation length, or the realized infimum distance that a point in Teichmüller space (under the Teichmüller metric) is translated after action by $\text{Mod}(\Sigma_{g,n})$. Thus, finding minimal dilatation maps extends to minimizing entropy in subsets of the mapping class group,

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the length of closed geodesics in moduli space, and the Teichmüller translation length.

There is extensive literature on the problem of minimizing dilatation over pA maps in $\mathrm{Mod}(\Sigma_{g,n})$ [AD10, FLM11, KT11, KT13, Pen91] as well as on minimizing dilatation of pseudo-Anosov braids on n strands, the case of the disc with n punctures [HK06, LT11]. We consider a specific class of pA maps related to *filling pairs* of curves in $\Sigma_{g,n}$. If α and β are representatives of isotopy classes of simple closed curves a and b on $\Sigma_{g,n}$ and are in minimal position (i.e., the geometric intersection number of a and b equals $|\alpha \cap \beta|$) we say α, β *fill* $\Sigma_{g,n}$ if the complement $\Sigma_{g,n} \setminus (\alpha \cup \beta)$ is a union of topological disks or punctured disks. To any such filling pair $\alpha \cup \beta$, let $\Gamma_{\alpha,\beta}$ be the subgroup generated by Dehn twists about α and β. Thurston showed that any infinite order element of $\Gamma_{\alpha,\beta}$ not conjugate to a power of T_α or T_β is pA (Theorem 1.3). Additionally, we call pseudo-Anosov elements of $\Gamma_{\alpha,\beta} \subset \mathrm{Mod}(\Sigma_{g,n})$ *Thurston pA maps*.

In this paper, we minimize dilatation over all Thurston pA elements in $\Gamma_{\alpha,\beta}$ for any genus g and number of punctures n and find the following:

Theorem 1.1. *For $g \neq 0, 2$, $n > 2$ let α, β be any filling pair on $\Sigma_{g,n}$ and let $i(\alpha, \beta)$ be the geometric intersection number of α and β. Then the minimal dilatation over Thurston pA maps in $\Gamma_{\alpha,\beta}$ is achieved by the product $T_\alpha \cdot T_\beta$. This dilatation equals*

$$\frac{1}{2}((i(\alpha,\beta)^2 + i(\alpha,\beta)\sqrt{(i(\alpha,\beta))^2 - 4} - 2).$$

We find that the minimum dilatation increases monotonically with the geometric intersection number $i(\alpha, \beta)$ for a filling pair α, β. Using realized minimums for intersection number given by Aougab, Huang, and Taylor ([AH15], Lemma 2.1-2.2, [AT14], Lemma 3.1, and summarized in section 3), we prove the following corollary giving a lower bound for the minimal dilatation Thurston pA map for all possible filling pairs.

Corollary 1.2. *The minimal dilatation over all Thurston pA mapping classes in $\Gamma_{\alpha,\beta}$ for all filling pairs α, β in $\Sigma_{g,n}$, $g \neq 0, 2$ is given for $n = 0$ by*

$$\frac{1}{2}((2g-1)^2 + (2g-1)\sqrt{(2g-1)^2 - 4} - 2)$$

and for $n \geq 1$ by

$$\frac{1}{2}((2g-1+n)^2 + (2g-1+n)\sqrt{(2g-1+n)^2 - 4} - 2).$$

Additionally, we have the following characterization:

Genus	*Punctures*	$i(\alpha,\beta)$	*Minimal Dilatation Thurston pA*
$g=0$	$n \geq 4$ *even*	$n-2$	$\frac{1}{2}((n-2)^2+(n-2)\sqrt{(n-2)^2-4}-2)$
$g=0$	n *odd*	$n-1$	$\frac{1}{2}((n-1)^2+(n-1)\sqrt{(n-1)^2-4}-2)$
$g=2$	$n \leq 2$	4	$7+4\sqrt{3}$
$g=2$	$n>2$	$2g+n-2$	$\frac{1}{2}((2g+n-2)^2+(2g+n-2)\sqrt{(2g+n-2)^2-4}-2)$

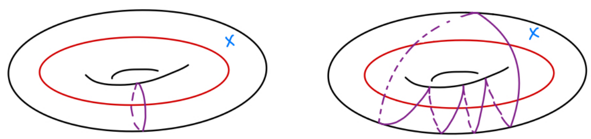

FIGURE 1. Two filling pairs on the punctured torus: the second pair has greater geometric intersection number and consequently corresponds to a higher dilatation pA. Moreover, the composition of Dehn twists about the curves on the first torus is the minimal dilatation pA mapping class.

