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ON INTEGRABILITY OF TRIGONOMETRIC SERIES WITH QUASI–MONOTONE COEFFICIENTS

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1. Introduction and results

Consider the formal sine and cosine series

$$g(x)=\sum_1^\infty a_n \sin nx \ \text{ and } \ f(x)=\frac{1}{2}a_0+\sum_1^\infty a_n \cos nx.$$

The following problem has been studied by many authors: if ψ is a given positive function, what hypothesis on $\{a_n\}$ are equivalent to $g\psi \in L(0,\pi)$ or to $f\psi \in L(0,\pi)$?

First assume that the sequence $\{a_n\}$ is monotonely decreasing to zero as $n\to\infty$.

Boas [4] proved that for $\psi(x)=x^{-\gamma}$, $0<\gamma<2$, the following holds

$$g\psi\in L(0,\pi)\Leftrightarrow \sum_1^\infty \frac{\psi(n)}{n}a_n<\infty. \tag{1}$$

Aljančić, Bojanić, Tomić [1] proved that assertion (1) holds true also for $\psi(x)=x^{-\gamma}L\left(\frac{1}{x}\right)$, $0<\gamma<2$, where L is a slowly varying function; and Aljančić [3] obtained a similiar result for $\psi(x)=K\left(\frac{1}{x}\right)$ where $K\in\mathcal{K}$ $(0<\underline{\rho},\overline{\rho}<2)$.

Here $\mathcal{K}(\underline{\rho},\overline{\rho})$ denotes the class of function defined on $I_a=[a,\infty)$, $a>0$, which are 0-*regularly varying* (0-RV) at infinity with lower index $\underline{\rho}$ and upper index $\overline{\rho}$ [2].

More generally, suppose that the sequence $\{a_n\}$ is *quasi-monotone*, i.e. $a_n\geq 0$ and for some $\alpha>0$

$$a_{n+1}\leq a_n\left(1+\frac{\alpha}{n}\right),$$

for n large enough [8].

Shah [6] and Yong [7] proved, for $\psi(x) = x^{-\gamma}$ and $\psi(x) = x^{-\gamma}L\left(\frac{1}{x}\right)$, $0 < \gamma < 2$, respectively, that formula (1) still remains true if $\{a_n\}$ is quasi-monotone and tending to zero as $n \to \infty$.

Moreover, Igari [5] and Yong [7] considered the integrability of ψg^p, where $p \geq 1$.

Similar results are valid for f.

In the present paper we combine the before mentioned results to prove (1) for both $\{a_n\}$ quasi-monotone and ψ 0-regularly varying.

THEOREM 1. *Let $\{a_n\}$ be quasi-monotone and $a_n \to 0$, $n \to \infty$, and let $K \in \mathcal{K}(1 - p < \underline{\rho}, \overline{\rho} < 1 + p)$, for $1 \leq p < \infty$. Then*

$$K\left(\frac{1}{x}\right) g^p(x) \in L(0, \pi) \Leftrightarrow \sum_1^\infty n^{p-2} K(n) {a_n}^p < \infty.$$

THEOREM 2. *Let $\{a_n\}$ be quasi-monotone and $a_n \to 0$, $n \to \infty$, and let $K \in \mathcal{K}(1 - p < \underline{\rho}, \overline{\rho} < 1)$, for $1 \leq p < \infty$. Then*

$$K\left(\frac{1}{x}\right) f^p(x) \in L(0, \pi) \Leftrightarrow \sum_1^\infty n^{p-2} K(n) {a_n}^p < \infty.$$

REMARK. These theorems give on one hand, a generalisation of Theorems 2 and 3 [3], in which the sequence $\{a_n\}$ is monotone and $p = 1$, and on the other hand, a generalisation of Theorems 1 and 2 [7], in which the function K is regularly varying and the sequence $\{a_n\}$ satisfies the further assumption

$$(*) \qquad 0 < M_1 \leq n^\beta L(n) a_n \leq M_2 < \infty, \quad \beta > 0,$$

where L is a slowly varying function and M_1, M_2 are constants.