1.1. Proof idea. To prove Theorem 1.1, we use a theorem due to Thurston, which gives a representation into $\mathrm{SL}_2(\mathbb{R})$ for the subset of $\mathrm{Mod}(\Sigma_{g,n})$ generated by twists about filling pairs of curves ([FM12], Section 14.1).[1]

Theorem 1.3 (Thurston's Construction)**.** *Suppose α,β are simple closed curves in $\Sigma_{g,n}$, $g,n \geq 0$ so that $\alpha \cup \beta$ fill $\Sigma_{g,n}$. Let $i(\alpha,\beta)$ denote geometric intersection number of α and β and let $\Gamma_{\alpha,\beta}$ be the subgroup generated by Dehn twists T_α and T_β about α and β, respectively. Then there is a representation $\rho : \Gamma_{\alpha,\beta} \to \mathrm{SL}_2(\mathbb{Z})$ given by*

$$T_\alpha \mapsto \begin{bmatrix} 1 & -i(\alpha,\beta) \\ 0 & 1 \end{bmatrix} \quad T_\beta \mapsto \begin{bmatrix} 1 & 0 \\ i(\alpha,\beta) & 1 \end{bmatrix}.$$

Moreover, ρ has the following properties:

(i) For $f \in \Gamma_{\alpha,\beta}$, f is periodic, reducible, or pseudo-Anosov if $\rho(f)$ is elliptic, parabolic, or hyperbolic, respectively.
(ii) Parabolic elements in $\rho(f)$ are exactly powers of T_α or T_β.
(iii) If $\rho(f)$ is hyperbolic then the dilatation of $[f] \in \mathrm{Mod}(\Sigma_g)$ is exactly the spectral radius of $\rho(f)$.

[1]The construction also generalizes for multicurves, or disjoint collections of simple closed curves. Here, we only present the Theorem 1.3 for two filling curves. See [FM12], Section 14.1 for the generalized version.

Using Thurston's representation, we minimize dilatation over all mapping classes in $\langle \rho(T_\alpha), \rho(T_\beta)\rangle \subseteq \mathrm{SL}_2(\mathbb{Z})$ to find the minimal dilatation mapping class in $\Gamma_{\alpha,\beta}$. Specifically, the smallest spectral radius matrices in the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ given by

$$\Lambda_n := \left\langle \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \right\rangle,\ n \geq 3$$

achieve the dilatations given in Corollary 1.2.

1.2. Comparison with prior literature. There are interesting comparisons between our bounds in $\Gamma_{\alpha,\beta}$ and universal bounds for the entire mapping class group.

Let $\ell(g,n) = \min\{\log \lambda(f) : f : \Sigma_{g,n} \to \Sigma_{g,n} \text{ pseudo-Anosov}\}$ denote the minimal dilatation for a pseudo-Anosov map f on an orientable surface $\Sigma_{g,n}$ of genus g with n punctures (i.e., the minimal topological entropy). Penner [Pen91] showed that for closed surfaces with $2g-2+n>0$ and $n \geq 0$

$$\ell_{g,n} \geq \frac{\log 2}{12g-12+4n}$$

and for $n=0$

$$\frac{\log 2}{12g-12} \leq \ell_{g,0} \leq \frac{\log 11}{g}.$$

Tsai [Tsa09] later proved that for any $g \geq 2$ and $n \geq 3$ there exists a constant c_g depending only on the genus such that

$$\frac{\log n}{c_g n} < \ell_{g,n} < \frac{c_g \log n}{n}$$

and gave an example of a map $f_{g,n} : \Sigma_{g,n} \to \Sigma_{g,n}$ with $\log \lambda(f) < c_g \log n/n$. In comparison, in the most general case of our bound ($g \neq 0, 2$ and $n=0$), the minimal dilatation in $\Gamma_{\alpha,\beta}$ increases monotonically with genus.