In the proof of the theorems we shall have need of the following lemmas. They are generalisations of Theorem 1 [5] to 0-regularly varying functions.

By C, possibly with subscripts, we denote a constant; a C may stay for different constants from one appearence to another.

LEMMA 1. *Let $K \in \mathcal{K}(\underline{\rho}, \overline{\rho} < p - 1)$, for $1 \leq p < \infty$.*

1) Let $f \geq 0$ be locally integrable on I_a and $F(t) = \int_a^t f(u)du$. Then

$$\int_a^\infty K(t) \left(\frac{F(t)}{t}\right)^p dt \leq C \int_a^\infty K(t) f^p(t) dt.$$

2) *Let* $c_n \geq 0$ *and* $A_n = \sum_{k=1}^{n} c_k$. *Then*

$$\sum_{1}^{\infty} K(n)\left(\frac{A_n}{n}\right)^p \leq C \sum_{1}^{\infty} K(n){c_n}^p.$$

Lemma 2. *Let* $K \in \mathcal{K}(p-1 < \underline{\rho}, \overline{\rho})$, *for* $1 \leq p < \infty$.

1) *Let* $f \geq 0$ *be locally integrable on* I_a *and* $G(t) = \int_t^\infty f(u)du$. *Then*

$$\int_a^\infty K(t)\left(\frac{G(t)}{t}\right)^p dt \leq C \int_a^\infty K(t) f^p(t) dt.$$

2) *Let* $c_n \geq 0$ *and* $B_n = \sum_{k=n}^{\infty} c_k$. *Then*

$$\sum_{1}^{\infty} K(n)\left(\frac{B_n}{n}\right)^p \leq C \sum_{1}^{\infty} K(n){c_n}^p.$$

Corollary 1. *Let* $K \in \mathcal{K}(\underline{\rho}, \overline{\rho} < 1-p)$, *for* $1 \leq p < \infty$. *Let* $f \geq 0$ *be locally integrable in* $(0, 1/a)$ *and* $G(t) = \int_t^{1/a} f(u)du$. *Then*

$$\int_0^{1/a} K\left(\frac{1}{t}\right)\left(\frac{G(t)}{t}\right)^p dt \leq C \int_0^{1/a} K\left(\frac{1}{t}\right) f^p(t) dt.$$

Corollary 2. *Let* $K \in \mathcal{K}(1-p < \underline{\rho}, \overline{\rho})$, *for* $1 \leq p < \infty$. *Let* $f \geq 0$ *be locally integrable in* $(0, 1/a)$ *and* $F(t) = \int_0^t f(u)du$. *Then*

$$\int_0^{1/a} K\left(\frac{1}{t}\right)\left(\frac{F(t)}{t}\right)^p dt \leq C \int_0^{1/a} K\left(\frac{1}{t}\right) f^p(t) dt.$$

2. Properties of 0-regularly varying functions

In this section we cite some properties of 0-regularly varying functions that are needed in the proof of the theorems. As before, K is assumed to belong to the class $\mathcal{K}(\underline{\rho}, \overline{\rho})$.

(i-1) if $\sigma < \underline{\rho}$ then $\int_a^x t^{-\sigma-1} K(t) dt \leq C_1 x^{-\sigma} K(x)$.

2) if $\tau > \overline{\rho}$ then $\int\limits_x^\infty t^{-\tau-1}K(t)dt \le C_2 x^{-\tau}K(x)$.