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2. Minimal spectral radii in Λ_n

Recall we defined Λ_n as

$$\Lambda_n = \left\langle \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \right\rangle,\ n \geq 3.$$

The *minimal dilatation* for any hyperbolic map in Λ_n (and thus pA maps in $\Gamma_{\alpha,\beta}$) is given by

$$\inf\{|\lambda(\alpha)| : |\lambda(\alpha)| > 2, \alpha \in \Lambda_n\},$$

where $\lambda(\alpha)$ is the spectral radius of α. Since Λ_n is discrete, this infimum must be realized.

We begin with case when $n = 1$. In this case, the solution is well-known since $\Lambda_1 \simeq SL_2(\mathbb{Z})$ (Theorem 2.5, [FM12]). So, the solution reduces to minimizing the roots of the characteristic polynomial equation

$$x^2 - \mathrm{tr}(\alpha)x + 1.$$

In $\mathrm{SL}_2(\mathbb{Z})$, eigenvalues grow monotonically as a function of trace; the smallest magnitude trace in the infimum is 3, so we have

$$x^2 - 3x + 1 = 0 \implies \lambda = \frac{3+\sqrt{5}}{2}.$$

Now, finding $\alpha = \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right]$ follows immediately from the conditions $w + z = 3, wz - xy = 1$: the solution is given by $\alpha = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Furthermore, α has two distinct real eigenvalues, so this solution is unique up to conjugacy.

For the general case, let

$$A = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}.$$

We will assume $n \neq 2$; later (Remark 3.5) we show that Λ_2 is not the representation given by the Thurston construction for any number g or n.

Theorem 2.1. *Fix $n > 2$. The minimal spectral radius in Λ_n is given by*

$$\frac{1}{2}(n^2 + n\sqrt{n^2-4} - 2)$$

corresponding to the matrix $\left[\begin{smallmatrix} 1-n^2 & -n \\ n & 1 \end{smallmatrix}\right]$.

Fix $n > 2$. In $\mathrm{SL}_2(\mathbb{Z})$, the spectral radius of a matrix α is given by the larger root of the characteristic polynomial

$$x^2 - \mathrm{tr}(\alpha)x + 1 = 0.$$

Explicitly, these solutions are

$$x = \frac{\mathrm{tr}(\alpha) \pm \sqrt{(\mathrm{tr}(\alpha))^2 - 4}}{2}.$$

We wish to minimize spectral radius over hyperbolic matrices, so we assume also that $|\,\mathrm{tr}(\alpha)| > 2$. For $A \in \mathrm{SL}_2(\mathbb{Z})$, it is also known that $\lambda(A)$ increases monotonically as a function of the magnitude of the trace; it follows that minimizing spectral radius is equivalent to minimizing trace magnitude. Here we minimize the latter and then compute the corresponding dilatation.

To begin, we show the following, which was observed initially by Chorna, Geller and Shpilrain (Theorem 4(a), [CGS17]):

Proposition 2.2. *Let $\alpha \in \Lambda_n$, $n > 2$. Then α has the form*

$$\begin{bmatrix} 1+k_1n^2 & k_2n \\ k_3n & 1+k_4n^2 \end{bmatrix} \quad k_i \in \mathbb{Z}.$$

Proof. For simplicity, we say that a matrix γ is *congruent*, denoted

$$\gamma \cong \begin{bmatrix} 1 \mod n^2 & 0 \mod n \\ 0 \mod n & 1 \mod n^2 \end{bmatrix},$$

if γ takes on the form

$$\gamma = \begin{bmatrix} 1+k_1n^2 & k_2n \\ k_3n & 1+k_4n^2 \end{bmatrix}, \quad k_i \in \mathbb{Z}. \tag{2.1}$$

Define $S \subseteq \mathrm{SL}_2(\mathbb{Z})$ as

$$S := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \cong \begin{bmatrix} 1 \mod n^2 & 0 \mod n \\ 0 \mod n & 1 \mod n^2 \end{bmatrix}\right\}.$$

We claim that S is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then, since $A, B \in \mathrm{SL}_2(\mathbb{Z})$, it would follow that every $\gamma \in \Lambda_n$ would take on the form given by 2.1.