(ii) There exist positive constants C_1, C_2 such that

$$C_1 n^{-2}K(n) \le \int\limits_{1/n}^{1/(n-1)} K\left(\frac{1}{x}\right)dx \le C_2 n^{-2}K(n).$$

(iii) If $\sigma < \rho$ then for g locally integrable in $(0,\infty)$

1) $\int\limits_a^\infty x^\sigma g(x)dx \le C_1 \int\limits_a^\infty g(x)K(x)dx$

2) $\int\limits_0^{1/a} x^{-\sigma} g(x)dx \le C_2 \int\limits_0^{1/a} g(x)K\left(\frac{1}{x}\right)dx.$

3. Proofs of the theorems

Proof of Theorem 1. To prove part $\Leftarrow$ consider

$$|g(x)| = \left|\sum_{k=1}^\infty a_k \sin kx\right| \le \sum_{k=1}^n a_k kx + \left|\sum_{k=n+1}^\infty a_k \sin kx\right|.$$

An application of Abel's transformation to the second sum yields

$$|g(x)| \le x\sum_{k=1}^n ka_k + \left|\frac{1}{2\sin\frac{x}{2}}\sum_{k=n}^\infty (a_k - a_{k+1})\left(\cos\left(n+\frac{1}{2}\right)x - \cos\left(k+\frac{1}{2}\right)x\right)\right| \le$$

$$\le x\sum_{k=1}^n ka_k + \frac{\pi}{x}\sum_{k=n}^\infty |a_k - a_{k+1}|.$$

If we put $\frac{\pi}{n} < x \le \frac{\pi}{n-1}$ then we have

$$|g(x)| \le \frac{\pi}{n-1}\sum_{k=1}^n ka_k + n\sum_{k=n}^\infty |a_k - a_{k+1}|. \tag{2}$$

The sequence $\{a_n\}$ being quasi-monotone, it follows by [8, p. 5]

$$\sum_{k=n}^\infty |a_k - a_{k+1}| \le a_n + 2\alpha\sum_{k=n}^\infty \frac{a_k}{k}. \tag{3}$$

Therefore we obtain by substituting (3) into (2)

$$|g(x)| \le \frac{\pi}{n-1}\sum_{k=1}^n ka_k + n\left(a_n + 2\alpha\sum_{k=n}^\infty \frac{a_k}{k}\right), \quad \frac{\pi}{n} < x \le \frac{\pi}{n-1},$$

and introducing the notation $A_n = \sum_{k=1}^{n} ka_k$, $B_n = \sum_{k=n}^{\infty} \frac{a_k}{k}$, we have

$$|g(x)| \leq C\left(\frac{1}{n}A_n + na_n + nB_n\right), \quad \frac{\pi}{n} < x \leq \frac{\pi}{n-1}. \tag{4}$$

Now, this implies that

$$\begin{aligned}\int_0^{\pi} |g(x)|^p K\left(\frac{1}{x}\right) dx &= \sum_{n=1}^{\infty} \int_{\pi/n}^{\pi/(n-1)} |g(x)|^p K\left(\frac{1}{x}\right) dx \leq \\ &\leq C\sum_{n=1}^{\infty}\left(\frac{1}{n}A_n + na_n + nB_n\right)^p \int_{\pi/n}^{\pi/(n-1)} K\left(\frac{1}{x}\right) dx \leq \\ &\leq C_1\sum_{n=1}^{\infty}\left(\frac{1}{n}A_n + na_n + nB_n\right)^p n^{-2}K(n)\end{aligned} \tag{5}$$

(where for the last inequality we have used property (ii) of 0-RV functions).