To prove the claim, consider the natural homomorphism $\varphi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/n^2\mathbb{Z})$ given by reduction modulo n^2. Then $S = \varphi^{-1}(S')$, where

$$S' := \left\{\begin{bmatrix} 1 & k_1n \\ k_2n & 1 \end{bmatrix} : k_1, k_2 \in \mathrm{SL}_2(\mathbb{Z}/n^2\mathbb{Z})\right\}.$$

We show that S' is a subgroup of $\mathrm{SL}_2(\mathbb{Z}/n^2\mathbb{Z})$. Define $N, M \in \mathrm{SL}_2(\mathbb{Z}/n^2\mathbb{Z})$ as

$$N = \begin{bmatrix} 1 & k_1n \\ k_2n & 1 \end{bmatrix}, M = \begin{bmatrix} 1 & k_3n \\ k_4n & 1 \end{bmatrix}.$$

Then we have:

$$\begin{aligned} NM^{-1} &= \begin{bmatrix} 1 & k_1n \\ k_2n & 1 \end{bmatrix}\begin{bmatrix} 1 & -k_3n \\ -k_4n & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1-k_1k_4n^2 & n(k_1-k_3) \\ n(k_2-k_4) & 1-k_2k_3n^2 \end{bmatrix} \\ &\equiv \begin{bmatrix} 1 & n(k_1-k_3) \\ n(k_2-k_4) & 1 \end{bmatrix} \in S'. \end{aligned}$$

It follows that S' is a subgroup of $\mathrm{SL}_2(\mathbb{Z}/n^2\mathbb{Z})$. Then $S = \varphi^{-1}(S')$, so S is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, giving the desired result. □

Proof of Theorem 2.1. By Proposition 2.2, it suffices to minimize trace over all matrices of the form

$$\alpha = \begin{bmatrix} k_1n^2+1 & k_2n \\ k_3n & k_4n^2+1 \end{bmatrix} \quad \text{such that } k_i \in \mathbb{Z}, (k_1n^2+1)(k_4n^2+1)-k_2k_3n^2 = 1. \tag{2.2}$$

Note that we impose the second constraint equation because $\alpha \in SL_2(\mathbb{Z})$, so we have the determinant

$$(k_1n^2+1)(k_4n^2+1) - k_2k_3n^2 = 1.$$

Rearranging the determinant equation gives $k_2k_3 = k_1k_4n^2 + (k_1+k_4) \in \mathbb{Z}$. Thus, given any fixed $k_1, k_4 \in \mathbb{Z}$, there always exists k_2, k_3 such that the matrix $\begin{bmatrix} 1+k_1n^2 & k_2n \\ k_3n & 1+k_4n^2 \end{bmatrix}$ is in $SL_2(\mathbb{Z})$.

For any α given by 2.2, $|\operatorname{tr}(\alpha)|$ is given by

$$|2 + n^2(k_1+k_4)|$$

which is smallest when $k_1 + k_4 = 0$. In this case, $\operatorname{tr}(\alpha) = 2$. Then α is not hyperbolic, so we disregard it. When $k_1 + k_4 = -1$, then $|\operatorname{tr}(\alpha)| = 2 - n^2$. For $k_1 + k_4 = 1$, then $|\operatorname{tr}(\alpha)| = 2 + n^2 > n^2 - 2$. Finally, for $|k_1 + k_4| > 1$, we have

$$|2 + n^2(k_1+k_4)| \in \{(k_1+k_4)n^2 - 2, (k_1+k_4)n^2 + 2\},$$

which in either case is greater in magnitude than $n^2 - 2$.

It is left to show that a matrix in Λ_n achieves the minimum trace of $n^2 - 2$. Choosing $k_1 = -1, k_4 = 0$ gives the matrix $\begin{bmatrix} 1-n^2 & k_2n \\ k_3n & 1 \end{bmatrix}$, which implies $k_2 = -n$ and $k_3 = n$. But, this matrix is equal to AB, given by $\begin{bmatrix} 1-n^2 & -n \\ n & 1 \end{bmatrix}$. Thus both AB and BA (which are conjugate) in Λ_n achieve the minimum dilatation of $\frac{1}{2}(n^2 + n\sqrt{n^2-4} - 2)$. □

To prove Theorem 1.1, note that for two filling curves α, β on $\Sigma_{g,n}$ where $i(\alpha,\beta) = n$, we have $\Lambda_n = \langle \rho(T_\alpha), \rho(T_\beta) \rangle$ achieves its smallest dilatation map with $\rho(T_\alpha) \cdot \rho(T_\beta)$, since $\rho(T_\alpha) \cdot \rho(T_\beta) = AB \in \Lambda_n$. This map corresponds to $T_\alpha \cdot T_\beta$ in the associated mapping class group.