An application of Minkowsky's inequality yields

$$\begin{aligned}\left[\int_0^{\pi} |g(x)|^p K\left(\frac{1}{x}\right) dx\right]^{1/p} &\leq C_1\left[\sum_{n=1}^{\infty}\left(\frac{1}{n}A_n + na_n + nB_n\right)^p n^{-2}K(n)\right]^{1/p} \leq \\ &\leq C_1\left[\sum_1^{\infty}\left(\frac{A_n}{n}\right)^p n^{-2}K(n)\right]^{1/p} + C_1\left[\sum_1^{\infty} n^p {a_n}^p n^{-2}K(n)\right]^{1/p} + \\ &+ C_1\left[\sum_1^{\infty} n^p {B_n}^p n^{-2}K(n)\right]^{1/p} = S_1 + S_2 + S_3.\end{aligned} \tag{6}$$

To estimate S_1 put $c_n = na_n$ and $K_1(n) = n^{-2}K(n)$. By hypothesis we have $\overline{\rho}(K) < 1+p$. Thus it follows that $\overline{\rho}(K_1) = \overline{\rho}(K) - 2 < 1+p-2 = p-1$, so that K_1 satisfies the assumptions of Lemma 1. Applying the second part of this lemma to S_1 we obtain

$$\begin{aligned}{S_1}^p &= \sum_1^{\infty} n^{-2}K(n)\left(\frac{An}{n}\right)^p = \sum_1^{\infty} K_1(n)\left(\frac{An}{n}\right)^p \leq C\sum_1^{\infty} K_1(n){c_n}^p = \\ &= C\sum_1^{\infty} n^{-2}K(n)n^p{a_n}^p.\end{aligned} \tag{7}$$

To estimate S_3 put $c_n = \frac{a_n}{n}$ and $K_2(n) = n^{2p-2}K(n)$. Since $\underline{\rho}(K) > 1-p$, it follows that $\underline{\rho}(K_2) = \underline{\rho}(K) + 2p - 2 > 1-p+2p-2 = p-1$; thus K_2 satisfies the assumptions of Lemma 2. Applying this lemma to S_3 we obtain

$$(8)\qquad \begin{aligned} {S_3}^p &= \sum_1^\infty n^{2p-2}K(n)\left(\frac{B_n}{n}\right)^p = \sum_1^\infty K_2(n)\left(\frac{B_n}{n}\right)^p \le C\sum_1^\infty K_2(n){c_n}^p = \\ &= C\sum_1^\infty n^{2p-2}K(n)\left(\frac{a_n}{n}\right)^p = C\sum_1^\infty n^{p-2}K(n){a_n}^p. \end{aligned}$$

Finally, from (6), (7) and (8) it follows that

$$\left[\int_0^\pi |g(x)|^p K\left(\frac{1}{x}\right)dx\right]^{1/p} \le C\left[\sum_1^\infty n^{p-2}K(n){a_n}^p\right]^{1/p}.$$

This establishes part $\Leftarrow$ of the theorem.

Remark. If, like in [7], the sequence $\{a_n\}$ is supposed to satisfy the additional assumption $(^*)$, the proof of part $\Leftarrow$ becomes somewhat shorter. Indeed, it is easily seen that in this case $B_n \le Ca_n$; therefore S_3 in (6) is estimated by S_2.

The proof of part $\Rightarrow$ is very similar to the corresponding part in Theorem 1 [7] (assumption $(^*)$ not being used at all for this part). The only difference is that we replace the class of regularly varying functions by the larger class of 0-regularly varying functions.

First note that $K\left(\frac{1}{x}\right) g^p(x) \in L(0,\pi)$ implies $g \in L(0,\pi)$. Indeed,

$$(9)\qquad \int_0^\pi |g(x)|dx \le \left(\int_0^\pi x^{-\beta p}|g(x)|^p dx\right)^{1/p}\left(\int_0^\pi x^{\beta p'}dx\right)^{1/p'}$$

by Hölder's inequality. Chose β such that $0 < \beta p < \underline{\rho}$ (this is possible, since $\rho > p-1 > 0$). For such β both integrals in (9) converge (the first by property (iii-2) of 0-RV functions).