3. Construction of filling curves

We exposit work given by Aougab-Huang-Taylor [AH15], [AT14] and Jeffreys [Jef19]. For a fixed surface $\Sigma_{g,n}$, our goal is to obtain a lower bound for the intersection number of a pair of filling curves and subsequently construct examples achieving these minima. We use lower bounds given by the filling permutations of Aougab-Huang [AH15] and Aougab-Taylor [AT14] and the generalized filling permutations of Jeffreys [Jef19], which gives us an algebraic way to describe "gluing patterns" of polygons.

The idea is to construct polygons whose sides are identified in such a way that, once glued, they form the surface $\Sigma_{g,n}$ with the glued sides becoming the filling curves α, β. Each polygon will correspond to a disk in the complement of $\alpha \cup \beta$ on $\Sigma_{g,n}$, so we can retroactively puncture the polygons to form $\Sigma_{g,n}$. Since we will "place" the punctures, our convention will be to treat them as marked points and thus exclude them from the Euler characteristic.

We begin with a general lower bound for the intersection number on any surface $\Sigma_{g,n}$ from Aougab-Huang ([AH15], Lemma 2.1).

Lemma 3.1. *Fix $g \geq 1, n \geq 0$. If α, β fill $\Sigma_{g,n}$, then $i(\alpha,\beta) \geq 2g-1$, where i denotes geometric intersection number.*

Proof. We model α, β as a 4-valent graph G (where vertices v are intersection points) since the complement $\Sigma_g \setminus (\alpha,\beta)$ is a union of topological discs D. The Euler characteristic of the graph must match that of $\Sigma_{g,n}$. We know

$$\sum_{v \in G} \deg_v(G) = 2|E| = 4|V| = 2i(\alpha,\beta).$$

Then we obtain

$$\chi(\Sigma_g) = 2 - 2g = |D| - 2i(\alpha,\beta) + i(\alpha,\beta)$$

and since $|D| \geq 1$, we have the result. □

This bound is only realized in the case when $n = 0$. For punctured surfaces, however, we can come very close. To construct an explicit example where equality is realized, we now introduce the notion of *filling permutations* from [AH15] and [Jef19].

Fix a surface $\Sigma_{g,n}$ and a filling pair α, β. We will label the subarcs of the curves (segments connecting two intersection points) in the following manner, beginning with the curve α. Fix an orientation for α and choose a starting intersection point $x_0 \in \alpha \cap \beta$. Travel in the direction of α until we reach an intersection point $x_1 \neq x_0$ and label the subarc of α joining x_0 to x_1 as α_1. Continue this process until we arrive back at x_0–this will occur since the curve α is closed–labeling the subarcs $\{\alpha_1, \dots, \alpha_m\}$; note that $m = i(\alpha,\beta)$. Repeat this process with β to obtain a labeling $\{\beta_1, \dots, \beta_m\}$.

Now, cutting the surface along $\alpha \cup \beta$, we obtain $n+2-2g$ polygons whose sides correspond to subarcs of α and β and whose vertices are intersection points in $\alpha \cap \beta$. Orient these polygons clockwise. Our goal is to describe the polygons algebraically in terms of permutations acting on their edges. First, note that since we cut along α and β to obtain these polygons, every subarc α_k of α will have an inverse α_k^{-1} with the opposite orientation; similarly for β. Define

$$A = \{\alpha_1, \beta_1, \dots, \alpha_m, \beta_m, \alpha^{-1}, \beta^{-1}, \dots, \alpha_m^{-1}, \beta_m^{-1}\}$$

and identity A with the set $\{1, \dots, 4m\}$. Label the sides of the polygons with the corresponding elements of A.

Now, we define the *filling permutation* of a polygon as $\sigma(j) = k$ if, and only if, traveling clockwise around the polygon the edge labeled by the jth element of A is followed by the edge labeled by the kth element of A. Each filling permutation will be a cycle in S_{4m}, the symmetric group on $4m$ elements, so since there are $n+2-2g$ polygons we have $n+2-2g$ corresponding cycles $\sigma \in S_{4m}$.

There are two more geometrically significant permutations we are interested in. Take $Q = Q_{\alpha,\beta} \in S_{4m}$ as $Q = (1, 2, \dots, 4m)^{2m}$. We note that Q acts on the edges by reversing their orientation, i.e., it sends j to k if and only if the jth and

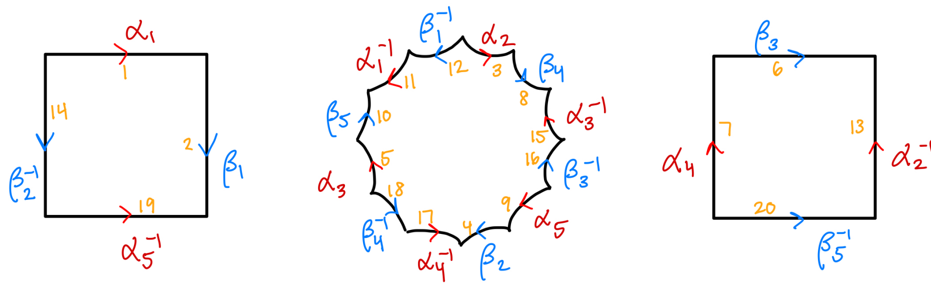

FIGURE 2. The polygons corresponding to a filling pair on $\Sigma_{2,3}$. The associated filling permutations are, from left to right, $(1,2,19,14)$, $(3,8,15,16,9,17,18,5,10,11,12)$, and $(6,13,20,7)$

kth elements of A are inverses of each other. Finally, define $\tau = \tau_{\alpha,\beta} \in S_{4m}$ as

$$\tau = (1,3,...,2m-1)(2,4,...,2m)(4m-1,4m-3,...,2m+1)(4m,4m-2,...,2m+2).$$

The first cycle represents sending α_i to α_{i+1}, the second β_i to β_{i+1}, the third α_k^{-1} to α_{k+1}^{-1} and the fourth β_k^{-1} to β_{k+1}^{-1}. In other words, τ moves each arc in α to the next arc of α with the same orientation, and similarly for β.

We will say that a permutation is *parity-respecting* if it sends even numbers to even numbers and odd numbers to odd numbers and *parity-reversing* if it sends even numbers to odd numbers and odd numbers to even numbers.

The following lemma from Jeffreys ([Jef19], Lemma 2.3) gives the conditions necessary to define a filling permutation on a surface $\Sigma_{g,n}$. We will subsequently construct the filling curves by finding a permutation that satisfies these hypotheses.

Lemma 3.2. *Let α, β be a filling pair on $\Sigma_{g,n}$ with $i(\alpha,\beta) = m \geq i(\alpha,\beta)$, the minimal intersection number. Then, $\sigma = \sigma_{\alpha,\beta}$ satisfies $\sigma Q \sigma = \tau$. Conversely, a parity-reversing permutation $\sigma \in S_{4m}$ consisting of $m+2-2g$ cycles and no more than n 2-cycles that satisfies the above relation defines a filling pair on $\Sigma_{g,n}$ with intersection number m.*

Now we have the necessary ingredients to compute the minimal realized number of intersection points on $\Sigma_{g,n}$; we closely follow the one given by [AT14], Lemma 3.1.

Proposition 3.3. *Suppose $g \neq 0, 2$ and $n = 0$. If α, β are minimally intersecting filling curves on $\Sigma_{g,n}$, then*

$$i(\alpha,\beta) = 2g - 1.$$

If $n \geq 1$, then

$$i(\alpha,\beta) = 2g + n - 2.$$

Proof. Using the same argument as in Lemma 3.1, we have that $i(\alpha,\beta) = 2g + n - 2 + |D|$ where $|D|$ is the number of topological disks in the complement of

$\alpha \cup \beta$ in $\Sigma_{g,n}$. Thus, we have the lower bounds and it is left to show that these bounds are realized. The first case is given explicitly by Lemma 3.2; for the second, we induct on n. When $n = 1$, then $2g-1 = 2g+n-2$. Thus, the filling curves given in Lemma 3.2, which have a single disk D in their complement, still fill $\Sigma_{g,1}$, obtained by puncturing D once.