Now since the integrability of g implies that a_n are the Fourier coefficients of g, it follows that

$$G(x) = \int_0^x g(t)dt = \sum_{k=1}^\infty \frac{a_k}{k}(1-\cos kx) = 2\sum_{k=1}^\infty \frac{a_k}{k}\sin^2\frac{kx}{2}.$$

Therefore

$$(10)\qquad \begin{aligned} G\left(\frac{\pi}{n}\right) &= 2\sum_{k=1}^\infty \frac{a_k}{k}\sin^2\frac{k\pi}{2n} \ge 2\sum_{k=[\frac{n}{2}]}^n \frac{a_k}{k}\sin^2\frac{k\pi}{2n} \ge \\ &\ge 2\sum_{k=[\frac{n}{2}]}^n \frac{a_k}{k}\left(\frac{2}{\pi}\frac{k\pi}{2n}\right)^2 \ge \frac{2}{4n}\sum_{k=[\frac{n}{2}]}^n a_k. \end{aligned}$$

Now according to the definitions of quasi-monotone sequences it follows that for $\frac{n}{2} < k \le n$

$$a_k \ge \frac{a_{k+1}}{1+\frac{\alpha}{k}} \ge \dots \ge \frac{a_k}{\left(1+\frac{\alpha}{k}\right)^{n-k}} \ge \frac{a_n}{\left(1+\frac{\alpha}{k}\right)^{k}} \ge a_n e^{-\alpha}.$$

Hence (10) implies that

$$G\left(\frac{\pi}{n}\right) \ge \frac{1}{2n} \sum_{\left[\frac{n}{2}\right]+1}^{n} a_k \ge \frac{1}{2n} a_n e^{-\alpha} n = C a_n. \tag{11}$$

Denoting $\psi(x) = \int_0^x |g(t)| dt$ we have

$$\sum_1^\infty n^{p-2} K(n) a_n{}^p \le C \sum_1^\infty n^{p-2} K(n) G^p\left(\frac{\pi}{n}\right) \le \qquad \text{[by (11)]}$$

$$\le C \sum_1^\infty n^{p-2} K(n) \psi^p\left(\frac{\pi}{n}\right) \le C_1 \sum_1^\infty \psi^p\left(\frac{\pi}{n}\right) \int_{\pi/n}^{\pi/(n-1)} \left(\frac{1}{x}\right)^p K\left(\frac{1}{x}\right) dx \qquad \text{[by (ii)]}$$

$$\le C_1 \sum_1^\infty \int_{\pi/n}^{\pi/(n-1)} \left(\frac{1}{x}\right)^p K\left(\frac{1}{x}\right) \psi^p(x) dx = C_1 \int_0^\pi K\left(\frac{1}{x}\right) \left(\frac{\psi(x)}{x}\right)^p dx \le$$

$$\le C_2 \int_0^\pi K\left(\frac{1}{x}\right) |g(x)|^p dx$$

by Corollary 2 (since $\underline{\rho}(K) > 1-p$). This completes the proof of Theorem 1.

Note that in the proof of part $\Rightarrow$ only condition $\underline{\rho}(K) > 1-p$ is used; therefore no restiction on the upper index is necessary for this part of the theorem.

Proof of Theorem 2. To prove part $\Leftarrow$ consider

$$|f(x)| = \left|\sum_{k=1}^\infty a_k \cos kx\right| \le \sum_{k=1}^n a_k + \left|\sum_{k=n+1}^\infty a_k \cos kx\right| \le$$

$$\le \sum_{k=1}^n a_k + \left|\frac{1}{2\sin\frac{x}{2}} \sum_{k=n}^\infty (a_k - a_{k+1}) \left(\sin\left(n+\frac{1}{2}\right)x - \sin\left(k+\frac{1}{2}\right)x\right)\right| \le$$

$$\le \sum_{k=1}^n a_k + \frac{\pi}{x} \sum_{k=n}^\infty |a_k - a_{k+1}|.$$

If we set $\frac{\pi}{n} < x \le \frac{\pi}{n-1}$ and apply (3) we obtain

$$|f(x)| \le \sum_{k=1}^n a_k + n\left(a_n + 2\alpha \sum_{k=n}^\infty \frac{a_k}{k}\right)$$