To begin constructing the filling pairs for surfaces with $n \geq 1$ punctures, we give an example for when $g = 1$; there is a formula for the intersection number of curves on the torus ([FM12], Section 1.2.3). Namely, if α is a (p, q) curve and β is an (r, s) curve, then

$$i(\alpha, \beta) = ps - qr.$$

Taking α to be a $(n, 1)$-curve and β to be a $(0, 1)$ gives two curves intersecting exactly n times. The complement of these two curves is n topological disks, and puncturing each gives $2g + n - 2 = n$ intersections on $\Sigma_{1,n}$.

Now, we describe the *double bigon method,* which begins with a filling pair α, β on $\Sigma_{g,n}$ and constructs a filling pair on $\Sigma_{g,n+2}$ with intersection number $i(\alpha, \beta)+2$. As before, let α, β be a filling pair on $\Sigma_{g,n}$, and orient and label them into subarcs $\alpha_1, \dots, \alpha_{i(\alpha,\beta)}$ and $\beta_1, \dots, \beta_{i(\alpha,\beta)}$. Suppose $i(\alpha_1, \beta_{i(\alpha,\beta)}) \neq 0$. Then pushing α_1 across $\beta_{i(\alpha,\beta)}$ and back over forms 2 bigons. Puncturing each of these bigons gives the same pair of filling curves on $\Sigma_{g,n+2}$ with intersection number $i(\alpha, \beta) + 2$. See Figure 3 for reference.

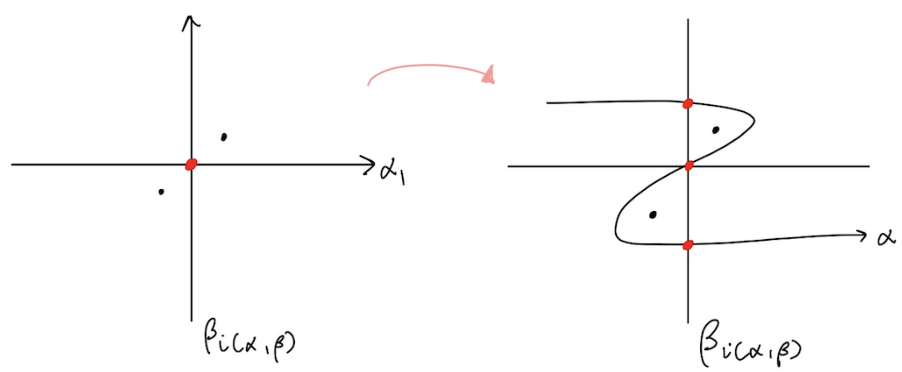

FIGURE 3. The "double bigon method." Given a pair of filling curves α, β on a surface $\Sigma_{g,n}$ with intersection number $i(\alpha, \beta)$, the same pair fills $\Sigma_{g,n+2}$ with intersection number $i(\alpha, \beta) + 2$.

Suppose $n = 2k + 1$ is odd and $g > 2$. Take a pair α, β which fill $\Sigma_{g,0}$, whose complement is connected, i.e. is a single topological disk, and such that $i(\alpha, \beta) = 2g - 1$ (we know such an α, β exist by Lemma 3.2). Then, puncturing this disk gives that α, β fill $\Sigma_{g,1}$. For the remaining $2k$ punctures, perform the double bigon method k times; each time will increase $i(\alpha, \beta)$ by 2 and will result in α, β filling $\Sigma_{g,2k+1}$ with intersection number

$$i(\alpha, \beta) + 2k = (2g - 1) + 2k = 2g - 2 + n.$$

For $n = 2k$ even, the same argument generalizes if there exists a filling pair α, β on $\Sigma_{g,0}$ intersecting $2g$ times; we refer the reader to [AT14], Lemma 3.1, for the construction of such a pair. □

A similar application of the double bigon method gives minimal intersection numbers for $\Sigma_{g,n}$ for $g = 0, 2$ (see [AT14], Lemma 3.1 and [Jef19] Theorem 3.3). We summarize the results as follows:

Genus	Punctures	$i(\alpha, \beta)$
$g = 0$	$n \geq 4$ even	$n - 2$
$g = 0$	$n \geq 4$ odd	$n - 1$
$g = 2$	$n \leq 2$	4
$g = 2$	$n > 2$	$2g + n - 2$

Remark 3.4. The case $g = 0, n < 4$ is not considered because the filling curves have intersection number zero: if there are two or fewer punctures then a single curve fills and if there are exactly three punctures then the filling pair does not intersect.

proof of Corollary 1.2. The proof follows immediately from plugging in the values from Proposition 3.3 into the matrices in Theorem 2.1 and applying Thurston's construction. Fix a surface $\Sigma_{g,n}$, $g \neq 0, 2$ and let $\{\alpha, \beta\}$ be a minimally intersecting filling pair, so that $i(\alpha, \beta) = i_{g,n}$.