or, introducing the notation $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=n}^{\infty} \frac{a_k}{k}$,

$$|f(x)| \le C(A_n + na_n + nB_n).$$

Now we have like in the proof of Theorem 1

$$\begin{aligned}
&\left[\int_0^\pi |f(x)|^p K\left(\frac{1}{x}\right) dx\right]^{1/p} + C\left[\sum_1^\infty (A_n + na_n + nB_n)^p n^{-2} K(n)\right]^{1/p} \le \\
&\le C\left[\sum_1^\infty {A_n}^p n^{-2} K(n)\right]^{1/p} + C\left[\sum_1^\infty n^p {a_n}^p n^{-2} K(n)\right]^{1/p} + \\
&+ C\left[\sum_1^\infty n^p {B_n}^p n^{-2} K(n)\right]^{1/p} = \overline{S}_1 + S_2 + S_3.
\end{aligned} \tag{12}$$

Since in estimating the sum S_3 in (8) only condition $\underline{\rho}(K) > 1 - p$ was used, it follows that (8) remains in this case also.

On the other hand, since $\overline{\rho}(K) < 1$, if we set $K_3(n) = n^{p-2}K(n)$ it follows that $\overline{\rho}(K_3) = \overline{\rho}(K) + p - 2 < 1 + p - 2 = p - 1$. Therefore Lemma 1.2 may be applied to K_3

$$\begin{aligned}
\overline{S_1}^p &= \sum_1^\infty n^{p-2} K(n) \left(\frac{A_n}{n}\right)^p = \sum_1^\infty K_3(n) \left(\frac{A_n}{n}\right)^p \le C \sum_1^\infty K_3(n) {a_n}^p = \\
&= C \sum_1^\infty n^{p-2} K(n) {a_n}^p.
\end{aligned} \tag{13}$$

Finally, from (12), (8) and (13) it follows that

$$\left[\int_0^\pi |f(x)|^p K\left(\frac{1}{x}\right) dx\right]^{1/p} \le C\left[\sum_1^\infty n^{p-2} K(n) {a_n}^p\right]^{1/p}$$

which establishes part $\Leftarrow$ of the theorem.

To prove the converse, first note that $f^p(x) K\left(\frac{1}{x}\right) \in L(0,\pi)$ implies $f \in L(0,\pi)$ (cf. (9)). Therefore

$$F(x) = \int_0^x f(t)dt = \sum_1^\infty \frac{a_k}{k} \sin kx.$$

Denote $\frac{a_k}{k}$ by b_k. It is easily seen that whenever $\{a_n\}$ is quasi-monotone, then so is $\{b_n\}$

$$b_{k+1} = \frac{a_{k+1}}{k+1} \le \frac{a_k}{k+1}\left(1 + \frac{\alpha}{k}\right) \le \frac{a_k}{k}\left(1 + \frac{\alpha}{k}\right) = b_k\left(1 + \frac{\alpha}{k}\right).$$

Thus we can apply part ⇒ of Theorem 1 to the function

$$F(x)=\sum_1^\infty b_k\sin kx$$

and conclude that the for 0-RV functions K_4 such that $\underline{\rho}(K_4)>1-p$ the following holds

$$\sum_1^\infty n^{p-2}K_4(n){b_n}^p\le C\int_0^\pi K_4\left(\frac{1}{x}\right)|F(x)|^p dx.$$

Putting $K_4(n)=n^pK(n)$ (for which $\underline{\rho}(K_4)=p+\underline{\rho}(K)>\underline{\rho}(K)>1-p$) we obtain

$$\sum_1^\infty n^{p-2}n^pK(n){b_n}^p\le C\int_0^\pi K\left(\frac{1}{x}\right)\frac{|F(x)|^p}{x^p}dx. \tag{14}$$

Therefore, since $|F(x)|\le\int_0^x|f(t)|dt\equiv\psi(x)$, relation (14) yields

$$\sum_1^\infty n^{p-2}K(n){a_n}^p\le C\int_0^\pi K\left(\frac{1}{x}\right)\frac{\psi^p(x)}{x^p}dx.$$

Finally we can apply Corollary 1 to the last integral to obtain

$$\sum_1^\infty n^{p-2}K(n){a_n}^p\le C_1\int_0^\pi K\left(\frac{1}{x}\right)|f(x)|^p dx$$

which completes the proof of the theorem.

4. Proofs of the lemmas

PROOF OF LEMMA 1. Since by hypothesis $\overline{\rho}<p-1$ and $p\ge 1$, it follows that $\overline{\rho}<0$. Thus it is possible to chose λ such that $\overline{\rho}<\lambda p<0$.

Now we have by Hölder's inequality

$$F(t)=\int_a^t f(u)du\le\left(\int_a^t u^{\lambda p}f^p(u)du\right)^{1/p}\left(\int_a^t u^{-\lambda p'}du\right)^{1/p'}\le$$
$$\le(Ct^{-\lambda p'+1})^{1/p'}\left(\int_a^t u^{\lambda p}f^p(u)du\right)^{1/p}$$

or

$$F^p(t) \le C_1 t^{-\lambda p + \frac{p}{p'}} \int_a^t u^{\lambda p} f^p(u) du.$$

This implies that

$$(15) \qquad \begin{aligned} \int_a^\infty K(t) \frac{F^p(t)}{t^p} dt &\le C_1 \int_a^\infty K(t) t^{-p} t^{-\lambda p + \frac{p}{p'}} \int_a^t u^{\lambda p} f^p(u) du \; dt = \\ &= C_1 \int_a^\infty u^{\lambda p} f^p(u) \int_u^\infty K(t)^{-\lambda p - 1} dt \; du. \end{aligned}$$

Now since λ is closen such that $\lambda p > \overline{\rho}$, we can apply property (i-2) of 0-RV functions to the inner integral in (15) and obtain

$$\int_a^\infty K(t) \frac{F^p(t)}{t^p} dt \le C_2 \int_a^\infty u^{\lambda p} f^p(u) K(u) u^{-\lambda p} du = C_2 \int_a^\infty f^p(u) K(u) du.$$

This establishes part 1) of the lemma. It is obvious that the proof of part 2) follows along the same lines.

The proof of Lemma 2 is very similar to the proof of Lemma 1 (except that property (i-1) of 0-RV functions has to be used instead of property (i-2)), and we therefore omit it.

Corollaries 1 and 2 follow from Lemma 1 and 2 respectively, by introducing an obvious change of variables in the integrals.

REFERENCES

[1] S. Aljančić, R. Bojanić, M. Tomić, *Sur l'intégrabilité de certaines séries trigonometriques,* Publ. Inst. Math. 8 (1955) 67–84.

[2] S. Aljančić, D. Aranđelović, 0*-regularly varying functions,* Publ. Inst. Math. 22 (36) (1977) 5–22.

[3] S. Aljančić, *Some applications of* 0*-regularly varying functions,* to appear.

[4] R.P. Boas, *Integrability theorems for trigonometric transforms,* Springer-Verlag Berlin 1967.

[5] S. Igari, *Some integrability theorems of trigonometric series and monotone decreasing functions,* Tôhoku Math. J. 12 (1960) 139–146.

[6] S.M. Shah, *Trigonometric series with quasi-monotone coefficients,* Proc. Amer. Math. Soc. 13 (1962) 266–273.

[7] C.-H. Yong, *Integrability theorems of power of trigonometric series with quasi-monotone coefficients,* Math. Zeitschr. 91 (1966) 280–293.

[8] C.-H. Yong, *Asymptotic behaviour of trigonometric series with modified monotone coefficients,* The Chinese University of Hong Kong, 1974.