Letting $A = \left[\begin{smallmatrix} 1 & -i(g,n) \\ 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ i(g,n) & 1 \end{smallmatrix}\right]$, by Thurston's Construction (Theorem 1.3) the Thurston pA maps in $\Gamma_{\alpha,\beta} \subset \text{Mod}(\Sigma_{g,n})$–the subset of the mapping class generated by Dehn twists T_α, T_β about the curves α and β–correspond to the hyperbolic elements of $\Lambda_{i(\alpha,\beta)} = \langle A, B \rangle$. Moreover, the spectral radius of the elements of $\Lambda_{i(\alpha,\beta)}$ correspond to the dilatation of the pA maps.

By Theorem 2.1, the minimal nonzero spectral radius in $\Lambda_{i(\alpha,\beta)}$ (and thus minimal dilatation in $\Gamma_{\alpha,\beta}$) is given by

$$\frac{1}{2}\left(i(\alpha, \beta)^2 + i(\alpha, \beta)\sqrt{i(\alpha, \beta)^2 - 4} - 2\right)$$

achieved by the hyperbolic matrix

$$AB = \begin{bmatrix} 1 & -i(\alpha, \beta) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ i(\alpha, \beta) & 1 \end{bmatrix} = \begin{bmatrix} 1 - i(\alpha, \beta)^2 & -i(\alpha, \beta) \\ i(\alpha, \beta) & 1 \end{bmatrix}.$$

By Thurston's Construction, AB represents the pA map $T_\alpha T_\beta$, the product of Dehn twists about α and β. The specific values for dilatation in Corollary 1.2 are obtained by substituting the corresponding values of $i_{g,n}$ from Proposition 3.3 for $i(\alpha, \beta)$. □

Remark 3.5. We note that the value of $n = 2$ is never realized for any $\Sigma_{g,n}$ justifying the exclusion of this value in Proposition 3.3.

4. Future directions

Throughout this paper, we exclusively explored the case where A and B are single curves α and β, respectively, but the problem of finding the minimal dilatation Thurston pA map extends to the general case of *multicurves* $A = \{\alpha_1, \dots, \alpha_k\}$, $B = \{\beta_1, \dots, \beta_\ell\}$ on $\Sigma_{g,n}$ (i.e. disjoint collections of simple closed curves). The *multitwist* about A and B are the products $\prod_{i=1}^{n} T_{\alpha_i}, \prod_{i=1}^{m} T_{\beta_i}$, respectively. We recall that the Thurston construction generalizes for multicurve systems which fill $\Sigma_{g,n}$ to obtain a representation $\rho : \Gamma_{A,B} \to \mathrm{SL}_2(\mathbb{Z})$ given by

$$T_A \mapsto \begin{bmatrix} 1 & -\mu^{1/2} \\ 0 & 1 \end{bmatrix} \quad T_B \mapsto \begin{bmatrix} 1 & 0 \\ \mu^{1/2} & 1 \end{bmatrix},$$

where μ is the square of the largest singular value of the $k \times \ell$ intersection matrix N whose (n, m) entry is given by

$$N_{n,m} = i(\alpha_n, \beta_m),$$

i.e., μ is the *Perron-Frobenius eigenvalue* of $N^T N$ (note that we must work with this matrix instead of N since the latter is not necessarily square). We refer the reader to [FM12], Section 14.1.2 for some background on Perron-Frobenius theory. Leininger [Lei04] derived several useful facts regarding minimal pseudo-Anosov dilatation elements in groups generated by multitwists.

However, as we noted after stating Corollary 1.2, twists about two filling curves become less representative of the entire mapping class group with increasing genus. Thus, another question to ask is whether $\lambda(\Gamma_{A,B})$ also increases monotonically with genus, particularly when A, B each consist of g multicurves, i.e. are twists about $2g$ filling curves more characteristic of $\Sigma_{g,n}$, particularly for large g.

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